Some New Approaches of Hermite-Hadamard Type Inequalities Pertaining to Generalized Convexity on Coordinates Using Hypergeometric Function

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Abstract. In this study, we develop a class of generalized fractional inequalities by employing (m, n)-polynomial (p_1, p_2)-convex functions defined on the coordinates. A novel integral identity for functions of two variables is established, serving as a key tool in our analysis. Furthermore, we derive new sort of Hermite-Hadamard-type inequality through generalized fractional operators. In addition, we present some new refinements of Hermite-Hadamard type inequality via (m, n)-polynomial (p_1, p_2)-convex functions with the help of hypergeometric functions. This framework provides a unified approach that encompasses several existing concepts, including (m, n)-polynomial harmonic convexity, (m, n)-polynomial convexity, classical harmonic convexity, and classical convexity, all obtained as specific instances of our results. Consequently, the findings presented here not only extend previously known inequalities but also recover a number of recent contributions in the literature as particular cases.

Key Words and Phrases: convex functions, Hermite-Hadamard inequality, Generalized fractional operators.

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1. Introduction

Convexity and its underlying assumptions have grown into an intriguing area of applied mathematics during the past century. Many academics have contributed their knowledge and understanding of this subject by offering revised versions of particular convex function inequalities. It goes without saying that the unique perspective on the concept of convexity consistently yields fresh concepts and significant, pertinent applications in any field of applied sciences. Many mathematical luminaries are constantly working to apply and appreciate the novel

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ideas for the enhancement and beautification of convexity theory. The main one is convex optimization, which makes extensive use of the convexity concept in applications. Applications of convex function theory are widespread, encompassing fields such as engineering [1], finance [2], economics [3], and optimization [4].

In the field of mathematical awareness, the research of integral inequalities in conjunction with convex analysis provides an intriguing and engaging field of inquiry. The strategies and literature of convexity and inequalities have recently attracted a lot of attention in both the present and the past due to their broad perspectives and applications. Of all the inequalities, the Hermite-Hadamard type is the one that is used the most. In practical mathematics, these convex function-based inequalities are fundamental and essential. Because of their crucial role and beneficial significance in many scientific fields, convexity and inequalities have thus been suggested as an intriguing field for researchers. In the fields of probability, information technology, stochastic processes, statistics, integral operator theory, optimization theory, and numerical integration, integral inequalities are remarkably useful. For the literature see [5, 6, 7, 8].

In many real processes, especially physical and biological ones, memory cannot be ignored, so classical calculus is not always enough. Researchers use fractional tools to describe slow diffusion, hereditary materials, and long-range interactions more accurately. Over recent years, interest in this area has grown because these models often match experimental data better than traditional ones. In our view, fractional calculus is becoming a key technique for problems that involve history-dependent behavior. The subject of fractional calculus has many applications in control systems, transform theory, nanotechnology, modeling, fluid flow, mathematical biology, epidemiology, optimal control and physics. For the literature, see [9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21, 22, 23, 24, 25, 26, 27, 28, 29].

The novelty of this work stems from establishing a broader and more flexible class of convexity, namely the (m, n)-polynomial (p_1, p_2)-convex functions on coordinates, within the setting of fractional calculus. Instead of relying on classical convex or harmonic convex structures, we introduce a unified perspective that accommodates a wide range of known convexity types as special cases. A fresh integral identity is derived for functions of two variables, which then allows the development of new Hermite-Hadamard inequalities through generalized fractional operators. This approach not only extends several recent contributions in the literature, but also offers a framework capable of producing many existing results as natural consequences, highlighting the theoretical value and originality of our method.

Recent developments in fractional calculus and generalized convexity have revealed strong links between new integral operators and inequalities. However, most existing works focus only on classical or limited forms of convexity, which

restricts the scope of analytical tools available for complex functions, especially in multivariable settings. This gap motivated our study: to explore a more versatile structure that can unify several well-known convexity classes and provide a stronger foundation for establishing fractional inequalities. By broadening the convexity framework, we aim to create an approach that can accommodate diverse functions and yield meaningful extensions of known inequality results.

The main aim of this research is to investigate new fractional inequalities via (m, n)-polynomial (p_1, p_2)-convexity on coordinates. We seek to establish a fresh integral identity for functions of two variables and use it to derive Hermite-Hadamard fractional operators. Through this new setting, we intend to unify and extend various known convexity concepts and demonstrate that many classical and recent inequalities appear as direct consequences of our findings.

The organization of this paper is as follows. In Section 2, we revisit several fundamental concepts and definitions that form the foundation for our subsequent analysis. In Section 3, we present new approach of Hermite-Hadamard inequality via (m, n)-polynomial (p_1, p_2)-convexity on coordinates. In Section 4, we investigate some refinements of Hermite-Hadamard type inequality. We provide a brief conclusion and propose some potential future research directions in the last Section 5.

2. Preliminaries

For the sake of reader interest, conciseness, utility, and worth, it seems worthwhile to revisit and focus on a few theorems, remarks, and definitions in this section. This section's foremost intent is to examine and explore specific associated notions and themes that are essential to our inquiry in later sections of this paper. Convexity and the generalized form of the H-H inequality are first examined and studied in the framework of classical calculus. We also went over the convexity on coordinates, generalized fractional integral operator (Riemann-Liouville and Hadamard fractional operator) and Gaussian hypergeometric function. We conclude this section by (m, n)-polynomial (p_1, p_2)-convex function.

Definition 1. *[30] The inequality*

$$\mathcal{F}\left(t\wp_1 + (1 - t)\wp_2\right) \le t\mathcal{F}\left(\wp_1\right) + (1 - t)\mathcal{F}\left(\wp_2\right) \tag{1}$$

is said to be convex, if it holds for all $\wp_1, \wp_2 \in I$ and $t \in [0, 1]$.

The most popular and frequently referenced inequality in the research is the Hermite-Hadamard inequality. Numerous mathematicians have approached the study of inequalities from various perspectives.

Theorem 1. *[31] If $\mathcal{F} : [\wp_1, \wp_2] \to \mathbb{R}$ is a convex function, then*

$$\mathcal{F}\left(\frac{\wp_1 + \wp_2}{2}\right) \leq \frac{1}{\wp_2 - \wp_1} \int_{\wp_1}^{\wp_2} \mathcal{F}(x)dx \leq \frac{\mathcal{F}(\wp_1) + \mathcal{F}(\wp_2)}{2}. \tag{2}$$

Despite having many types and generalizations, the idea of a convexity has also been applied to different spaces. Dragomir touches on the problem of transferring convex functions to multiple dimensions in [32]. Due to its extensive use in numerous inequalities and applications in various mathematical fields, particularly convex programming, this modification is highly appealing.

Definition 2. *Let us consider a bi-dimensional interval $\Delta = [\wp_1, \wp_2] \times [\wp_3, \wp_4]$ in $\mathbb{R}^2$ with $\wp_1 < \wp_2$ and $\wp_3 < \wp_4$. A function $\mathcal{F} : \Delta \to \mathbb{R}$ will be called convex on the coordinates if the partial mappings $\mathcal{F}_y : [\wp_1, \wp_2] \to \mathbb{R}$, $\mathcal{F}_y(u) = \mathcal{F}(u, y)$, and $\mathcal{F}_x : [\wp_3, \wp_4] \to \mathbb{R}$, $\mathcal{F}_x(v) = \mathcal{F}(x, v)$ are convex defined for all $y \in [\wp_3, \wp_4]$ and $x \in [\wp_1, \wp_2]$.*

Recall that the mapping $\mathcal{F} : \Delta \to \mathbb{R}$ is convex on Δ if the following inequality holds:

$$\mathcal{F}(\lambda x + (1 - \lambda)z, \mu y + (1 - \mu)w) \leq \lambda \mathcal{F}(x, y) + (1 - \lambda)\mathcal{F}(z, w)$$

for all $(x, y), (z, w) \in \Delta$ and $\lambda, \mu \in [0, 1]$.

Theorem 2. *Suppose that $\mathcal{F} : \Delta = [\wp_1, \wp_2] \times [\wp_3, \wp_4] \to \mathbb{R}$ is convex on the coordinates on Δ. Then one has the inequalities*

$$\mathcal{F}\left(\frac{\wp_1 + \wp_2}{2}, \frac{\wp_3 + \wp_4}{2}\right)$$
$$\leq \frac{1}{2}\left[\frac{1}{\wp_2 - \wp_1}\int_{\wp_1}^{\wp_2}\mathcal{F}\left(x, \frac{\wp_3 + \wp_4}{2}\right)dx + \frac{1}{\wp_4 - \wp_3}\int_{\wp_3}^{\wp_4}\mathcal{F}\left(\frac{\wp_1 + \wp_2}{2}, y\right)dy\right]$$
$$\leq \frac{1}{(\wp_2 - \wp_1)(\wp_4 - \wp_3)}\int_{\wp_1}^{\wp_2}\int_{\wp_3}^{\wp_4}\mathcal{F}(x, y)\,dy\,dx$$
$$\leq \frac{1}{4}\left[\frac{1}{\wp_2 - \wp_1}\int_{\wp_1}^{\wp_2}\mathcal{F}(x, \wp_3)\,dx + \frac{1}{\wp_2 - \wp_1}\int_{\wp_1}^{\wp_2}\mathcal{F}(x, \wp_4)\,dx\right.$$
$$\left. + \frac{1}{\wp_4 - \wp_3}\int_{\wp_3}^{\wp_4}\mathcal{F}(\wp_1, y)\,dy + \frac{1}{\wp_4 - \wp_3}\int_{\wp_3}^{\wp_4}\mathcal{F}(\wp_2, y)\,dy\right]$$
$$\leq \frac{\mathcal{F}(\wp_1, \wp_3) + \mathcal{F}(\wp_1, \wp_4) + \mathcal{F}(\wp_2, \wp_3) + \mathcal{F}(\wp_2, \wp_4)}{4}.$$

The above inequalities are sharp.

Definition 3. *[33] Let $\Delta \in \Re$ be an interval. Then a real value $\mathcal{F} : \Delta \to \Re$ is said to be harmonically convex on Δ if the inequality*

$$\mathcal{F}\left(\frac{\wp_1 \wp_2}{t\wp_1 + (1-t)\wp_2}, \frac{uw}{su + (1-s)w} \right)$$

$$\leq ts\mathcal{F}(\wp_2, w) + t(1-s)\mathcal{F}(\wp_2, u) + (1-t)s\mathcal{F}(u, w) + (1-t)(1-s)\mathcal{F}(\wp_1, u)$$

holds $\forall \wp_1, \wp_2, u, w \in \Delta$ and $t, s \in [0,1]$.

