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Existence Results for a Perturbed Nonlinear Coupled System with Hybrid Instantaneous and Non-instantaneous Impulses

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Abstract. This research paper investigates a coupled system of second-order differential equations characterized by mixed impulsive effects, both instantaneous and non-instantaneous. Our approach employs variational methods and critical point theory to establish the existence and multiplicity of weak solutions. Specifically, we reframe the problem of determining solution existence and multiplicity as an equivalent task of minimizing an energy functional within a suitable function space. The critical points of this energy function correspond to solutions of the impulsive problem at hand. By imposing distinct growth conditions on the nonlinearities and impulsive functions, we rigorously demonstrate the existence of at least one solution and infinitely many solutions for the considered problem.

Key Words and Phrases: impulsive differential equations, instantaneous impulses, non-instantaneous impulses, variational methods, energy functional, critical point theory.

2010 Mathematics Subject Classifications: 34A37, 35A15, 74G65, 35B38

1. Introduction

Impulsive differential equations capture abrupt changes in dynamic systems, representing phenomena like population shifts, power outages, neuronal firing patterns, and economic shifts due to policy changes. In the literature, two predominant types of impulses are recognized. The first type involves instantaneous impulses, where the state change occurs rapidly relative to the overall process duration. Extensive coverage of detailed discussions of the basic theory is available in monographs such as [5, 9, 10, 16, 17, 26]. However, instantaneous impulses have limitations, failing to model certain real-life phenomena like earthquakes and tsunamis. Recognizing these limitations, Hernández and O'Regan

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introduced non-instantaneous impulses in [15]. Unlike instantaneous impulses, non-instantaneous impulses entail an impulsive action that initiates at a predetermined point and persists over a finite time interval. This concept has significantly contributed to the mathematical study of such equations, as evidenced by works like the monograph [1].

Many phenomena necessitate the examination of coupled systems of differential equations within diverse domains like chemistry, physics and biology. This is because a single equation frequently fails to capture the complexity of real-world dynamics. In this paper, we focus on a coupled system of differential equations that incorporates two types of impulses: instantaneous and non-instantaneous. We direct our attention to this coupled system, exploring

$$\begin{cases} -u''(t)+g_i(t)u(t) &= \partial_u f_i(t,u(t),v(t)), \quad t\in(s_i,t_{i+1}],\ i=\overline{0,m},\\ -v''(t)+h_i(t)v(t) &= \partial_v f_i(t,u(t),v(t)), \quad t\in(s_i,t_{i+1}],\ i=\overline{0,m},\\ u'(t_i^+)-u'(t_i^-) &= I_i(u(t_i)), \quad i=\overline{1,m},\\ v'(t_i^+)-v'(t_i^-) &= J_i(v(t_i)), \quad i=\overline{1,m},\\ u'(t) &= u'(t_i^+), \quad t\in(t_i,s_i],\ i=\overline{1,m},\\ v'(t) &= v'(t_i^+), \quad t\in(t_i,s_i],\ i=\overline{1,m},\\ u'(s_i^+) &= u'(s_i^-), \quad i=\overline{1,m},\\ v'(s_i^+) &= v'(s_i^-), \quad i=\overline{1,m},\\ u(0) &= u(T)\ =\ 0,\\ v(0) &= v(T)\ =\ 0, \end{cases} \tag{1}$$

here $0=s_0<t_1<s_1<t_2<s_2<\ldots<t_m<s_m<t_{m+1}=T$. The functions g_i and h_i belong to $L^\infty(s_i,t_{i+1}]$, while the partial derivatives of $f_i(t,u,v)$ with respect to u and v, denoted by $\partial_u f_i$ and $\partial_v f_i$ respectively, are Carathéodory functions defined on $(s_i,t_{i+1}]\times\mathbb{R}^2$. Moreover, $u'(t_i^\pm)=\lim_{t\to t_i^\pm}u'(t)$ and $v'(t_i^\pm)=\lim_{t\to t_i^\pm}v'(t)$, with I_i and J_i being continuous functions on $\mathbb{R}$. In these equations, instantaneous impulses occur at points t_i, whereas non-instantaneous impulses persist over intervals $(t_i,s_i]$ while maintaining a constant derivative.

The existence of solutions to impulsive problems has been explored using various classical tools, including fixed point theory, the theory of analytic semigroups, the comparison method, topological degree theory, and variational methods. A plethora of research has investigated this area, as evidenced by works such as [8, 24, 20, 30, 29, 4, 6]. Notably, the first researchers to use variational methods to solve boundary value problems for impulsive differential equations were Tian and Ge [27] and Nieto and O'Regan [23]. The variational structure of general non-instantaneous impulsive problems was initially studied by Bai and Nieto [3]. Building upon this foundation, Tian and Zhang [28] incorporated instantaneous impulses into non-instantaneous impulsive differential equations. They expanded linear terms into nonlinear ones to address second-order differential equations featuring simultaneous instantaneous and non-instantaneous impulses. Subsequently, through the application of Ekeland's variational principle, they established the existence of classical solutions to the formulated problem. Additionally, Zhang and Liu [35], Zhou, Deng, and Wang [37], and Chen, Gu, and Ma [7] investigated fractional differential equations with both instantaneous and non-instantaneous impulses, extending the results of [28] and obtaining solutions using variational methods. For recent works, see, for example, [19, 32, 34, 36] and the references therein.

In the work of Wu and Liu [31], they examine the subsequent instantaneous impulsive system

$$\begin{cases} -u''(t)+g(t)u(t) &=& \partial_u f(u(t),v(t)), & \text{a.e. } t\in[0,T],\\ -v''(t)+h(t)v(t) &=& \partial_v f(u(t),v(t)), & \text{a.e. } t\in[0,T],\\ u'\left(t_i^+\right)-u'\left(t_i^-\right) &=& I_i\left(u\left(t_i\right)\right), & i=\overline{1,m},\\ v'\left(t_i^+\right)-v'\left(t_i^-\right) &=& J_i\left(v\left(t_i\right)\right), & i=\overline{1,m},\\ u(0) &=& u(T)\ =\ 0, &\\ v(0) &=& v(T)\ =\ 0, & \end{cases} \tag{2}$$

where $0=t_0<t_1<\cdots<t_m<t_{m+1}=T$. Here, g and h belong to $L^\infty[0,T]$, and $\partial_u f,\partial_v f:\mathbb{R}^2\to\mathbb{R}$, $I_i,J_i:\mathbb{R}\to\mathbb{R}$ are continuous functions. They established the existence of at least one nontrivial solution to problem (2) using variational methods. For further insights into recent developments on instantaneous impulsive differential systems, readers are invited to consult [12, 18, 14, 13, 2, 11].

In contrast, Nesraoui, Dellal, Nieto, and Ouahab [22] delved into a system of differential equations featuring non-instantaneous impulses

$$\begin{cases} -u''(t) &=& \partial_u f_i\big(t,u(t)-u(t_{i+1}),v(t)-v(t_{i+1})\big),\ t\in(s_i,t_{i+1}],\ i=\overline{0,m},\\ -v''(t) &=& \partial_v f_i\big(t,u(t)-u(t_{i+1}),v(t)-v(t_{i+1})\big),\ t\in(s_i,t_{i+1}],\ i=\overline{0,m},\\ u'(t) &=& \alpha_i,\quad t\in(t_i,s_i],\ i=\overline{1,m},\\ v'(t) &=& \beta_i,\quad t\in(t_i,s_i],\ i=\overline{1,m},\\ u'(s_i^+) &=& u'(s_i^-),\quad i=\overline{1,m},\\ v'(s_i^+) &=& v'(s_i^-),\quad i=\overline{1,m},\\ u'(0^+) &=& \alpha_0,\\ v'(0^+) &=& \beta_0,\\ u(0) &=& u(T)\ =\ v(0)\ =\ v(T). \end{cases} \tag{3}$$

In this scenario, the impulses commence suddenly at points t_i and maintain a constant derivative (α_i and β_i represent given constants) over the interval $(t_i,s_i]$. The nonlinear functions $\partial_u f_i$ and $\partial_v f_i$ are Carathéodory defined on $(s_i,t_{i+1}]\times\mathbb{R}^2$. The authors established the existence of at least one solution to problem (3).

In the reference [33], Yao investigated the following impulsive differential system, which encompasses both instantaneous and non-instantaneous impulses

$$\begin{cases} -u''(t) &=& \partial_u f_i(t,u(t),v(t)), & t\in(s_i,t_{i+1}]\,, i=\overline{0,m}, \\ -v''(t) &=& \partial_v f_i(t,u(t),v(t)), & t\in(s_i,t_{i+1}]\,, i=\overline{0,m}, \\ u'\left(t_i^+\right)-u'\left(t_i^-\right) &=& I_i\left(u\left(t_i\right)\right), & i=\overline{1,m}, \\ v'\left(t_i^+\right)-v'\left(t_i^-\right) &=& J_i\left(v\left(t_i\right)\right), & i=\overline{1,m}, \\ u'(t) &=& u'\left(t_i^+\right), & t\in(t_i,s_i]\,, i=\overline{1,m}, \\ v'(t) &=& v'\left(t_i^+\right), & t\in(t_i,s_i]\,, i=\overline{1,m}, \\ u'\left(s_i^+\right) &=& u'\left(s_i^-\right), & i=\overline{1,m}, \\ v'\left(s_i^+\right) &=& v'\left(s_i^-\right), & i=\overline{1,m}, \\ u(0) &=& u(T) \;=\; 0, & \\ v(0) &=& v(T) \;=\; 0. & \end{cases} \tag{4}$$

In this problem, instantaneous impulses take place at the points t_i, while non-instantaneous impulses persist over the intervals $(t_i, s_i]$. Utilizing variational methods, the author derived new existence results for solutions.

In this paper, we leverage the insights gleaned from the works related to (2), (3), and (4) to establish connections and amalgamate their findings to address the problem outlined in (1). Specifically, by eliminating non-instantaneous impulses from problem (1), it bears striking resemblance to the one delineated in (2). Similarly, by omitting perturbation terms and instantaneous impulses, (1) closely mirrors (3). Furthermore, disregarding perturbation terms in problem (1) aligns it with (4).

Building upon these parallels, we integrate and extend prior research, leveraging their insights and outcomes to investigate the variational structure of the problem outlined in (1). We demonstrate that the quest to ascertain the existence and multiplicity of solutions can effectively be reframed as an equivalent problem of minimizing some energy functional within a suitable function space. In this context, the critical points of this energy function correspond to solutions of the impulsive problem under consideration. Assuming distinct growth conditions for the nonlinearities and impulsive functions, we establish the existence of at least one solution and infinitely many solutions for (1).

2. Assumptions and main results

Throughout the paper, we make the following set of assumptions.

(A) $\forall i=\overline{0,m}:\ \nu_i>-\lambda_1,$
where $\nu_i := \min\left\{ess\inf_{t\in(s_i,t_{i+1}]} g_i(t), ess\inf_{t\in(s_i,t_{i+1}]} h_i(t)\right\}$ and $\lambda_1 := \frac{\pi^2}{T^2}$

is the first eigenvalue of the Dirichlet problem

$$-u''(t) = \lambda u(t),\ t \in (0,T),\ u(0) = u(T) = 0. \tag{5}$$

(B_1) There exist $a_i, b_i > 0$, and $\gamma_i, \mu_i \in [0,1)$ for $i = \overline{0,m}$, such that for all $(t,x,y) \in (s_i, t_{i+1}] \times \mathbb{R}^2$:

$$|\partial_x f_i(t,x,y)| \leq a_i + b_i|x|^{\gamma_i},\ |\partial_y f_i(t,x,y)| \leq a_i + b_i|y|^{\mu_i}.$$

(B_2) There exist $c_i, d_i > 0$, and $\eta_i \in [0,1), i = \overline{1,m}$, such that for all $x \in \mathbb{R}$

$$|I_i(x)|,\ |J_i(x)| \leq c_i + d_i|x|^{\eta_i}.$$

(C_1) (I) There exist $\alpha > 2$ such that the impulsive functions I_i and $J_i, i = \overline{1,m}$, satisfy for all $x \in \mathbb{R}$

$$0 \leq I_i(x)x \leq \alpha \int_0^x I_i(t)dt, \quad 0 \leq J_i(x)x \leq \alpha \int_0^x J_i(t)dt.$$