Katugampola introduced the new fractional integral to generalize the fractional Riemann-Liouville and Hadamard integrals in one single way (see [34, 35, 36, 37]).

Definition 4. *Let $\alpha, \gamma, \rho_1, \rho_2 > 0$. Then the left and right sided generalized fractional integrals of the function on Δ are defined by*

$$I^{\alpha,\gamma}_{\wp_1^+, \wp_3^+} \mathcal{F}(x, y) =$$

$$= \frac{\rho_1^{1-\alpha} \rho_2^{1-\gamma}}{\Gamma(\alpha)\Gamma(\gamma)} \int_{\wp_1}^{x} \int_{\wp_3}^{y} \frac{t^{\rho_1 - 1} s^{\rho_2 - 1}}{(x^{\rho_1} - t^{\rho_1})^{1-\alpha}(y^{\rho_2} - s^{\rho_2})^{1-\gamma}} \mathcal{F}(t, s)\, ds\, dt, \quad x > \wp_1, y > \wp_3,$$

$$I^{\alpha,\gamma}_{\wp_1^+, \wp_4^-} \mathcal{F}(x, y) =$$

$$= \frac{\rho_1^{1-\alpha} \rho_2^{1-\gamma}}{\Gamma(\alpha)\Gamma(\gamma)} \int_{\wp_1}^{x} \int_{y}^{\wp_4} \frac{t^{\rho_1 - 1} s^{\rho_2 - 1}}{(x^{\rho_1} - t^{\rho_1})^{1-\alpha}(s^{\rho_2} - y^{\rho_2})^{1-\gamma}} \mathcal{F}(t, s)\, ds\, dt, \quad x > \wp_1, y < \wp_4,$$

$$I^{\alpha,\gamma}_{\wp_2^-, \wp_3^+} \mathcal{F}(x, y) =$$

$$= \frac{\rho_1^{1-\alpha} \rho_2^{1-\gamma}}{\Gamma(\alpha)\Gamma(\gamma)} \int_{x}^{\wp_2} \int_{\wp_3}^{y} \frac{t^{\rho_1 - 1} s^{\rho_2 - 1}}{(t^{\rho_1} - x^{\rho_1})^{1-\alpha}(y^{\rho_2} - s^{\rho_2})^{1-\gamma}} \mathcal{F}(t, s)\, ds\, dt, \quad x < \wp_2, y > \wp_3,$$

$$I^{\alpha,\gamma}_{\wp_2^-, \wp_4^-} \mathcal{F}(x, y) =$$

$$= \frac{\rho_1^{1-\alpha} \rho_2^{1-\gamma}}{\Gamma(\alpha)\Gamma(\gamma)} \int_{x}^{\wp_2} \int_{y}^{\wp_4} \frac{t^{\rho_1 - 1} s^{\rho_2 - 1}}{(t^{\rho_1} - x^{\rho_1})^{1-\alpha}(s^{\rho_2} - y^{\rho_2})^{1-\gamma}} \mathcal{F}(t, s)\, ds\, dt, \quad x < \wp_2, y < \wp_4.$$

Remark 1. *Definition 4 leads to the conclusion that*
(1) If $\rho_1 = \rho_2 = 1$, then we get the Riemann–Liouville fractional integral operators

$$\left(I^{\alpha,\gamma}_{\wp_1^+, \wp_3^+} \mathcal{F} \right)(x, y) = \frac{1}{\Gamma(\alpha)\Gamma(\gamma)} \int_{\wp_1}^{x} \int_{\wp_3}^{y} (x - t)^{\alpha - 1}(y - s)^{\gamma - 1} \mathcal{F}(t, s)\, ds\, dt,$$

$$\left(I^{\alpha,\gamma}_{\wp_1^+,\wp_4^-}\mathcal{F}\right)(x,y) = \frac{1}{\Gamma(\alpha)\Gamma(\gamma)} \int_{\wp_1}^{x} \int_{y}^{\wp_4} (x-t)^{\alpha-1}(s-y)^{\gamma-1}\mathcal{F}(t,s)\, ds\, dt,$$

$$\left(I^{\alpha,\gamma}_{\wp_2^-,\wp_3^+}\mathcal{F}\right)(x,y) = \frac{1}{\Gamma(\alpha)\Gamma(\gamma)} \int_{x}^{\wp_2} \int_{\wp_3}^{y} (t-x)^{\alpha-1}(y-s)^{\gamma-1}\mathcal{F}(t,s)\, ds\, dt,$$

$$\left(I^{\alpha,\gamma}_{\wp_2^-,\wp_4^-}\mathcal{F}\right)(x,y) = \frac{1}{\Gamma(\alpha)\Gamma(\gamma)} \int_{x}^{\wp_2} \int_{y}^{\wp_4} (t-x)^{\alpha-1}(s-y)^{\gamma-1}\mathcal{F}(t,s)\, ds\, dt.$$

(2) If $\rho_1, \rho_2 \to 0$, then we obtain the Hadamard fractional integral operators

$$\left(I^{\alpha,\gamma}_{\wp_1^+,\wp_3^+}\mathcal{F}\right)(x,y) = \frac{1}{\Gamma(\alpha)\Gamma(\gamma)} \int_{\wp_1}^{x} \int_{\wp_3}^{y} \left(\log\frac{x}{t}\right)^{\alpha-1} \left(\log\frac{y}{s}\right)^{\alpha-1} \frac{\mathcal{F}(t,s)}{ts}\, ds\, dt,$$

$$\left(I^{\alpha,\gamma}_{\wp_1^+,\wp_4^-}\mathcal{F}\right)(x,y) = \frac{1}{\Gamma(\alpha)\Gamma(\gamma)} \int_{\wp_1}^{x} \int_{y}^{\wp_4} \left(\log\frac{x}{t}\right)^{\alpha-1} \left(\log\frac{s}{y}\right)^{\alpha-1} \frac{\mathcal{F}(t,s)}{ts}\, ds\, dt,$$

$$\left(I^{\alpha,\gamma}_{\wp_2^-,\wp_3^+}\mathcal{F}\right)(x,y) = \frac{1}{\Gamma(\alpha)\Gamma(\gamma)} \int_{x}^{\wp_2} \int_{\wp_3}^{y} \left(\log\frac{t}{x}\right)^{\alpha-1} \left(\log\frac{y}{s}\right)^{\alpha-1} \frac{\mathcal{F}(t,s)}{ts}\, ds\, dt,$$

$$\left(I^{\alpha,\gamma}_{\wp_2^-,\wp_4^-}\mathcal{F}\right)(x,y) = \frac{1}{\Gamma(\alpha)\Gamma(\gamma)} \int_{x}^{\wp_2} \int_{y}^{\wp_4} \left(\log\frac{t}{x}\right)^{\alpha-1} \left(\log\frac{s}{y}\right)^{\alpha-1} \frac{\mathcal{F}(t,s)}{ts}\, ds\, dt.$$

Now, we also need to introduce the Beta function β and Gaussian hypergeometric function $_2F_1$ defined by

$$\beta(x,y) = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)} = \int_0^1 t^{x-1}(1-t)^{y-1}dt \quad (x,y>0)$$

and

$$_2F_1(x,y;z;u) = \frac{1}{\beta(y,z-y)} \int_0^1 t^{y-1}(1-t)^{z-y-1}(1-ut)^{-x}dt \quad (z>y>0, |x|<1)$$

respectively, where $\Gamma(u) = \int_0^\infty e^{-t}t^{u-1}dt$ is the Euler gamma function.

Saad et al. [38], for the first time defined the (m,n)-polynomial (p_1,p_2)-convex function on coordinates.

Definition 5. *Let $m, n \in N$, $p_1, p_2 > 0$ and $\Delta \subseteq \Re$ be a (p_1, p_2)- convex interval. Then the non-negative real valued function $\mathcal{F} : \Delta \to [0, \infty) \times [0, \infty)$ is said to be (m,n)-polynomial (p_1, p_2)-convex function if the inequality*

$$\mathcal{F}\left([t\wp_1^{p_1} + (1-t)\wp_2^{p_1}]^{\frac{1}{p_1}}, [su^{p_2} + (1-s)w^{p_2}]^{\frac{1}{p_2}}\right)$$

$$\leq \frac{1}{n}\sum_{i=1}^{n}[1 - (1-t)^i]\frac{1}{m}\sum_{j=1}^{m}[1 - (1-s)^j]\mathcal{F}(\wp_1, u)$$

$$+\frac{1}{n}\sum_{i=1}^{n}[1 - (1-t)^i]\frac{1}{m}\sum_{j=1}^{m}[1 - s^j]\mathcal{F}(\wp_1, w)$$

$$+\frac{1}{n}\sum_{i=1}^{n}[1 - t^i]\frac{1}{m}\sum_{j=1}^{m}[1 - (1-s)^j]\mathcal{F}(\wp_2, u) + \frac{1}{n}\sum_{i=1}^{n}[1 - t^i]\frac{1}{m}\sum_{j=1}^{m}[1 - s^j]\mathcal{F}(\wp_2, w),$$

holds.

Remark 2. *Definition 5 leads to the following conclusions:*

1. *If $m = n = p_1 = p_2 = 1$, then Definition 5 reduces to convex function.*

2. *If $m = n = 1$, then Definition 5 reduces to p_1, p_2-convex function.*

3. *If $p_1 = p_2 = -1$, then Definition 5 reduces to (m, n)-polynomial harmonically convex function.*

4. *If $m = n = 1$ and $p_1 = p_2 = -1$, then Definition 5 reduces to harmonically convex function.*

5. *If $m = n = 1$, then Definition 5 reduces to (p_1, p_2)- harmonically convex function.*

Example 1. *Assume that $\mathcal{F}_\alpha : (0, \infty) \times (0, \infty) \to \mathbb{R}$ is a family of (m, n)-polynomial (p_1, p_2)-convex functions, and $\mathcal{F}(x, y) = \sup \mathcal{F}_\alpha(x, y)$. If*

$$K = \{u \in [\wp_1, \wp_4] \subset (0, \infty), \ v \in [\wp_3, \wp_4] \subset (0, \infty)\}$$

is nonempty, then K is a bi-dimensional interval and f is an (m, n)-polynomial (p_1, p_2)-convex function on K.