(II) There exist $\delta_i > 0$ for $i = \overline{1,m}$, such that for all $x \in \mathbb{R}$

$$\int_0^x I_i(t)dt,\ \int_0^x J_i(t)dt \leq \delta_i|x|^{\alpha}.$$

(C_2) For all $i = \overline{0,m}$, the functions f_i satisfy

(i) There exist $R > 0, K > 0$, and $2 < \alpha < \beta$ such that for a.e. $t \in (s_i, t_{i+1}]$ and for all $x, y \in \mathbb{R}$

$$|x| + |y| \geq R \Longrightarrow f_i(t,x,y) \geq K(|x|^{\beta} + |y|^{\beta}).$$

(ii) For all $t \in (s_i, t_{i+1}]$ and $x, y \in \mathbb{R}$, we have

$$\alpha|f_i(t,x,y)| \leq x\partial_x f_i(t,x,y) + y\partial_y f_i(t,x,y).$$

(iii) For all $t \in (s_i, t_{i+1}]$ and $x, y \in \mathbb{R}$, we have

$$\partial_x f_i(t,x,y),\ \partial_y f_i(t,x,y) = o(|x| + |y|) \text{ as } |x| + |y| \longrightarrow 0.$$

(D_1) For all $i = \overline{0,m}$, the functions $\partial_x f_i(t,x,y)$ and $\partial_y f_i(t,x,y)$ are both odd in terms of the variables x and y.

(D_2) For all $i = \overline{1,m}$, I_i and J_i are odd functions.

(E) For all $i = \overline{1,m}$, I_i and J_i are non-decreasing.

The following theorems encapsulate the key findings presented in this study.

Theorem 1. *Assume that the conditions* (A), (B_1) *and* (B_2) *are fulfilled. Under these assumptions, the problem (1) possesses at least one weak solution.*

Theorem 2. *Let* (A) *and* $(C_1), (C_2)$ *be true. In such a scenario, the problem (1) possesses at least one weak solution.*

The subsequent results relate to the existence of an infinite number of solutions.

Theorem 3. *Assume that the conditions* (A), $(C_1), (C_2)$ *and* $(D_1), (D_2)$ *are satisfied. Under these assumptions, the problem (1) admits an infinite number of weak solutions.*

Theorem 4. *Under the assumptions* (A), (B_2), (C_2), (D_1), (D_2) *and* (E), *the problem (1) exhibits an infinite set of weak solutions.*

3. Prerequisites

This section provides a concise overview of key tools in differential calculus and critical point theory that will be utilized throughout this paper. For a more comprehensive review, we recommend referring to the sources [21, 25].

We use the symbol X to represent a real Banach space.

Definition 1. *(Weakly lower semicontinuous) A functional* $F : X \longrightarrow \mathbb{R}$ *is said to be weakly lower semicontinuous if, for any sequence* $(x_j) \subset X$ *such that* $x_j \rightharpoonup x$ *weakly in* X, *we have*

$$\liminf_{j \longrightarrow \infty} F(x_j) \geq F(x).$$

Remark 1. *[21, Th. 1.2.] If* F *is continuous and convex on* X, *then* F *is weakly lower semicontinuous. As an illustrative application, consider the norm, which is a weakly lower semicontinuous function. Specifically, for any sequence* $(x_j) \subset X$ *converging weakly to* x, *the following inequality holds:*

$$\liminf_{j \longrightarrow \infty} \|x_j\|_X \geq \|x\|_X.$$

Definition 2. *(Minimizing sequence) A sequence* $(x_j) \subset X$ *is termed a minimizing sequence for the functional* $F : X \longrightarrow \mathbb{R}$ *if it satisfies the following condition:*

$$\inf_{x \in X} F(x) = \lim_{j \to \infty} F(x_j).$$

Definition 3 (Coercive). *A functional $F : X \longrightarrow \mathbb{R}$ is said to be coercive if for all $x \in X$,*

$$F(x) \longrightarrow +\infty \quad as \quad \|x\|_X \longrightarrow +\infty.$$

The following theorem represents a classical result in the calculus of variations, ensuring the existence of a critical point for a given functional F defined on a reflexive Banach space.

Theorem 5. *[21, Th. 1.1.] In a reflexive Banach space X, if a function F is weakly lower semicontinuous and possesses a sequence that minimizes it and is bounded, then F reaches its minimum on X.*

Remark 2. *[21, Page 4] The existence of a bounded minimizing sequence is particularly guaranteed when F is coercive.*

The Mountain Pass Theorem is a fundamental result in the calculus of variations, establishing the existence of a critical point for a functional F defined on a Banach space X, subject to certain conditions. Let B_r denote the open ball in X with a radius r centered at the origin, and let ∂B_r denote its boundary. The Palais-Smale condition (a compactness condition), an essential concept for the above-mentioned theorem, is defined as follows:

Definition 4 (Palais-Smale condition (PS)). *[21, Def. 6.3.] The Palais-Smale condition is satisfied if, for any sequence (u_j) in X with $\big(F(u_j)\big)$ bounded and $F'(u_j)$ (the derivative of F at u_j) tending to zero, there exists a convergent subsequence.*

Theorem 6 (Mountain Pass Theorem). *[25, Page 4] Assume that F satisfies the following conditions:*

1. *$F \in \mathcal{C}^1(X, \mathbb{R})$ with $F(0) = 0$.*
2. *F satisfies the Palais-Smale condition (PS).*
3. *There exists $r, \rho > 0$ such that $F_{|\partial B_r} \geq \rho$.*
4. *There is an $u_0 \in X \backslash B_r$ such that $F(u_0) \leq 0$.*

Then F possesses a critical value.

Theorem 7 (Symmetric Mountain Pass Theorem). *[25, Page 5] Assume that F is defined on an infinite-dimensional real Banach space X, with the following conditions:*

1. *$F \in \mathcal{C}^1(X, \mathbb{R})$ with $F(0) = 0$.*

2. *F satisfies (PS).*

3. *F is an even functional.*

4. *$X = Y \bigoplus Z$, with Y being of finite dimension.*

5. *There exists $r, \rho > 0$ such that $F_{|\partial B_r \cap Z} \geq \rho$.*

6. *Whenever W is a finite-dimensional subspace of X, there exists a specific constant $\varsigma = \varsigma(W)$ such that $F(u) \leq 0$ everywhere on $W \backslash B_\varsigma$.*

Then, F attains a sequence of critical values that is unbounded.

4. Functional framework

In this section, we introduce the functional space, denoted as H, and its appropriate norm, which lays the groundwork for our exploration of solutions.

Define $H_0^1(0,T)$ as the Sobolev space equipped with inner norms

$$\|u\|_1 = \left(\int_0^T \left[|u(t)|^2 + \left|u'(t)\right|^2 \right] dt \right)^{\frac{1}{2}} \quad \text{and} \quad \|u\|_2 = \left(\int_0^T \left|u'(t)\right|^2 dt \right)^{\frac{1}{2}}.$$

Poincaré's inequality, given by

$$\forall u \in H_0^1(0,T): \ \lambda_1 \int_0^T u^2(t)dt \leq \int_0^T \left|u'(t)\right|^2 dt, \tag{6}$$

easily implies the equivalence of the norms $\|\cdot\|_1$ and $\|\cdot\|_2$, where $\lambda_1 = \dfrac{\pi^2}{T^2}$ denotes the first eigenvalue associated with problem (5).

Let $H = H_0^1(0,T) \times H_0^1(0,T)$. In this Hilbert space H, for any $(u,v) \in H$, we define the norm as

$$\|(u,v)\| = \left(\|u\|_2^2 + \|v\|_2^2 \right)^{\frac{1}{2}}.$$

Under the assumption (A), we introduce the norm

$$\|(u,v)\|_H = \left[\int_0^T \left|u'(t)\right|^2 dt + \sum_{i=0}^{m} \int_{s_i}^{t_{i+1}} g_i(t) u^2(t) dt + \int_0^T \left|v'(t)\right|^2 dt + \sum_{i=0}^{m} \int_{s_i}^{t_{i+1}} h_i(t) v^2(t) dt \right]^{\frac{1}{2}}.$$

Note that this norm is well-defined. For any $w \in H_0^1(0,T)$, by applying assumption (A) and the use of Poincaré's inequality (6), we can show the following:

$$\int_0^T \left|w'(t)\right|^2 dt + \sum_{i=0}^{m} \int_{s_i}^{t_{i+1}} g_i(t) w^2(t) dt \geq 0.$$

Lemma 1. *If assumption (A) is satisfied, the norms $\|\cdot\|$ and $\|\cdot\|_H$ in the Sobolev space H are equivalent.*

Proof. Assuming (A) is satisfied, we have $\nu_i > -\lambda_1$, allowing us to find $\zeta_i \in (0,1)$ such that $\nu_i \geq -\lambda_1(1-\zeta_i)$. This implies $\nu_i \geq -\lambda_1(1-\zeta)$ for $i = \overline{0,m}$, where $\zeta = \min\{\zeta_i,\ i = \overline{0,m}\}$. Thus, for all $(u,v) \in H$, we get

$$\|(u,v)\|_H^2 \geq \int_0^T \left|u'(t)\right|^2 dt - \lambda_1(1-\zeta) \int_0^T u^2(t) dt + \int_0^T \left|v'(t)\right|^2 dt - \lambda_1(1-\zeta) \int_0^T v^2(t) dt.$$

Applying Poincaré's inequality (6), we obtain

$$\begin{aligned}\|(u,v)\|_H^2 &\geq \int_0^T \left|u'(t)\right|^2 dt - (1-\zeta) \int_0^T \left|u'(t)\right|^2 dt + \int_0^T \left|v'(t)\right|^2 dt - (1-\zeta) \int_0^T \left|v'(t)\right|^2 dt \\ &= \zeta \int_0^T \left|u'(t)\right|^2 dt + \zeta \int_0^T \left|v'(t)\right|^2 dt,\end{aligned}$$

leading to $\|(u,v)\|_H^2 \geq \zeta \|(u,v)\|^2$.

On the other hand, considering $|g|_\infty = \max\left\{\|g_i\|_\infty,\ i = \overline{0,m}\right\}$ and $|h|_\infty = \max\left\{\|h_i\|_\infty,\ i = \overline{0,m}\right\}$ we get

$$\|(u,v)\|_H^2 \leq \int_0^T \left|u'(t)\right|^2 dt + |g|_\infty \int_0^T u^2(t) dt + \int_0^T \left|v'(t)\right|^2 dt + |h|_\infty \int_0^T v^2(t) dt.$$

Referring to (6), we additionally derive

$$\|(u,v)\|_H^2 \leq \left(1 + \frac{|g|_\infty}{\lambda_1}\right) \int_0^T \left|u'(t)\right|^2 dt + \left(1 + \frac{|h|_\infty}{\lambda_1}\right) \int_0^T \left|v'(t)\right|^2 dt,$$

hence $\|(u,v)\|_H^2 \leq \left(1 + \frac{1}{\lambda_1} \max\left\{|g|_\infty, |h|_\infty\right\}\right) \|(u,v)\|^2$.

Thus, the norms $\|\cdot\|$ and $\|\cdot\|_H$ are equivalent. ◄

We conclude this section by defining $\mathcal{C}[0,T]$ as the set of all continuous functions on $[0,T]$ with the natural norm $\|u\|_\infty = \max_{t\in[0,T]} |u(t)|$.

Lemma 2. *There exists a positive constant γ such that for any $(u,v) \in H$, the following inequalities hold:*

$$\|u\|_\infty,\ \|v\|_\infty \leq \gamma \left\|(u,v)\right\|_H.$$

Proof. Utilizing the continuity of the injection from $H_0^1(0,T)$ into $\mathcal{C}[0,T]$, and incorporating Lemma 1, we establish the desired result. ◄

5. Proof of the main results

Below, we prove the core results stated earlier in this paper.