Example 2. *Let us consider the function*

$$f(x, y) = x^2 y^3, \quad x, y \geq 0$$

with parameters $m = n = 2$ and $p_1 = p_2 = 1$. Take

$$x = 1, \quad y = 2, \quad u = 1, \quad w = 3, \quad t = s = 0.5.$$

Step 1: Compute the left-hand side (LHS):

$$LHS = f([tx^{p_1} + (1-t)y^{p_1}]^{1/p_1}, [su^{p_2} + (1-s)w^{p_2}]^{1/p_2})$$

$$[tx^{p_1} + (1-t)y^{p_1}]^{1/p_1} = 0.5 \cdot 1 + 0.5 \cdot 2 = 1.5$$

$$[su^{p_2} + (1-s)w^{p_2}]^{1/p_2} = 0.5 \cdot 1 + 0.5 \cdot 3 = 2$$

$$LHS = f(1.5, 2) = (1.5)^2 \cdot (2)^3 = 2.25 \cdot 8 = 18$$

Step 2: Compute the right-hand side (RHS) weighted sum:

$$RHS = \frac{1}{n} \sum_{i=1}^{n} [1 - (1-t)^i] \frac{1}{m} \sum_{j=1}^{m} [1 - (1-s)^j] f(corners)$$

Step 2a: Compute sums of weights:

$$\frac{1}{n} \sum_{i=1}^{2} [1 - (1-t)^i] = \frac{1}{2}\left(1 - 0.5 + 1 - 0.25\right) = 0.625$$

$$\frac{1}{m} \sum_{j=1}^{2} [1 - (1-s)^j] = \frac{1}{2}\left(1 - 0.5 + 1 - 0.25\right) = 0.625$$

Step 2b: Evaluate function at corners:

$$f(1,1) = 1, \quad f(1,3) = 27, \quad f(2,1) = 4, \quad f(2,3) = 108$$

Step 2c: Compute weighted contributions:

$$0.625 \cdot 0.625 \cdot f(1,1) = 0.390625$$

$$0.625 \cdot (1 - 0.625) \cdot f(1,3) = 6.328125$$

$$(1 - 0.625) \cdot 0.625 \cdot f(2,1) = 0.9375$$

$$(1 - 0.625) \cdot (1 - 0.625) \cdot f(2,3) = 15.1875$$

Step 2d: Sum contributions:

$$RHS = 0.390625 + 6.328125 + 0.9375 + 15.1875 \approx 22.84375$$

Step 3: Verify the inequality:

$$LHS = 18 \leq RHS \approx 22.84$$

Hence, the (m,n)-polynomial (p_1, p_2)-convexity inequality is satisfied numerically.

3. New Approach of Hermite-Hadamard type inequality pertaining to generalized convexity mapping

The Hermite-Hadamard inequality is one of the most elegant and influential results connected with convex functions. It provides a sharp estimate for the value of a convex function at the midpoint of an interval in comparison with its average value over that interval. This simple geometric principle has opened the door to a wide range of analytical techniques and has inspired extensive research in modern inequality theory. Beyond its theoretical beauty, the inequality plays a meaningful role in applied settings, including numerical approximation, optimization methods, and integral estimation problems. Many generalized versions have since appeared in the literature, especially in the context of fractional calculus and generalized convexity, reflecting its importance and flexibility in contemporary mathematical analysis.

Theorem 3. *Let $m, n \in \mathbb{N}$, $p_1, p_2 > 0$, and $\eta_1, \eta_2 > 0$. Suppose that $\Delta \subset (0, \infty) \times (0, \infty)$ is a (p_1, p_2)–convex domain, and let $\wp_1, \wp_2, \wp_3, \wp_4 \in \Delta$ satisfy $\wp_1 < \wp_2$ and $\wp_3 < \wp_4$. Assume that $\mathcal{F} : \Delta \to \mathbb{R}$ is an (m, n)–polynomial (p_1, p_2)–convex function and $\mathcal{F} \in L_1(\Delta)$. Then, the following inequality holds:*

$$\frac{1}{4}\left(\frac{2^n n}{2^n(n-1)+1}\right)\left(\frac{2^m m}{2^m(m-1)+1}\right)\mathcal{F}\left(\left[\frac{\wp_1^{p_1}+\wp_2^{p_1}}{2}\right]^{\frac{1}{p_1}}, \left[\frac{\wp_3^{p_2}+\wp_4^{p_2}}{2}\right]^{\frac{1}{p_2}}\right)$$

$$\leq \frac{p_1^{\eta_1} p_2^{\eta_2} \Gamma(\eta_1+1)\Gamma(\eta_2+1)}{4\left(\wp_2^{p_1}-\wp_1^{p_1}\right)^{\eta_1}\left(\wp_4^{p_2}-\wp_3^{p_2}\right)^{\eta_2}}$$

$$\times \left[{}^{p_1, p_2} I_{\wp_1^+, \wp_3^+}^{\eta_1, \eta_2} \mathcal{F}(\wp_2, \wp_4) + {}^{p_1, p_2} I_{\wp_1^+, \wp_4^-}^{\eta_1, \eta_2} \mathcal{F}(\wp_2, \wp_3) + \right.$$

$$\left. + {}^{p_1, p_2} I_{\wp_2^-, \wp_3^+}^{\eta_1, \eta_2} \mathcal{F}(\wp_1, \wp_4) + {}^{p_1, p_2} I_{\wp_2^-, \wp_4^-}^{\eta_1, \eta_2} \mathcal{F}(\wp_1, \wp_3)\right]$$

$$\leq \frac{\eta_1 \eta_2}{4}\left[\frac{\mathcal{F}(\wp_1, \wp_3) + \mathcal{F}(\wp_1, \wp_4) + \mathcal{F}(\wp_2, \wp_3) + \mathcal{F}(\wp_2, \wp_4)}{mn}\right]$$

$$\times \sum_{i=1}^{n}\left[\frac{\eta_1 + 2i}{\eta_1(\eta_1+i)} - \beta(i+1, \eta_1)\right]\sum_{j=1}^{m}\left[\frac{\eta_2 + 2j}{\eta_2(\eta_2+j)} - \beta(j+1, \eta_2)\right]$$

Proof. Let $x, y, u, w \in \Delta$ and $t = s = \frac{1}{2}$ in Definition 5. Then it follows

$$\mathcal{F}\left(\left[\frac{x^{p_1}+y^{p_1}}{2}\right]^{\frac{1}{p_1}}, \left[\frac{u^{p_2}+w^{p_2}}{2}\right]^{\frac{1}{p_2}}\right) \leq \frac{1}{n}\sum_{i=1}^{n}\left(1 - (\tfrac{1}{2})^i\right)\frac{1}{m}\sum_{j=1}^{m}\left(1 - (\tfrac{1}{2})^j\right)$$

$$\times \left[\mathcal{F}(x,u) + \mathcal{F}(x,w) + \mathcal{F}(y,u) + \mathcal{F}(y,w) \right]. \tag{3}$$

Putting $x^{p_1} = t\wp_1^{p_1} + (1-t)\wp_2^{p_1}$, $y^{p_1} = t\wp_2^{p_1} + (1-t)\wp_1^{p_1}$, $u^{p_2} = s\wp_3^{p_2} + (1-s)\wp_4^{p_2}$, $w^{p_2} = s\wp_4^{p_2} + (1-s)\wp_3^{p_2}$ leads to

$$\mathcal{F}\left(\left[\frac{x^{p_1} + y^{p_1}}{2} \right]^{\frac{1}{p_1}}, \left[\frac{u^{p_2} + w^{p_2}}{2} \right]^{\frac{1}{p_2}} \right) \leq \frac{1}{n} \sum_{i=1}^{n} \left(1 - (\tfrac{1}{2})^i \right) \frac{1}{m} \sum_{j=1}^{m} \left(1 - (\tfrac{1}{2})^j \right) \times$$

$$\left[\mathcal{F}\left([t\wp_1^{p_1} + (1-t)\wp_2^{p_1}]^{\frac{1}{p_1}}, [s\wp_3^{p_2} + (1-s)\wp_4^{p_2}]^{\frac{1}{p_2}} \right) \right.$$

$$+ \mathcal{F}\left([t\wp_1^{p_1} + (1-t)\wp_2^{p_1}]^{\frac{1}{p_1}}, [s\wp_4^{p_2} + (1-s)\wp_3^{p_2}]^{\frac{1}{p_2}} \right) +$$

$$+ \mathcal{F}\left([t\wp_2^{p_1} + (1-t)\wp_1^{p_1}]^{\frac{1}{p_1}}, [s\wp_3^{p_2} + (1-s)\wp_4^{p_2}]^{\frac{1}{p_2}} \right) +$$

$$\left. + \mathcal{F}\left([t\wp_2^{p_1} + (1-t)\wp_1^{p_1}]^{\frac{1}{p_1}}, [s\wp_4^{p_2} + (1-s)\wp_3^{p_2}]^{\frac{1}{p_2}} \right) \right]. \tag{4}$$

Multiplying both sides of (4) by $\frac{\eta_1 \eta_2}{4} t^{\eta_1 - 1} s^{\eta_2 - 1}$ and integrating the obtained inequality with respect to t, s over $(0,1)$ give

$$\frac{1}{4} \left(\frac{2^{n+1} n}{2^n(n-1)+1} \right) \left(\frac{2^{m+1} m}{2^m(m-1)+1} \right) \mathcal{F}\left(\left[\frac{\wp_1^{p_1} + \wp_2^{p_1}}{2} \right]^{\frac{1}{p_1}}, \left[\frac{\wp_3^{p_2} + \wp_4^{p_2}}{2} \right]^{\frac{1}{p_2}} \right)$$

$$\leq \frac{\eta_1 \eta_2}{4} \left[\int_0^1 \int_0^1 t^{\eta_1 - 1} s^{\eta_2 - 1} \mathcal{F}\left(\left[t\wp_1^{p_1} + (1-t)\wp_2^{p_1} \right]^{\frac{1}{p_1}}, \left[s\wp_3^{p_2} + (1-s)\wp_4^{p_2} \right]^{\frac{1}{p_2}} \right) dt\,ds \right.$$

$$+ \int_0^1 \int_0^1 t^{\eta_1 - 1} s^{\eta_2 - 1} \mathcal{F}\left(\left[t\wp_1^{p_1} + (1-t)\wp_2^{p_1} \right]^{\frac{1}{p_1}}, \left[s\wp_4^{p_2} + (1-s)\wp_3^{p_2} \right]^{\frac{1}{p_2}} \right) dt\,ds$$

$$+ \int_0^1 \int_0^1 t^{\eta_1 - 1} s^{\eta_2 - 1} \mathcal{F}\left(\left[t\wp_2^{p_1} + (1-t)\wp_1^{p_1} \right]^{\frac{1}{p_1}}, \left[s\wp_3^{p_2} + (1-s)\wp_4^{p_2} \right]^{\frac{1}{p_2}} \right) dt\,ds$$

$$\left. + \int_0^1 \int_0^1 t^{\eta_1 - 1} s^{\eta_2 - 1} \mathcal{F}\left(\left[t\wp_2^{p_1} + (1-t)\wp_1^{p_1} \right]^{\frac{1}{p_1}}, \left[s\wp_4^{p_2} + (1-s)\wp_3^{p_2} \right]^{\frac{1}{p_2}} \right) dt\,ds \right]$$

$$\leq \frac{\eta_1 \eta_2}{4} \left[\frac{p_1 p_2}{(\wp_2^{p_1} - \wp_1^{p_1})(\wp_4^{p_2} - \wp_3^{p_2})} \right.$$

$$\times \int_{\wp_1}^{\wp_2} \int_{\wp_3}^{\wp_4} \left(\frac{\wp_2^{p_1} - x^{p_1}}{\wp_2^{p_1} - \wp_1^{p_1}} \right)^{\eta_1 - 1} \left(\frac{\wp_4^{p_2} - y^{p_2}}{\wp_4^{p_2} - \wp_3^{p_2}} \right)^{\eta_2 - 1} \frac{\mathcal{F}(x,y)}{x^{1-p_1} y^{1-p_2}} dx\,dy$$

$$+ \frac{p_1 p_2}{(\wp_2^{p_1} - \wp_1^{p_1})(\wp_4^{p_2} - \wp_3^{p_2})}$$

$$\times \int_{\wp_1}^{\wp_2} \int_{\wp_3}^{\wp_4} \left(\frac{\wp_2^{p_1} - x^{p_1}}{\wp_2^{p_1} - \wp_1^{p_1}}\right)^{\eta_1 - 1} \left(\frac{y^{p_2} - \wp_4^{p_2}}{\wp_4^{p_2} - \wp_3^{p_2}}\right)^{\eta_2 - 1} \frac{\mathcal{F}(x,y)}{x^{1-p_1} y^{1-p_2}} dx dy$$

$$+ \frac{p_1 p_2}{(\wp_2^{p_1} - \wp_1^{p_1})(\wp_4^{p_2} - \wp_3^{p_2})}$$

$$\times \int_{\wp_1}^{\wp_2} \int_{\wp_3}^{\wp_4} \left(\frac{x^{p_1} - \wp_2^{p_1}}{\wp_2^{p_1} - \wp_1^{p_1}}\right)^{\eta_1 - 1} \left(\frac{\wp_4^{p_2} - y^{p_2}}{\wp_4^{p_2} - \wp_3^{p_2}}\right)^{\eta_2 - 1} \frac{\mathcal{F}(x,y)}{x^{1-p_1} y^{1-p_2}} dx dy$$

$$+ \frac{p_1 p_2}{(\wp_2^{p_1} - \wp_1^{p_1})(\wp_4^{p_2} - \wp_3^{p_2})}$$

$$\left. \times \int_{\wp_1}^{\wp_2} \int_{\wp_3}^{\wp_4} \left(\frac{x^{p_1} - \wp_2^{p_1}}{\wp_2^{p_1} - \wp_1^{p_1}}\right)^{\eta_1 - 1} \left(\frac{y^{p_2} - \wp_4^{p_2}}{\wp_4^{p_2} - \wp_3^{p_2}}\right)^{\eta_2 - 1} \frac{\mathcal{F}(x,y)}{x^{1-p_1} y^{1-p_2}} dx dy \right].$$

Now,

$$\frac{1}{4}\left(\frac{2^{n+1}n}{2^n(n-1)+1}\right)\left(\frac{2^{m+1}m}{2^m(m-1)+1}\right)\mathcal{F}\left(\left[\frac{\wp_1^{p_1} + \wp_2^{p_1}}{2}\right]^{\frac{1}{p_1}}, \left[\frac{\wp_3^{p_2} + \wp_4^{p_2}}{2}\right]^{\frac{1}{p_2}}\right)$$

$$\leq \frac{p_1^{\eta_1} p_2^{\eta_2} \Gamma(\eta_1+1)\Gamma(\eta_2+1)}{4(\wp_2^{p_1} - \wp_1^{p_1})^{\eta_1}(\wp_4^{p_2} - \wp_3^{p_2})^{\eta_2}}\left[{}^{p_1,p_2}I_{\wp_1^+,\wp_3^+}^{\eta_1,\eta_2}\mathcal{F}(\wp_2,\wp_4) + {}^{p_1,p_2}I_{\wp_1^+,\wp_4^-}^{\eta_1,\eta_2}\mathcal{F}(\wp_2,\wp_3)\right.$$

$$\left. + {}^{p_1,p_2}I_{\wp_2^-,\wp_3^+}^{\eta_1,\eta_2}\mathcal{F}(\wp_1,\wp_4) + {}^{p_1,p_2}I_{\wp_2^-,\wp_4^-}^{\eta_1,\eta_2}\mathcal{F}(\wp_1,\wp_3)\right],$$

which completes the proof of first inequality. For the proof of second inequality of the (m,n)-Polynomial (p_1, p_2)-convex functions on Δ, we get

$$\mathcal{F}\left([t\wp_1^{p_1} + (1-t)\wp_2^{p_1}]^{\frac{1}{p_1}}, [s\wp_3^{p_2} + (1-s)\wp_4^{p_2}]^{\frac{1}{p_2}}\right)$$

$$\leq \frac{1}{n}\sum_{i=1}^{n}[1-(1-t)^i]\frac{1}{m}\sum_{j=1}^{m}[1-(1-s)^j]\mathcal{F}(\wp_1,\wp_3)$$

$$+ \frac{1}{n}\sum_{i=1}^{n}[1-(1-t)^i]\frac{1}{m}\sum_{j=1}^{m}[1-s^j]\mathcal{F}(\wp_1,\wp_4)$$

$$+ \frac{1}{n}\sum_{i=1}^{n}[1-t^i]\frac{1}{m}\sum_{j=1}^{m}[1-(1-s)^j]\mathcal{F}(\wp_2,\wp_3)$$

$$+\frac{1}{n}\sum_{i=1}^{n}[1-t^i]\frac{1}{m}\sum_{j=1}^{m}[1-s^j]\mathcal{F}(\wp_2,\wp_4),$$

$$\mathcal{F}\left([t\wp_1^{p_1}+(1-t)\wp_2^{p_1}]^{\frac{1}{p_1}},[s\wp_4^{p_2}+(1-s)\wp_3^{p_2}]^{\frac{1}{p_2}}\right)$$

$$\leq\frac{1}{n}\sum_{i=1}^{n}[1-(1-t)^i]\frac{1}{m}\sum_{j=1}^{m}[1-(1-s)^j]\mathcal{F}(\wp_1,\wp_4)$$

$$+\frac{1}{n}\sum_{i=1}^{n}[1-(1-t)^i]\frac{1}{m}\sum_{j=1}^{m}[1-s^j]\mathcal{F}(\wp_1,\wp_3)$$

$$+\frac{1}{n}\sum_{i=1}^{n}[1-t^i]\frac{1}{m}\sum_{j=1}^{m}[1-(1-s)^j]\mathcal{F}(\wp_2,\wp_4)$$

$$+\frac{1}{n}\sum_{i=1}^{n}[1-t^i]\frac{1}{m}\sum_{j=1}^{m}[1-s^j]\mathcal{F}(\wp_2,\wp_3),$$

$$\mathcal{F}\left([t\wp_2^{p_1}+(1-t)\wp_1^{p_1}]^{\frac{1}{p_1}},[s\wp_3^{p_2}+(1-s)\wp_4^{p_2}]^{\frac{1}{p_2}}\right)$$

$$\leq\frac{1}{n}\sum_{i=1}^{n}[1-(1-t)^i]\frac{1}{m}\sum_{j=1}^{m}[1-(1-s)^j]\mathcal{F}(\wp_2,\wp_3)$$

$$+\frac{1}{n}\sum_{i=1}^{n}[1-(1-t)^i]\frac{1}{m}\sum_{j=1}^{m}[1-s^j]\mathcal{F}(\wp_2,\wp_4)$$

$$+\frac{1}{n}\sum_{i=1}^{n}[1-t^i]\frac{1}{m}\sum_{j=1}^{m}[1-(1-s)^j]\mathcal{F}(\wp_1,\wp_3)$$

$$+\frac{1}{n}\sum_{i=1}^{n}[1-t^i]\frac{1}{m}\sum_{j=1}^{m}[1-s^j]\mathcal{F}(\wp_1,\wp_4),$$

$$\mathcal{F}\left([t\wp_2^{p_1}+(1-t)\wp_1^{p_1}]^{\frac{1}{p_1}},[s\wp_4^{p_2}+(1-s)\wp_3^{p_2}]^{\frac{1}{p_2}}\right)$$

$$\leq\frac{1}{n}\sum_{i=1}^{n}[1-(1-t)^i]\frac{1}{m}\sum_{j=1}^{m}[1-(1-s)^j]\mathcal{F}(\wp_2,\wp_4)$$

$$+\frac{1}{n}\sum_{i=1}^{n}[1-(1-t)^i]\frac{1}{m}\sum_{j=1}^{m}[1-s^j]\mathcal{F}(\wp_2,\wp_3)$$

$$+\frac{1}{n}\sum_{i=1}^{n}[1-t^i]\frac{1}{m}\sum_{j=1}^{m}[1-(1-s)^j]\mathcal{F}(\wp_1,\wp_4)$$

$$+\frac{1}{n}\sum_{i=1}^{n}[1-t^i]\frac{1}{m}\sum_{j=1}^{m}[1-s^j]\mathcal{F}(\wp_1,\wp_3).$$