5.1. Variational formula

For each $(\varphi, \psi) \in H$, we have

$$-\int_0^T u''(t)\varphi(t)dt = \int_0^T u'(t)\varphi'(t)dt + \sum_{i=1}^m \big(u'(t_i^+) - u'(t_i^-)\big)\varphi(t_i)$$

$$+\sum_{i=1}^m \big(u'(s_i^+) - u'(s_i^-)\big)\varphi(s_i) = \int_0^T u'(t)\varphi'(t)dt + \sum_{i=1}^m I_i\big(u(t_i)\big)\varphi(t_i). \tag{7}$$

On the other hand,

$$-\int_0^T u''(t)\varphi(t)\ dt = \sum_{i=0}^m \int_{s_i}^{t_{i+1}} \partial_u f_i\big(t, u(t), v(t)\big)\varphi(t)dt -$$

$$\sum_{i=0}^m \int_{s_i}^{t_{i+1}} g_i(t)u(t)\varphi(t)dt - \sum_{i=1}^m \int_{t_i}^{s_i} \frac{d}{dt}\big(u'(t_i^+)\big)\varphi(t)dt =$$

$$= \sum_{i=0}^m \int_{s_i}^{t_{i+1}} \partial_u f_i\big(t, u(t), v(t)\big)\varphi(t)dt - \sum_{i=0}^m \int_{s_i}^{t_{i+1}} g_i(t)u(t)\varphi(t)dt. \tag{8}$$

Consequently, combining (7) and (8), we derive

$$\int_0^T u'(t)\varphi'(t)dt + \sum_{i=1}^m I_i\big(u(t_i)\big)\varphi(t_i)$$
$$= \sum_{i=0}^m \int_{s_i}^{t_{i+1}} \partial_u f_i\big(t, u(t), v(t)\big)\varphi(t)dt - \sum_{i=0}^m \int_{s_i}^{t_{i+1}} g_i(t)u(t)\varphi(t)dt. \tag{9}$$

Likewise, we secure

$$\int_0^T v'(t)\psi'(t)dt + \sum_{i=1}^m J_i\big(v(t_i)\big)\psi(t_i)$$
$$= \sum_{i=0}^m \int_{s_i}^{t_{i+1}} \partial_v f_i\big(t, u(t), v(t)\big)\psi(t)dt - \sum_{i=0}^m \int_{s_i}^{t_{i+1}} h_i(t)v(t)\psi(t)dt. \tag{10}$$

Combining (9) and (10), we get

$$\begin{aligned}\int_0^T u'(t)\varphi'(t)dt + \int_0^T v'(t)\psi'(t)dt + \sum_{i=1}^m I_i\big(u(t_i)\big)\varphi(t_i) + \sum_{i=1}^m J_i\big(v(t_i)\big)\psi(t_i)\\ = \sum_{i=0}^m \int_{s_i}^{t_{i+1}} \partial_u f_i\big(t,u(t),v(t)\big)\varphi(t)dt + \sum_{i=0}^m \int_{s_i}^{t_{i+1}} \partial_v f_i\big(t,u(t),v(t)\big)\psi(t)dt\\ - \sum_{i=0}^m \int_{s_i}^{t_{i+1}} g_i(t)u(t)\varphi(t)dt - \sum_{i=0}^m \int_{s_i}^{t_{i+1}} h_i(t)v(t)\varphi(t)dt. \qquad (11)\end{aligned}$$

The equality (11) leads us to the introduction of a weak solution for problem (1).

Definition 5. *We say $(u,v) \in H$ is a weak solution to problem* (1) *if equation (11) is satisfied for every $(\varphi,\psi) \in H$.*

Now, consider the energy functional $\Phi : H \longrightarrow \mathbb{R}$ corresponding to problem (1), defined as

$$\begin{aligned}\Phi(u,v) &= \frac{1}{2}\int_0^T |u'(t)|^2 dt + \frac{1}{2}\int_0^T |v'(t)|^2 dt + \sum_{i=1}^m \int_0^{u(t_i)} I_i(s)ds + \sum_{i=1}^m \int_0^{v(t_i)} J_i(s)ds\\ &- \sum_{i=0}^m \int_{s_i}^{t_{i+1}} f_i\big(t,u(t),v(t)\big)dt + \frac{1}{2}\sum_{i=0}^m \int_{s_i}^{t_{i+1}} g_i(t)u^2(t)dt + \frac{1}{2}\sum_{i=0}^m \int_{s_i}^{t_{i+1}} h_i(t)v^2(t)dt,\\ &= \frac{1}{2}\|(u,v)\|_H^2 + \sum_{i=1}^m \int_0^{u(t_i)} I_i(s)ds + \sum_{i=1}^m \int_0^{v(t_i)} J_i(s)ds - \sum_{i=0}^m \int_{s_i}^{t_{i+1}} f_i\big(t,u(t),v(t)\big)dt. \qquad (12)\end{aligned}$$

Given the continuity of $\partial_u f_i$, $\partial_v f_i$, I_i, J_i, we easily deduce that $\Phi \in \mathcal{C}^1(H,\mathbb{R})$. The derivative of Φ at $(u,v) \in H$ is, for any $(\varphi,\psi) \in H$, expressed as

$$\begin{aligned}\Phi'(u,v)(\varphi,\psi) &= \int_0^T u'(t)\varphi'(t)dt + \int_0^T v'(t)\psi'(t)dt\\ &+ \sum_{i=1}^m I_i\big(u(t_i)\big)\varphi(t_i) + \sum_{i=1}^m J_i\big(v(t_i)\big)\psi(t_i)\\ &- \sum_{i=0}^m \int_{s_i}^{t_{i+1}} \partial_u f_i\big(t,u(t),v(t)\big)\varphi(t)dt - \sum_{i=0}^m \int_{s_i}^{t_{i+1}} \partial_v f_i\big(t,u(t),v(t)\big)\psi(t)dt\\ &+ \sum_{i=0}^m \int_{s_i}^{t_{i+1}} g_i(t)u(t)\varphi(t)dt + \sum_{i=0}^m \int_{s_i}^{t_{i+1}} h_i(t)v(t)\varphi(t)dt. \qquad (13)\end{aligned}$$

This confirms that the weak solutions of problem (1) correspond to critical points of Φ.

5.2. Proof of Theorem 1

Lemma 3. *The functional Φ defined by (12) is weakly lower semicontinuous and coercive.*

Proof. To demonstrate the weak lower semicontinuity of Φ, consider a sequence $\big((u_j, v_j)\big) \subset H$ such that $(u_j, v_j) \rightharpoonup (u, v)$. Utilizing the compactness of the injection from $H_0^1(0,T)$ into $\mathcal{C}[0,T]$, we observe uniform convergence of (u_j) and (v_j) to u and v, respectively, over the interval $[0,T]$. Given that $\liminf_{j\to\infty} \|(u_j, v_j)\|_H \geq \|(u,v)\|_H$ (see Remark 1), we can conclude that

$$\liminf_{j\longrightarrow\infty} \Phi(u_j, v_j) \geq \frac{1}{2}\|(u,v)\|_H^2 + \sum_{i=1}^{m} \int_0^{u(t_i)} I_i(s)ds + \sum_{i=1}^{m} \int_0^{v(t_i)} J_i(s)ds$$

$$-\sum_{i=0}^{m} \int_{s_i}^{t_{i+1}} f_i\big(t, u(t), v(t)\big)dt = \Phi(u,v).$$

This implies that the functional Φ is weakly lower semicontinuous.