By adding the above four inequalities, we have

$$\mathcal{F}\left([t\wp_1^{p_1}+(1-t)\wp_2^{p_1}]^{\frac{1}{p_1}},[s\wp_3^{p_2}+(1-s)\wp_4^{p_2}]^{\frac{1}{p_2}}\right)$$

$$+\mathcal{F}\left([t\wp_1^{p_1}+(1-t)\wp_2^{p_1}]^{\frac{1}{p_1}},[s\wp_4^{p_2}+(1-s)\wp_3^{p_2}]^{\frac{1}{p_2}}\right)$$

$$+\mathcal{F}\left([t\wp_2^{p_1}+(1-t)\wp_1^{p_1}]^{\frac{1}{p_1}},[s\wp_3^{p_2}+(1-s)\wp_4^{p_2}]^{\frac{1}{p_2}}\right)$$

$$+\mathcal{F}\left([t\wp_2^{p_1}+(1-t)\wp_1^{p_1}]^{\frac{1}{p_1}},[s\wp_4^{p_2}+(1-s)\wp_3^{p_2}]^{\frac{1}{p_2}}\right)$$

$$\leq\left[\mathcal{F}(\wp_1,\wp_3)+\mathcal{F}(\wp_1,\wp_4)+\mathcal{F}(\wp_2,\wp_3)+\mathcal{F}(\wp_2,\wp_4)\right]$$

$$\times\left[\left(\frac{1}{n}\sum_{i=1}^{n}[1-(1-t)^i]+\frac{1}{n}\sum_{i=1}^{n}[1-t^i]\right)\left(\frac{1}{m}\sum_{j=1}^{m}[1-(1-s)^j]+\frac{1}{m}\sum_{j=1}^{m}[1-s^j]\right)\right] \quad (5)$$

Multiplying both sides of (5) by $\frac{\eta_1\eta_2}{4}t^{\eta_1-1}s^{\eta_2-1}$ and integrating the obtained inequality with respect to t and s over $(0,1)$, we obtain

$$\frac{p_1^{\eta_1}p_2^{\eta_2}\Gamma(\eta_1+1)\Gamma(\eta_2+1)}{4(\wp_2^{p_1}-\wp_1^{p_1})^{\eta_1}(\wp_4^{p_2}-\wp_3^{p_2})^{\eta_2}}\left[{}^{p_1,p_2}I^{\eta_1,\eta_2}_{\wp_1^+,\wp_3^+}\mathcal{F}(\wp_2,\wp_4)+{}^{p_1,p_2}I^{\eta_1,\eta_2}_{\wp_1^+,\wp_4^-}\mathcal{F}(\wp_2,\wp_3)\right.$$

$$\left.+{}^{p_1,p_2}I^{\eta_1,\eta_2}_{\wp_2^-,\wp_3^+}\mathcal{F}(\wp_1,\wp_4)+{}^{p_1,p_2}I^{\eta_1,\eta_2}_{\wp_2^-,\wp_4^-}\mathcal{F}(\wp_1,\wp_3)\right]$$

$$\leq\frac{\eta_1\eta_2}{4}\sum_{i=1}^{n}\left[\frac{\eta_1+2i}{\eta_1(\eta_1+i)}-\beta(i+1,\eta_1)\right]\frac{1}{m}\sum_{j=1}^{m}\left[\frac{\eta_2+2j}{\eta_2(\eta_2+j)}-\beta(j+1,\eta_2)\right]$$

$$\times\left[\frac{\mathcal{F}(\wp_1,\wp_3)+\mathcal{F}(\wp_1,\wp_4)+\mathcal{F}(\wp_2,\wp_3)+\mathcal{F}(\wp_2,\wp_4)}{4}\right].$$

Remark 3. *If in Theorem 3 we let $m=n=1$, then the result converts to the the inequalities (2.1) in [39].*

Remark 4. *If in Theorem 3 then we let $p_1=p_2=1$, m=n=1. Then the result converts to the the inequalities in [40].*

Remark 5. *If $p_1 = p_2 = 1$ in Theorem 3, then we get the following Hermite-Hadamard type inequalities for Riemann-Liouville fractional integrals by using (m,n) polynomial coordinate convex functions:*

$$\frac{1}{4}\left(\frac{2^n n}{2^n(n-1)+1}\right)\left(\frac{2^m m}{2^m(m-1)+1}\right)\mathcal{F}\left(\frac{\wp_1 + \wp_2}{2}, \frac{\wp_3 + \wp_4}{2}\right)$$

$$\leq \frac{\Gamma(\eta_1+1)\Gamma(\eta_2+1)}{4(\wp_2-\wp_1)^{\eta_1}(\wp_4-\wp_3)^{\eta_2}} \times$$

$$\left[I^{\eta_1,\eta_2}_{\wp_1^+,\wp_3^+}\mathcal{F}(\wp_2,\wp_4) + I^{\eta_1,\eta_2}_{\wp_1^+,\wp_4^-}\mathcal{F}(\wp_2,\wp_3) + I^{\eta_1,\eta_2}_{\wp_2^-,\wp_3^+}\mathcal{F}(\wp_1,\wp_4) + I^{\eta_1,\eta_2}_{\wp_2^-,\wp_4^-}\mathcal{F}(\wp_1,\wp_3)\right]$$

$$\leq \frac{\eta_1\eta_2}{4}\left[\frac{\mathcal{F}(\wp_1,\wp_3) + \mathcal{F}(\wp_1,\wp_4) + \mathcal{F}(\wp_2,\wp_3) + \mathcal{F}(\wp_2,\wp_4)}{mn}\right]$$

$$\times \sum_{i=1}^{n}\left[\frac{\eta_1 + 2i}{\eta_1(\eta_1+i)} - \beta(i+1,\eta_1)\right]\sum_{j=1}^{m}\left[\frac{\eta_2 + 2j}{\eta_2(\eta_2+j)} - \beta(j+1,\eta_2)\right].$$

Remark 6. *If $\eta_1 = \eta_2 = 1$, in Theorem 3, then we get the following Hermite-Hadamard type inequalities for (m,n)-polynomial (p_1, p_2) convex function on coordinates:*

$$\frac{1}{4}\left(\frac{2^n n}{2^n(n-1)+1}\right)\left(\frac{2^m m}{2^m(m-1)+1}\right)\mathcal{F}\left(\left[\frac{\wp_1^{p_1} + \wp_2^{p_1}}{2}\right]^{\frac{1}{p_1}}, \left[\frac{\wp_3^{p_2} + \wp_4^{p_2}}{2}\right]^{\frac{1}{p_2}}\right)$$

$$\leq \frac{p_1 p_2}{(\wp_2^{p_1} - \wp_1^{p_1})(\wp_4^{p_2} - \wp_3^{p_2})}\int_{\wp_1}^{\wp_2}\int_{\wp_3}^{\wp_4} x^{p_1-1}y^{p_2-1}\mathcal{F}(x,y)\,dx\,dy$$

$$\leq \left[\frac{\mathcal{F}(\wp_1,\wp_3) + \mathcal{F}(\wp_1,\wp_4) + \mathcal{F}(\wp_2,\wp_3) + \mathcal{F}(\wp_2,\wp_4)}{4mn}\right]$$

$$\times\left(\sum_{i=1}^{n}\left[\frac{1+2i}{1+i} - \beta(i+1,1)\right]\sum_{j=1}^{m}\left[\frac{1+2j}{1+j} - \beta(j+1,1)\right]\right).$$

Remark 7. *If $\eta_1 = \eta_2 = 1$, m=n=1, then Theorem 3 reduces to the result for coordinate convex functions, in [41].*

Remark 8. *If $p_1 = p_2 = \eta_1 = \eta_2 = 1$ in Theorem 3, then we get the following Hermite-Hadamard type inequalities for (m,n)-polynomial convex function on coordinates:*

$$\frac{1}{4}\left(\frac{2^n n}{2^n(n-1)+1}\right)\left(\frac{2^m m}{2^m(m-1)+1}\right)\mathcal{F}\left(\frac{\wp_1 + \wp_2}{2}, \frac{\wp_3 + \wp_4}{2}\right)$$

$$\leq \frac{1}{(\wp_2 - \wp_1)(\wp_4 - \wp_3)} \int_{\wp_1}^{\wp_2} \int_{\wp_3}^{\wp_4} \mathcal{F}(x, y)\, dx\, dy$$

$$\leq \left[\frac{\mathcal{F}(\wp_1, \wp_3) + \mathcal{F}(\wp_1, \wp_4) + \mathcal{F}(\wp_2, \wp_3) + \mathcal{F}(\wp_2, \wp_4)}{4} \right] \frac{1}{n} \sum_{i=1}^{n} \frac{2i}{i+1} \cdot \frac{1}{m} \sum_{j=1}^{m} \frac{2j}{j+1}.$$

Remark 9. *If $p_1 = p_2 = \eta_1 = \eta_2 = m = n = 1$, then Theorem 3 reduces to the result for coordinate convex functions, in [32].*

4. Refinements of Hermite-Hadamard type inequality pertaining to generalized convexity

The refinements of the Hermite-Hadamard inequality provide a profound enhancement in understanding the behavior of convex and generalized convex functions. By introducing additional structural parameters and integrating fractional and hypergeometric operators, these refinements offer sharper bounds and more precise estimations of integral means, surpassing the limitations of classical formulations. In this section, first we investigate Lemma 1. On the basis of this lemma, we present the refinement of Hermite-Hadamard inequaity with the support of newly introduced concept. Namely, (m, n)-polynomial (p_1, p_2) convexity on the coordinates and hypergeometric functions.

Lemma 1. *Let $p_1, p_2, \eta_1, \eta_2 > 0$, $\mathcal{F} : \Delta \subset (0, \infty) \times (0, \infty) \to \Re$ be a differentiable function on Δ and $\wp_1, \wp_2, \wp_3, \wp_4 \in \Delta^\circ$ with $\wp_1 < \wp_2$, $\wp_3 < \wp_4$ such that $\frac{\partial^2 \mathcal{F}}{\partial t \partial s} \in L_1(\Delta)$. Then we have*

$$\frac{\mathcal{F}(\wp_1, \wp_3) + \mathcal{F}(\wp_1, \wp_4) + \mathcal{F}(\wp_2, \wp_3) + \mathcal{F}(\wp_2, \wp_4)}{4} - A$$

$$+ \frac{p_1^{\eta_1} p_2^{\eta_2} \Gamma(\eta_1 + 1) \Gamma(\eta_2 + 1)}{4(\wp_2^{p_1} - \wp_1^{p_1})^{\eta_1} (\wp_4^{p_2} - \wp_3^{p_2})^{\eta_2}} \left[\, ^{p_1,p_2} I_{\wp_1^+, \wp_3^+}^{\eta_1, \eta_2} \mathcal{F}(\wp_2, \wp_4) + \, ^{p_1,p_2} I_{\wp_1^+, \wp_4^-}^{\eta_1, \eta_2} \mathcal{F}(\wp_2, \wp_3) \right.$$

$$\left. + \, ^{p_1,p_2} I_{\wp_2^-, \wp_3^+}^{\eta_1, \eta_2} \mathcal{F}(\wp_1, \wp_4) + \, ^{p_1,p_2} I_{\wp_2^-, \wp_4^-}^{\eta_1, \eta_2} \mathcal{F}(\wp_1, \wp_3) \right]$$

$$= \frac{(\wp_2^{p_1} - \wp_1^{p_1})(\wp_4^{p_2} - \wp_3^{p_2})}{4 p_1 p_2}$$

$$\times \left[\int_0^1 \int_0^1 \frac{t^{\eta_1} s^{\eta_2}}{[t\wp_1^{p_1} + (1-t)\wp_2^{p_1}]^{1 - \frac{1}{p_1}} [s\wp_3^{p_2} + (1-s)\wp_4^{p_2}]^{1 - \frac{1}{p_2}}} \times \right.$$

$$\left. \frac{\partial^2 \mathcal{F}}{\partial t \partial s} \left([t\wp_1^{p_1} + (1-t)\wp_2^{p_1}]^{\frac{1}{p_1}}, [s\wp_3^{p_2} + (1-s)\wp_4^{p_2}]^{\frac{1}{p_2}} \right) dt ds \right.$$

$$-\int_0^1 \int_0^1 \frac{t^{\eta_1} s^{\eta_2}}{[t\wp_1^{p_1} + (1-t)\wp_2^{p_1}]^{1-\frac{1}{p_1}}[s\wp_4^{p_2} + (1-s)\wp_3^{p_2}]^{1-\frac{1}{p_2}}} \times$$

$$\frac{\partial^2 \mathcal{F}}{\partial t \partial s}\left([t\wp_1^{p_1} + (1-t)\wp_2^{p_1}]^{\frac{1}{p_1}}, [s\wp_4^{p_2} + (1-s)\wp_3^{p_2}]^{\frac{1}{p_2}}\right) dt ds$$

$$-\int_0^1 \int_0^1 \frac{t^{\eta_1} s^{\eta_2}}{[t\wp_2^{p_1} + (1-t)\wp_1^{p_1}]^{1-\frac{1}{p_1}}[s\wp_3^{p_2} + (1-s)\wp_4^{p_2}]^{1-\frac{1}{p_2}}} \times$$

$$\frac{\partial^2 \mathcal{F}}{\partial t \partial s}\left([t\wp_2^{p_1} + (1-t)\wp_1^{p_1}]^{\frac{1}{p_1}}, [s\wp_3^{p_2} + (1-s)\wp_4^{p_2}]^{\frac{1}{p_2}}\right) dt ds$$

$$-\int_0^1 \int_0^1 \frac{t^{\eta_1} s^{\eta_2}}{[t\wp_2^{p_1} + (1-t)\wp_1^{p_1}]^{1-\frac{1}{p_1}}[s\wp_4^{p_2} + (1-s)\wp_3^{p_2}]^{1-\frac{1}{p_2}}}$$

$$\frac{\partial^2 \mathcal{F}}{\partial t \partial s}\left([t\wp_2^{p_1} + (1-t)\wp_1^{p_1}]^{\frac{1}{p_1}}, [s\wp_4^{p_2} + (1-s)\wp_3^{p_2}]^{\frac{1}{p_2}}\right) dt ds \Bigg],$$

where

$$A = \frac{p_2^{\eta_2}}{4} \frac{\Gamma(\eta_2 + 1)}{(\wp_4^{p_2} - \wp_3^{p_2})^{\eta_2}}$$

$$\times \left[{}^{p_2} I_{\wp_4^-}^{\eta_2} \mathcal{F}(\wp_1, \wp_3) + {}^{p_2} I_{\wp_3^+}^{\eta_2} \mathcal{F}(\wp_1, \wp_4) + {}^{p_2} I_{\wp_4^-}^{\eta_2} \mathcal{F}(\wp_2, \wp_3) + {}^{p_2} I_{\wp_3^+}^{\eta_2} \mathcal{F}(\wp_2, \wp_4)\right]$$

$$+ \frac{p_1^{\eta_1}}{4} \frac{\Gamma(\eta_1 + 1)}{(\wp_2^{p_1} - \wp_1^{p_1})^{\eta_1}}$$

$$\times \left[{}^{p_1} I_{\wp_2^-}^{\eta_1} \mathcal{F}(\wp_1, \wp_3) + {}^{p_1} I_{\wp_1^+}^{\eta_1} \mathcal{F}(\wp_2, \wp_3) + {}^{p_1} I_{\wp_2^-}^{\eta_1} \mathcal{F}(\wp_1, \wp_4) + {}^{p_1} I_{\wp_1^+}^{\eta_1} \mathcal{F}(\wp_2, \wp_4)\right].$$

Proof. Integrating by parts yields

$$i = \int_0^1 \int_0^1 \frac{t^{\eta_1} s^{\eta_2}}{[t\wp_1^{p_1} + (1-t)\wp_2^{p_1}]^{1-\frac{1}{p_1}}[s\wp_3^{p_2} + (1-s)\wp_4^{p_2}]^{1-\frac{1}{p_2}}}$$

$$\times \frac{\partial^2 \mathcal{F}}{\partial t \partial s}\left(\left[t\wp_1^{p_1} + (1-t)\wp_2^{p_1}\right]^{\frac{1}{p_1}}, \left[s\wp_3^{p_2} + (1-s)\wp_4^{p_2}\right]^{\frac{1}{p_2}}\right) dt ds$$

$$= \int_0^1 \frac{t^{\eta_1}}{[t\wp_1^{p_1} + (1-t)\wp_2^{p_1}]^{1-\frac{1}{p_1}}} \left[\int_0^1 \frac{s^{\eta_2}}{[s\wp_3^{p_2} + (1-s)\wp_4^{p_2}]^{1-\frac{1}{p_2}}}\right.$$

$$\times \frac{\partial^2 \mathcal{F}}{\partial t \partial s}\left(\left[t\wp_1^{p_1} + (1-t)\wp_2^{p_1}\right]^{\frac{1}{p_1}}, \left[s\wp_3^{p_2} + (1-s)\wp_4^{p_2}\right]^{\frac{1}{p_2}}\right) ds \Bigg] dt$$

$$= \int_0^1 \frac{t^{\eta_1}}{[t\wp_1^{p_1} + (1-t)\wp_2^{p_1}]^{1-\frac{1}{p_1}}}$$

$$\times \left\{ \left| \frac{p_2 s^{\eta_2}}{(\wp_3^{p_2} - \wp_4^{p_2})} \frac{\partial \mathcal{F}}{\partial t} \left([t\wp_1^{p_1} + (1-t)\wp_2^{p_1}]^{\frac{1}{p_1}}, [s\wp_3^{p_2} + (1-s)\wp_4^{p_2}]^{\frac{1}{p_2}} \right) \right|_0^1 \right.$$

$$\left. - \frac{p_2 \eta_2}{(\wp_3^{p_2} - \wp_4^{p_2})} \int_0^1 s^{\eta_2-1} \frac{\partial \mathcal{F}}{\partial t} \left([t\wp_1^{p_1} + (1-t)\wp_2^{p_1}]^{\frac{1}{p_1}}, [s\wp_3^{p_2} + (1-s)\wp_4^{p_2}]^{\frac{1}{p_2}} \right) ds \right\} dt$$

$$= \frac{p_2}{\wp_3^{p_2} - \wp_4^{p_2}} \left[\int_0^1 \frac{t^{\eta_1}}{[t\wp_1^{p_1} + (1-t)\wp_2^{p_1}]^{1-\frac{1}{p_1}}} \frac{\partial \mathcal{F}}{\partial t} \left([t\wp_1^{p_1}+(1-t)\wp_2^{p_1}]^{\frac{1}{p_1}}, \wp_3 \right) dt - \eta_2 \int_0^1 s^{\eta_2-1} \right.$$

$$\left. \times \left\{ \int_0^1 \frac{t^{\eta_1}}{[t\wp_1^{p_1} + (1-t)\wp_2^{p_1}]^{1-\frac{1}{p_1}}} \frac{\partial \mathcal{F}}{\partial t} \left([t\wp_1^{p_1}+(1-t)\wp_2^{p_1}]^{\frac{1}{p_1}}, [s\wp_3^{p_2}+(1-s)\wp_4^{p_2}]^{\frac{1}{p_2}} \right) dt \right\} ds \right]$$

$$= \frac{p_1 p_2}{(\wp_1^{p_1} - \wp_2^{p_1})(\wp_3^{p_2} - \wp_4^{p_2})} \mathcal{F}(\wp_1, \wp_3)$$

$$- \frac{p_1 p_2 \eta_1}{(\wp_1^{p_1} - \wp_2^{p_1})(\wp_3^{p_2} - \wp_4^{p_2})} \int_0^1 t^{\eta_1-1} \mathcal{F} \left([t\wp_1^{p_1} + (1-t)\wp_2^{p_1}]^{\frac{1}{p_1}}, \wp_3 \right) dt$$

$$- \frac{p_1 p_2 \eta_2}{(\wp_1^{p_1} - \wp_2^{p_1})(\wp_3^{p_2} - \wp_4^{p_2})} \int_0^1 s^{\eta_2-1} \mathcal{F} \left(\wp_1, [s\wp_3^{p_2} + (1-s)\wp_4^{p_2}]^{\frac{1}{p_2}} \right) ds$$

$$+ \frac{p_1 p_2 \eta_1 \eta_2}{(\wp_1^{p_1} - \wp_2^{p_1})(\wp_3^{p_2} - \wp_4^{p_2})}$$

$$\times \int_0^1 \int_0^1 t^{\eta_1-1} s^{\eta_2-1} \mathcal{F} \left([t\wp_1^{p_1} + (1-t)\wp_2^{p_1}]^{\frac{1}{p_1}}, [s\wp_3^{p_2} + (1-s)\wp_4^{p_2}]^{\frac{1}{p_2}} \right) dt ds.$$