On the other hand, for any $(u,v) \in H$, by $(B_1), (B_2)$ and Lemma 2, we have

$$\begin{aligned}\Phi(u,v) \quad \geq \quad & \frac{1}{2}\|(u,v)\|_H^2 - 2m\max_i\{c_i\}\gamma\|(u,v)\|_H - 2\max_i\{d_i\}\sum_{i=1}^{m}\gamma^{\eta_i+1}\|(u,v)\|_H^{\eta_i+1} \\ & -2T\max_i\{a_i\}\gamma\|(u,v)\|_H - T\max_i\big\{b_i\big\}\sum_{i=0}^{m}\gamma^{\gamma_i+1}\|(u,v)\|_H^{\gamma_i+1} \\ & -T\max_i\big\{b_i\big\}\sum_{i=0}^{m}\gamma^{\mu_i+1}\|(u,v)\|_H^{\mu_i+1}.\end{aligned}$$

Since $\gamma_i + 1, \mu_i + 1, \eta_i + 1 < 2$, it follows that $\lim_{\|(u,v)\|_H\to\infty} \Phi(u,v) = \infty$. In simpler terms, Φ exhibits coercive behavior. ◄

By Lemma 3 and Remark 2, it follows that the functional Φ fulfills all the requirements outlined in Theorem 5. Consequently, Φ possesses a minimum within H, which constitutes a critical point of Φ. Thus, problem (1) admits at least one solution.

5.3. Proof of Theorem 2

It is evident that $\Phi \in \mathcal{C}^1(H, \mathbb{R})$, and since from (C_2)-(ii) we have $f_i(t,0,0) = 0$, it follows that $\Phi(0,0) = 0$. The proof is structured across three segments.

Firstly, we establish that Φ fulfills the (PS) condition. Take a sequence $\big((u_j,v_j)\big)\subset H$ with $\big(\Phi(u_j,v_j)\big)$ bounded and $\Phi'(u_j,v_j)\to 0$. Utilizing (12), (13), (C_1)-(I), and (C_2)-(ii) we conclude that

$$\alpha\Phi(u_j,v_j)-\Phi'(u_j,v_j)(u_j,v_j)\geq\Big(\frac{\alpha}{2}-1\Big)\|(u_j,v_j)\|_H^2.$$

Assuming $\alpha>2$, we deduce that the sequence $\big((u_j,v_j)\big)$ is bounded in H. Furthermore, leveraging the reflexivity of $H_0^1(0,T)$ and the compact injection from $H_0^1(0,T)$ into $\mathcal{C}[0,T]$, and if required, by passing to a subsequence, we can establish the existence of $\big((u_j,v_j)\big)\in H$ such that
$(u_j,v_j)\rightharpoonup(u,v)$ in H, $u_j\to u$, $v_j\to v$ uniformly in $\mathcal{C}[0,T]$, as $j\to+\infty$. Therefore, as $j\to+\infty$, we acquire

$$\begin{aligned}
&\sum_{i=1}^{m}\Big(I_i\big(u_j(t_i)\big)-I_i\big(u(t_i)\big)\Big)\Big(u_j(t_i)-u(t_i)\Big)\to 0,\\
&\sum_{i=1}^{m}\Big(J_i\big(v_j(t_i)\big)-J_i\big(v(t_i)\big)\Big)\Big(v_j(t_i)-v(t_i)\Big)\to 0,\\
&\sum_{i=0}^{m}\int_{s_i}^{t_{i+1}}\Big(\partial_u f_i(t,u_j,v_j)-\partial_u f_i(t,u,v)\Big)\big(u_j-u\big)dt\to 0,\\
&\sum_{i=0}^{m}\int_{s_i}^{t_{i+1}}\Big(\partial_v f_i(t,u_j,v_j)-\partial_v f_i(t,u,v)\Big)\big(v_j-v\big)dt\to 0.
\end{aligned}\tag{14}$$

Moreover, by (13), we have

$$\begin{aligned}
&\big(\Phi'(u_j,v_j)-\Phi'(u,v)\big)(u_j-u,v_j-v)\\
&=\|(u_j-u,v_j-v)\|_H^2+\sum_{i=1}^{m}\Big(I_i\big(u_j(t_i)\big)-I_i\big(u(t_i)\big)\Big)\Big(u_j(t_i)-u(t_i)\Big)\\
&\quad+\sum_{i=1}^{m}\Big(J_i\big(v_j(t_i)\big)-J_i\big(v(t_i)\big)\Big)\Big(v_j(t_i)-v(t_i)\Big)\\
&\quad-\sum_{i=0}^{m}\int_{s_i}^{t_{i+1}}\Big(\partial_u f_i(t,u_j,v_j)-\partial_u f_i(t,u,v)\Big)\big(u_j-u\big)dt\\
&\quad-\sum_{i=0}^{m}\int_{s_i}^{t_{i+1}}\Big(\partial_v f_i(t,u_j,v_j)-\partial_v f_i(t,u,v)\Big)\big(v_j-v\big)dt.
\end{aligned}\tag{15}$$

Given that $\Phi'(u_j,v_j)\to 0$ and $(u_j,v_j)\rightharpoonup(u,v)$, we can infer that

$$\big(\Phi'(u_j,v_j)-\Phi'(u,v)\big)(u_j-u,v_j-v)\to 0,\qquad \text{as } j\to+\infty.\tag{16}$$

Hence, (14), (15), and (16) lead to $\|(u_j - u, v_j - v)\|_H \to 0$ as $j \to +\infty$. In other words, $(u_j, v_j) \to (u, v)$ in H, indicating the satisfaction of the (PS) condition by Φ.

Secondly, we confirm that Φ meets the requirement stated in assumption 3 of Theorem 6. The Sobolev embedding theorem asserts the existence of a positive constant $\kappa > 0$ such that, for any $(u, v) \in H$, the following inequality holds:

$$\|u\|_{L^2}^2 + \|v\|_{L^2}^2 \leq \kappa \|(u, v)\|_H^2. \tag{17}$$

Utilizing conditions (C_2)-(ii) and (C_2)-(iii), we obtain

$$f_i(t, u, v) = o\left(|u|^2 + |v|^2\right), \quad \text{as } |u| + |v| \to 0,$$

Let $\varepsilon = \frac{1}{4\kappa}$. Then there exists $\tau > 0$ such that for any $(u, v) \in H$, the inequality

$$f_i(t, u, v) \leq \frac{1}{4\kappa}\left(|u|^2 + |v|^2\right), \tag{18}$$

holds whenever $|u| + |v| < \tau$.
Furthermore, based on condition (C_1)-(I), we deduce

$$\int_0^{u(t_i)} I_i(s)ds \geq 0 \quad \text{and} \quad \int_0^{v(t_i)} J_i(s)ds \geq 0. \tag{19}$$

It is evident that the condition $\|(u, v)\|_H \leq \frac{\tau}{2\gamma}$, with γ defined in Lemma 2, implies $\|u\|_\infty, \|v\|_\infty < \frac{\tau}{2}$. Consequently, this leads to $|u(t)| + |v(t)| < \tau$ for all $t \in [0, T]$. Hence, employing (12), (17), (18), and (19), we can conclude that

$$\Phi(u, v) \geq \frac{1}{2}\|(u, v)\|_H^2 - \frac{1}{4\kappa}\int_0^T \left(|u|^2 + |v|^2\right) dt \geq \frac{1}{4}\|(u, v)\|_H^2.$$

We set $r = \frac{\tau}{2\gamma}$ and $\rho = \frac{\tau^2}{16\gamma^2}$. With this selection, we can ensure that $\Phi(u, v) \geq \rho > 0$ for any $(u, v) \in \partial B_r$.

Finally, we demonstrate the satisfaction of assumption 4 of Theorem 6. Consider any $(u, v) \in H \backslash (0, 0)$ and define $\chi_i = \left\{t \in (s_i, t_{i+1}] \ / \ |u(t)| + |v(t)| \neq 0\right\}$, $\chi_i^c = \left\{t \in (s_i, t_{i+1}] \ / \ |u(t)| + |v(t)| = 0\right\}$. Let Υ be a positive real number considered large enough. Since (C_2)-(ii) implies that $f_i(t, 0, 0) = 0$, we can then deduce, utilizing (C_2)-(i), that

$$\int_{s_i}^{t_{i+1}} f_i(t, \Upsilon u, \Upsilon v)dt =$$

$$= \int_{\chi_i} f_i(t, \Upsilon u, \Upsilon v)dt + \int_{\chi_i^c} f_i(t, 0, 0)dt \geq K\int_{\chi_i}\left(|\Upsilon u|^\beta + |\Upsilon v|^\beta\right) dt. \tag{20}$$

Next, using (C_1)-(II) and Lemma 2, and taking (20) into account, we get

$$\Phi(\Upsilon u,\Upsilon v)\leq\frac{\Upsilon^2}{2}\|(u,v)\|_H^2+2\gamma^\alpha\|(u,v)\|_H^\alpha\Upsilon^\alpha\sum_{i=1}^m\delta_i-K\Upsilon^\beta\sum_{i=0}^m\int_{\chi_i}\left(|u|^\beta+|v|^\beta\right)dt. \tag{21}$$

Consider any element $(u,v)\in H$, where $\|(u,v)\|_H=\sqrt{2}$. Applying (21), we have

$$\Phi(\Upsilon u,\Upsilon v)\leq\Upsilon^2+\left(2(\sqrt{2}\gamma)^\alpha\sum_{i=1}^m\delta_i\right)\Upsilon^\alpha-\left(K\sum_{i=0}^m\int_{\chi_i}\left(|u|^\beta+|v|^\beta\right)dt\right)\Upsilon^\beta.$$

Given that $2<\alpha<\beta$, this inequality implies that $\lim_{\Upsilon\to+\infty}\Phi(\Upsilon u,\Upsilon v)=-\infty$. Consequently, there exists $\Upsilon_0\in\mathbb{R}_+^\star$ with $\Upsilon_0>r$ such that $\Phi\left(\Upsilon_0 u,\Upsilon_0 v\right)\leq 0$. According to Theorem 6, there is at least one solution for problem (1).

5.4. Proof of Theorem 3

To prove Theorem 3, we utilize Theorem 7. From the proof of Theorem 2, we conclude that $\Phi\in C^1(H,\mathbb{R})$ and $\Phi(0,0)=0$, fulfilling the (PS) condition. Moreover, the conditions (D_1) and (D_2) together signify that Φ exhibits even symmetry.

The equation (5) yields eigenvalues expressed as a sequence of positive numbers $\lambda_n=\left(\frac{n\pi}{T}\right)^2$ for $n=1,2,\ldots$. Denote E_n as the feature space associated with λ_n. Consequently, we have $H_0^1(0,T)=\overline{\bigoplus_{n=1}^{+\infty}E_n}$ and $H=\overline{\bigoplus_{n=1}^{+\infty}E_n\times E_n}$.

Now, let $Y=\bigoplus_{n=1}^{2}E_n\times E_n$ and $Z=\overline{\bigoplus_{n=3}^{+\infty}E_n\times E_n}$. This implies $H=Y\bigoplus Z$, where Y is of finite dimension. As observed in the proof of Theorem 2, there exist constants r and $\rho>0$ such that $\Phi(u,v)\geq\rho$ for any $(u,v)\in\partial B_r\cap Z$. Additionally, following a similar argument as in the proof of Theorem 2, for any (u,v) in a finite-dimensional subspace $W\subset H$, we can deduce that $\lim_{\Upsilon\to+\infty}\Phi(\Upsilon u,\Upsilon v)=-\infty$. Consequently, there exists $\varsigma=\varsigma(W)$ such that $\Phi(\Upsilon u,\Upsilon v)\leq 0$ on $W\backslash B_\varsigma$.

By applying Theorem 7, we can assert that problem (1) admits an infinite number of solutions.