By change of variables, we get

$$= \frac{p_1 p_2}{(\wp_1^{p_1} - \wp_2^{p_1})(\wp_3^{p_2} - \wp_4^{p_2})} \mathcal{F}(\wp_1, \wp_3)$$

$$- \frac{p_1 p_2 \eta_1}{(\wp_1^{p_1} - \wp_2^{p_1})(\wp_3^{p_2} - \wp_4^{p_2})} \frac{p_1}{\wp_1^{p_1} - \wp_2^{p_1}} \int_{\wp_2}^{\wp_1} \left(\frac{\wp_2^{p_1} - x^{p_1}}{\wp_2^{p_1} - \wp_1^{p_1}} \right)^{\eta_1-1} \frac{\mathcal{F}(x, \wp_3)}{x^{p_1-1}} dx$$

$$+ \frac{p_1 p_2 \eta_2}{(\wp_1^{p_1} - \wp_2^{p_1})(\wp_3^{p_2} - \wp_4^{p_2})} \frac{p_2}{\wp_3^{p_2} - \wp_4^{p_2}} \int_{\wp_4}^{\wp_3} \left(\frac{\wp_4^{p_2} - y^{p_2}}{\wp_4^{p_2} - \wp_3^{p_2}} \right)^{\eta_2-1} \frac{\mathcal{F}(\wp_1, y)}{y^{p_2-1}} dy$$

$$+ \frac{p_1 p_2 \eta_1 \eta_2}{(\wp_1^{p_1} - \wp_2^{p_1})(\wp_3^{p_2} - \wp_4^{p_2})} \frac{p_1 p_2}{(\wp_1^{p_1} - \wp_2^{p_1})(\wp_3^{p_2} - \wp_4^{p_2})}$$

$$\times \int_{\wp_2}^{\wp_1} \int_{\wp_4}^{\wp_3} \left(\frac{\wp_2^{p_1} - x^{p_1}}{\wp_2^{p_1} - \wp_1^{p_1}} \right)^{\eta_1-1} \left(\frac{\wp_4^{p_2} - y^{p_2}}{\wp_4^{p_2} - \wp_3^{p_2}} \right)^{\eta_2-1} \frac{\mathcal{F}(x, y)}{x^{p_1-1} y^{p_2-1}} dx dy$$

Multiplying by $\frac{(\wp_2^{p_1} - \wp_1^{p_1})(\wp_4^{p_2} - \wp_3^{p_2})}{4p_1 p_2}$, we get

$$= \frac{\mathcal{F}(\wp_1, \wp_3)}{4} - \frac{p_1^{\eta_1}}{4} \frac{\Gamma(\eta_1 + 1)}{(\wp_2^{p_1} - \wp_1^{p_1})^{\eta_1}}{}^{p_1}I_{\wp_1^+}^{\eta_1} \mathcal{F}(\wp_2, \wp_3) - \frac{p_2^{\eta_2}}{4} \frac{\Gamma(\eta_2 + 1)}{(\wp_4^{p_2} - \wp_3^{p_2})^{\eta_2}}{}^{p_2}I_{\wp_3^+}^{\eta_2} \mathcal{F}(\wp_1, \wp_4)$$

$$+ \frac{p_1^{\eta_1} p_2^{\eta_2}}{4} \frac{\Gamma(\eta_1 + 1)\Gamma(\eta_2 + 1)}{(\wp_2^{p_1} - \wp_1^{p_1})^{\eta_1}(\wp_4^{p_2} - \wp_3^{p_2})^{\eta_2}}{}^{p_1 p_2}I_{\wp_1^+, \wp_3^+}^{\eta_1, \eta_2} \mathcal{F}(\wp_2, \wp_4).$$

Let

$$j = -\frac{(\wp_2^{p_1} - \wp_1^{p_1})(\wp_4^{p_2} - \wp_3^{p_2})}{4p_1 p_2} \int_0^1 \int_0^1 \frac{t^{\eta_1} s^{\eta_2}}{[t\wp_1^{p_1} + (1-t)\wp_2^{p_1}]^{1-\frac{1}{p_1}}[s\wp_4^{p_2} + (1-s)\wp_3^{p_2}]^{1-\frac{1}{p_2}}}$$

$$\times \frac{\partial^2 \mathcal{F}}{\partial t \partial s}\left(\left[t\wp_1^{p_1} + (1-t)\wp_2^{p_1}\right]^{\frac{1}{p_1}}, \left[s\wp_3^{p_2} + (1-s)\wp_4^{p_2}\right]^{\frac{1}{p_2}}\right) dt ds$$

Integration by parts and change of variables yield

$$= \frac{\mathcal{F}(\wp_1, \wp_4)}{4} - \frac{p_1^{\eta_1}}{4} \frac{\Gamma(\eta_1 + 1)}{(\wp_2^{p_1} - \wp_1^{p_1})^{\eta_1}}{}^{p_1}I_{\wp_1^+}^{\eta_1} \mathcal{F}(\wp_2, \wp_4) - \frac{p_2^{\eta_2}}{4} \frac{\Gamma(\eta_2 + 1)}{(\wp_4^{p_2} - \wp_3^{p_2})^{\eta_2}}{}^{p_2}I_{\wp_3^+}^{\eta_2} \mathcal{F}(\wp_1, \wp_3)$$

$$+ \frac{p_1^{\eta_1} p_2^{\eta_2}}{4} \frac{\Gamma(\eta_1 + 1)\Gamma(\eta_2 + 1)}{(\wp_2^{p_1} - \wp_1^{p_1})^{\eta_1}(\wp_4^{p_2} - \wp_3^{p_2})^{\eta_2}}{}^{p_1 p_2}I_{\wp_1^+, \wp_3^+}^{\eta_1, \eta_2} \mathcal{F}(\wp_2, \wp_3).$$

Similarly,

$$k = -\frac{(\wp_2^{p_1} - \wp_1^{p_1})(\wp_4^{p_2} - \wp_3^{p_2})}{4p_1 p_2} \int_0^1 \int_0^1 \frac{t^{\eta_1} s^{\eta_2}}{[t\wp_2^{p_1} + (1-t)\wp_1^{p_1}]^{1-\frac{1}{p_1}}[s\wp_3^{p_2} + (1-s)\wp_4^{p_2}]^{1-\frac{1}{p_2}}}$$

$$\times \frac{\partial^2 \mathcal{F}}{\partial t \partial s}\left(\left[t\wp_2^{p_1} + (1-t)\wp_1^{p_1}\right]^{\frac{1}{p_1}}, \left[s\wp_3^{p_2} + (1-s)\wp_4^{p_2}\right]^{\frac{1}{p_2}}\right) dt ds$$

$$= \frac{\mathcal{F}(\wp_1, \wp_4)}{4} - \frac{p_1^{\eta_1}}{4} \frac{\Gamma(\eta_1 + 1)}{(\wp_2^{p_1} - \wp_1^{p_1})^{\eta_1}}{}^{p_1}I_{\wp_1^+}^{\eta_1} \mathcal{F}(\wp_1, \wp_3) - \frac{p_2^{\eta_2}}{4} \frac{\Gamma(\eta_2 + 1)}{(\wp_4^{p_2} - \wp_3^{p_2})^{\eta_2}}{}^{p_2}I_{\wp_3^+}^{\eta_2} \mathcal{F}(\wp_2, \wp_4)$$

$$+ \frac{p_1^{\eta_1} p_2^{\eta_2}}{4} \frac{\Gamma(\eta_1 + 1)\Gamma(\eta_2 + 1)}{(\wp_2^{p_1} - \wp_1^{p_1})^{\eta_1}(\wp_4^{p_2} - \wp_3^{p_2})^{\eta_2}}{}^{p_1 p_2}I_{\wp_1^+, \wp_3^+}^{\eta_1, \eta_2} \mathcal{F}(\wp_1, \wp_4).$$

Finally,

$$l = -\frac{(\wp_2^{p_1} - \wp_1^{p_1})(\wp_4^{p_2} - \wp_3^{p_2})}{4p_1 p_2} \int_0^1 \int_0^1 \frac{t^{\eta_1} s^{\eta_2}}{[t\wp_2^{p_1} + (1-t)\wp_1^{p_1}]^{1-\frac{1}{p_1}}[s\wp_4^{p_2} + (1-s)\wp_3^{p_2}]^{1-\frac{1}{p_2}}}$$

$$\times \frac{\partial^2 \mathcal{F}}{\partial t \partial s}\left(\left[t\wp_2^{p_1} + (1-t)\wp_1^{p_1}\right]^{\frac{1}{p_1}}, \left[s\wp_4^{p_2} + (1-s)\wp_3^{p_2}\right]^{\frac{1}{p_2}}\right) dt ds$$

$$
= \frac{\mathcal{F}(\wp_1, \wp_4)}{4} - \frac{p_1^{\eta_1}}{4} \frac{\Gamma(\eta_1 + 1)}{(\wp_2^{p_1} - \wp_1^{p_1})^{\eta_1}} {}^{p_1}I_{\wp_1^+}^{\eta_1} \mathcal{F}(\wp_1, \wp_4) - \frac{p_2^{\eta_2}}{4} \frac{\Gamma(\eta_2 + 1)}{(\wp_4^{p_2} - \wp_3^{p_2})^{\eta_2}} {}^{p_2}I_{\wp_3^+}^{\eta_2} \mathcal{F}(\wp_2, \wp_3)
$$

$$
+ \frac{p_1^{\eta_1} p_2^{\eta_2}}{4} \frac{\Gamma(\eta_1 + 1)\Gamma(\eta_2 + 1)}{(\wp_2^{p_1} - \wp_1^{p_1})^{\eta_1}(\wp_4^{p_2} - \wp_3^{p_2})^{\eta_2}} {}^{p_1 p_2}I_{\wp_1^+, \wp_3^+}^{\eta_1, \eta_2} \mathcal{F}(\wp_2, \wp_3)
$$

By summing i, j, k, l, we obtain the left-hand side of. Lemma 1.