5.5. Proof of Theorem 4

We begin by noting that $\Phi\in\mathcal{C}^1(H,\mathbb{R})$ with $\Phi(0,0)=0$ is an even function. Our first task is to establish that Φ fulfills the (PS) condition. From the proof of Theorem 2 and utilizing (12), (13), (B_2), (C_2)-(ii), and Lemma 2, we get the inequality

$$\alpha\Phi\left(u_j,v_j\right)-\Phi'\left(u_j,v_j\right)\left(u_j,v_j\right)\geq\left(\frac{\alpha}{2}-1\right)\left\|\left(u_j,v_j\right)\right\|_H^2$$
$$-2(\alpha+1)\left(\sum_{i=1}^m c_i\gamma\left\|\left(u_j,v_j\right)\right\|_H+\sum_{i=1}^m d_i\gamma^{\eta_i+1}\left\|\left(u_j,v_j\right)\right\|_H^{\eta_i+1}\right).$$

This confirms that $\big((u_j,v_j)\big)$ is bounded in H. The remainder of the proof, demonstrating the satisfaction of the (PS) condition, closely parallels that in Theorem 2.

Secondly, considering the odd and non-decreasing nature of $I_i, J_i, i=\overline{1,m}$, we conclude that $\int_0^{u(t_i)} I_i(s)ds\geq 0$ and $\int_0^{v(t_i)} J_i(s)ds\geq 0$. By employing similar reasoning as in the proofs of Theorems 2 and 3, we can readily confirm the satisfaction of condition 5 in Theorem 7.

Finally, the proof of condition 6 of Theorem 7 mirrors that in Theorem 3, with assumption (B_2) replacing (C_1)-(II). Thus, according to Theorem 7, problem (1) possesses infinitely many solutions.

Examples

Following are two examples demonstrating the practical use of our findings.

Example 1. *Examine the problem defined for $T=1$ involving both instantaneous and non-instantaneous impulses:*

$$\begin{cases} -u''(t)+\left(\dfrac{1}{t_{i+1}-s_i}(t-s_i)+1\right)u(t) \\ \qquad = e^{-t^2}-\sin\big(u(t)\big)\cos\big(v(t)\big)+|u(t)|^{\frac12},\ t\in(s_i,t_{i+1}],\ i=0,1, \\ -v''(t)+\Big((t-s_i)+(t-s_i)^2\Big)v(t) \\ \qquad = 2e^{-t}-\cos\big(u(t)\big)\sin\big(v(t)\big)+|v(t)|^{\frac13},\ t\in(s_i,t_{i+1}],\ i=0,1, \\ u'(t_1^+)-u'(t_1^-)\ =\ \sin\big(u(t_1)\big)+|u(t_1)|^{\frac56}, \\ v'(t_i^+)-v'(t_i^-)\ =\ 2\cos\big(v(t_1)\big)+|v(t_1)|^{\frac56}, \\ u'(t)\ =\ u'(t_1^+),\ v'(t)\ =\ v'(t_1^+),\quad t\in(t_1,s_1], \\ u'(s_1^+)\ =\ u'(s_1^-),\ v'(s_1^+)\ =\ v'(s_1^-), \\ u(0)\ =\ u(1)\ =v(0)\ =\ v(1)\ =\ 0. \end{cases} \tag{22}$$

First, for $i=0,1$, let $g_i(t)=\frac{1}{t_{i+1}-s_i}(t-s_i)+1$, $\quad h_i(t)=(t-s_i)+(t-s_i)^2$, which leads to $\nu_i=0>-\lambda_1$. Thus, assumption (A) holds.

Next, for $i=0,1$, considering

$$f_i(t,u,v)=e^{-t^2}u+2e^{-t}v+\cos(u)\cos(v)+\frac{2}{3}u|u|^{\frac12}+\frac{3}{4}v|v|^{\frac13},$$

we have $\partial_u f_i(t,u,v) = e^{-t^2} - \sin(u)\cos(v) + |u|^{\frac{1}{2}}$, $\partial_v f_i(t,u,v) = 2e^{-t} - \cos(u)\sin(v) + |v|^{\frac{1}{3}}$, *with* $a_i = 3$, $b_i = 1$, $\gamma_i = \frac{1}{2}$, *and* $\mu_i = \frac{1}{3}$. *Thus,* (B_1) *holds.*

Finally, $I_1(u) = \sin(u) + |u|^{\frac{5}{6}}$, $J_1(v) = 2\cos(v) + |v|^{\frac{5}{6}}$, *and* $c_1 = 2$, $d_1 = 1$, $\eta_1 = \frac{5}{6}$, *satisfying* (B_2).

Therefore, by Theorem 1, the impulsive problem (22) has at least one solution.

Example 2. *Consider the problem defined for* $T = 1$*:*

$$\begin{cases} -u''(t) + \left(\dfrac{1}{t_{i+1} - s_i}(t - s_i) + 1\right)u(t) = \big(u(t)\big)^5, \ t \in (s_i, t_{i+1}], \ i = 0,1, \\ -v''(t) + \left((t - s_i) + (t - s_i)^2\right)v(t) = \big(v(t)\big)^5, \ t \in (s_i, t_{i+1}], \ i = 0,1, \\ u'(t_1^+) - u'(t_1^-) = \big(u(t_1)\big)^3, \ v'(t_i^+) - v'(t_i^-) = \big(v(t_1)\big)^3, \\ u'(t) = u'(t_1^+), \ v'(t) = v'(t_1^+), \quad t \in (t_1, s_1], \\ u'(s_1^+) = u'(s_1^-), \ v'(s_1^+) = v'(s_1^-), \\ u(0) = u(1) = v(0) = v(1) = 0. \end{cases} \tag{23}$$

Obviously, condition (A) *holds. By defining* $f_i(t,u,v) = \frac{u^6+v^6}{6}$, *we obtain* $\partial_u f_i(t,u,v) = u^5$ *and* $\partial_v f_i(t,u,v) = v^5$. *Let* $I_1(u) = u^3$ *and* $J_1(v) = v^3$. *Choosing* $\alpha = 4, \delta_i = \frac{1}{4}, K = \frac{1}{6}$, *and* $\beta = 6$, *after simple calculations, we verify that conditions* (C_1) *and* (C_2) *are satisfied. Hence, problem* (23) *has at least one solution by Theorem 2. Furthermore, conditions* (D_1) *and* (D_2) *are met. Utilizing Theorem 3, it can be concluded that problem* (23) *possesses an infinite number of solutions.*

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