Remark 10. *If $p_1 = p_2 = 1$, then*

$$
\frac{\mathcal{F}(\wp_1, \wp_3) + \mathcal{F}(\wp_1, \wp_4) + \mathcal{F}(\wp_2, \wp_3) + \mathcal{F}(\wp_2, \wp_4)}{4} - A + \frac{\Gamma(\eta_1 + 1)\Gamma(\eta_2 + 1)}{4(\wp_2 - \wp_1)^{\eta_1}(\wp_4 - \wp_3)^{\eta_2}}
$$

$$
\times \left[I_{\wp_1^+, \wp_3^+}^{\eta_1, \eta_2} \mathcal{F}(\wp_2, \wp_4) + I_{\wp_1^+, \wp_4^-}^{\eta_1, \eta_2} \mathcal{F}(\wp_2, \wp_3) + I_{\wp_2^-, \wp_3^+}^{\eta_1, \eta_2} \mathcal{F}(\wp_1, \wp_4) + I_{\wp_2^-, \wp_4^-}^{\eta_1, \eta_2} \mathcal{F}(\wp_1, \wp_3) \right]
$$

$$
= \frac{(\wp_2 - \wp_1)(\wp_4 - \wp_3)}{4} \times
$$

$$
\left[\int_0^1 \int_0^1 \frac{t^{\eta_1} s^{\eta_2}}{[t\wp_1 + (1 - t)\wp_2][s\wp_3 + (1 - s)\wp_4]} \frac{\partial^2 \mathcal{F}}{\partial t \partial s} \left(t\wp_1 + (1 - t)\wp_2, s\wp_3 + (1 - s)\wp_4 \right) dt ds \right.
$$

$$
- \int_0^1 \int_0^1 \frac{t^{\eta_1} s^{\eta_2}}{[t\wp_1 + (1 - t)\wp_2][s\wp_4 + (1 - s)\wp_3]} \frac{\partial^2 \mathcal{F}}{\partial t \partial s} \left(t\wp_1 + (1 - t)\wp_2, s\wp_4 + (1 - s)\wp_3 \right) dt ds
$$

$$
- \int_0^1 \int_0^1 \frac{t^{\eta_1} s^{\eta_2}}{[t\wp_2 + (1 - t)\wp_1][s\wp_3 + (1 - s)\wp_4]} \frac{\partial^2 \mathcal{F}}{\partial t \partial s} \left(t\wp_2 + (1 - t)\wp_1, s\wp_3 + (1 - s)\wp_4 \right) dt ds
$$

$$
\left. + \int_0^1 \int_0^1 \frac{t^{\eta_1} s^{\eta_2}}{[t\wp_2 + (1 - t)\wp_1][s\wp_4 + (1 - s)\wp_3]} \frac{\partial^2 \mathcal{F}}{\partial t \partial s} \left(t\wp_2 + (1 - t)\wp_1, s\wp_4 + (1 - s)\wp_3 \right) dt ds \right],
$$

Where

$$
A = \frac{\Gamma(\eta_2 + 1)}{4(\wp_4 - \wp_3)^{\eta_2}} \left[I_{\wp_4^-}^{\eta_2} \mathcal{F}(\wp_1, \wp_3) + I_{\wp_3^+}^{\eta_2} \mathcal{F}(\wp_1, \wp_4) + I_{\wp_4^-}^{\eta_2} \mathcal{F}(\wp_2, \wp_3) + I_{\wp_3^+}^{\eta_2} \mathcal{F}(\wp_2, \wp_4) \right]
$$

$$
+ \frac{\Gamma(\eta_1 + 1)}{4(\wp_2 - \wp_1)^{\eta_1}} \left[I_{\wp_2^-}^{\eta_1} \mathcal{F}(\wp_1, \wp_3) + I_{\wp_1^+}^{\eta_1} \mathcal{F}(\wp_2, \wp_3) + I_{\wp_2^-}^{\eta_1} \mathcal{F}(\wp_1, \wp_4) + I_{\wp_1^+}^{\eta_1} \mathcal{F}(\wp_2, \wp_4) \right].
$$

Remark 11. *If $\eta_1 = \eta_2 = 1$, then*

$$
\frac{\mathcal{F}(\wp_1, \wp_3) + \mathcal{F}(\wp_1, \wp_4) + \mathcal{F}(\wp_2, \wp_3) + \mathcal{F}(\wp_2, \wp_4)}{4} - A
$$

$$
+ \frac{1}{(\wp_2^{p_1} - \wp_1^{p_1})(\wp_4^{p_2} - \wp_3^{p_2})} \int_0^1 \int_0^1 \frac{\mathcal{F}(x, y)}{x^{1 - p_1} y^{1 - p_2}} \, dx dy
$$

$$= \frac{(\wp_2^{p_1} - \wp_1^{p_1})(\wp_4^{p_2} - \wp_3^{p_2})}{4p_1p_2}$$

$$\times \left[\int_0^1 \int_0^1 \frac{ts}{[t\wp_1^{p_1} + (1-t)\wp_2^{p_1}]^{1-\frac{1}{p_1}} [s\wp_3^{p_2} + (1-s)\wp_4^{p_2}]^{1-\frac{1}{p_2}}} \times \right.$$

$$\frac{\partial^2 \mathcal{F}}{\partial t \partial s} \left([t\wp_1^{p_1} + (1-t)\wp_2^{p_1}]^{\frac{1}{p_1}}, [s\wp_3^{p_2} + (1-s)\wp_4^{p_2}]^{\frac{1}{p_2}} \right) dt ds$$

$$- \int_0^1 \int_0^1 \frac{ts}{[t\wp_1^{p_1} + (1-t)\wp_2^{p_1}]^{1-\frac{1}{p_1}} [s\wp_4^{p_2} + (1-s)\wp_3^{p_2}]^{1-\frac{1}{p_2}}} \times$$

$$\frac{\partial^2 \mathcal{F}}{\partial t \partial s} \left([t\wp_1^{p_1} + (1-t)\wp_2^{p_1}]^{\frac{1}{p_1}}, [s\wp_4^{p_2} + (1-s)\wp_3^{p_2}]^{\frac{1}{p_2}} \right) dt ds$$

$$- \int_0^1 \int_0^1 \frac{ts}{[t\wp_2^{p_1} + (1-t)\wp_1^{p_1}]^{1-\frac{1}{p_1}} [s\wp_3^{p_2} + (1-s)\wp_4^{p_2}]^{1-\frac{1}{p_2}}} \times$$

$$\frac{\partial^2 \mathcal{F}}{\partial t \partial s} \left([t\wp_2^{p_1} + (1-t)\wp_1^{p_1}]^{\frac{1}{p_1}}, [s\wp_3^{p_2} + (1-s)\wp_4^{p_2}]^{\frac{1}{p_2}} \right) dt ds$$

$$+ \int_0^1 \int_0^1 \frac{ts}{[t\wp_2^{p_1} + (1-t)\wp_1^{p_1}]^{1-\frac{1}{p_1}} [s\wp_4^{p_2} + (1-s)\wp_3^{p_2}]^{1-\frac{1}{p_2}}} \times$$

$$\left. \frac{\partial^2 \mathcal{F}}{\partial t \partial s} \left([t\wp_2^{p_1} + (1-t)\wp_1^{p_1}]^{\frac{1}{p_1}}, [s\wp_4^{p_2} + (1-s)\wp_3^{p_2}]^{\frac{1}{p_2}} \right) dt ds \right],$$

where

$$A = -\frac{p_1}{2(\wp_2^{p_1} - \wp_1^{p_1})} \left[\int_{\wp_1}^{\wp_2} \frac{\mathcal{F}(x, \wp_3)}{x^{1-p_1}} dx + \int_{\wp_1}^{\wp_2} \frac{\mathcal{F}(x, \wp_4)}{x^{1-p_1}} dx \right]$$

$$- \frac{p_2}{2(\wp_4^{p_2} - \wp_3^{p_2})} \left[\int_{\wp_3}^{\wp_4} \frac{\mathcal{F}(\wp_1, y)}{y^{1-p_2}} dy + \int_{\wp_3}^{\wp_4} \frac{\mathcal{F}(\wp_2, y)}{y^{1-p_2}} dy \right].$$

Remark 12. *If $p_1 = p_2 = \eta_1 = \eta_2 = 1$, then*

$$\frac{\mathcal{F}(\wp_1, \wp_3) + \mathcal{F}(\wp_1, \wp_4) + \mathcal{F}(\wp_2, \wp_3) + \mathcal{F}(\wp_2, \wp_4)}{4} - A$$

$$+ \frac{1}{(\wp_2 - \wp_1)(\wp_4 - \wp_3)} \int_{\wp_1}^{\wp_2} \int_{\wp_3}^{\wp_4} \mathcal{F}(x, y) \, dx dy$$

$$= \frac{(\wp_2 - \wp_1)(\wp_4 - \wp_3)}{4} \left[\int_0^1 \int_0^1 ts \frac{\partial^2 \mathcal{F}}{\partial t \partial s} \left(t\wp_1 + (1-t)\wp_2, s\wp_3 + (1-s)\wp_4 \right) dt ds \right.$$

$$- \int_0^1 \int_0^1 ts \frac{\partial^2 \mathcal{F}}{\partial t \partial s} \left(t\wp_1 + (1-t)\wp_2, s\wp_4 + (1-s)\wp_3 \right) dt ds$$

$$-\int_0^1\int_0^1 ts\frac{\partial^2\mathcal{F}}{\partial t\partial s}\Big(t\wp_2+(1-t)\wp_1,s\wp_4+(1-s)\wp_3\Big)\,dtds$$

$$-\int_0^1\int_0^1 ts\frac{\partial^2\mathcal{F}}{\partial t\partial s}\Big(t\wp_2+(1-t)\wp_1,s\wp_4+(1-s)\wp_3\Big)\,dtds\bigg],$$

where

$$A=-\frac{1}{2}\left[\frac{1}{\wp_2-\wp_1}\int_{\wp_1}^{\wp_2}\Big[\mathcal{F}(x,\wp_3)+\mathcal{F}(x,\wp_4)\Big]dx\right.$$

$$\left.+\frac{1}{\wp_4-\wp_3}\int_{\wp_3}^{\wp_4}\Big[\mathcal{F}(\wp_1,y)+\mathcal{F}(\wp_2,y)\Big]dy\right].$$

Theorem 4. *Let* $m,n\in N$, $p_1,p_2>0$, $\eta_1,\eta_2>0$, $\Delta^\circ\subset(0,\infty)\times(0,\infty)\to\Re$ *be a differentiable function on* Δ°, $\wp_1,\wp_2,\wp_3,\wp_4\in\Delta^\circ$ *with* $\wp_1<\wp_2$, $\wp_3<\wp_4$ *such that* $\frac{\partial^2\mathcal{F}}{\partial t\partial s}\in L_1(\Delta)$ *and* $\left|\frac{\partial^2\mathcal{F}}{\partial t\partial s}\right|$ *be a* (m,n)-*polynomial* (p_1,p_2)-*convex function on* Δ°. *Then we have*

$$\left|\frac{\mathcal{F}(\wp_1,\wp_3)+\mathcal{F}(\wp_1,\wp_4)+\mathcal{F}(\wp_2,\wp_3)+\mathcal{F}(\wp_2,\wp_4)}{4}-A\right.$$

$$+\frac{p_1^{\eta_1}p_2^{\eta_2}\Gamma(\eta_1+1)\Gamma(\eta_2+1)}{4(\wp_2^{p_1}-\wp_1^{p_1})^{\eta_1}(\wp_4^{p_2}-\wp_3^{p_2})^{\eta_2}}\bigg[{}^{p_1,p_2}I^{\eta_1,\eta_2}_{\wp_1^+,\wp_3^+}\mathcal{F}(\wp_2,\wp_4)+{}^{p_1,p_2}I^{\eta_1,\eta_2}_{\wp_1^+,\wp_4^-}\mathcal{F}(\wp_2,\wp_3)$$

$$\left.+{}^{p_1,p_2}I^{\eta_1,\eta_2}_{\wp_2^-,\wp_3^+}\mathcal{F}(\wp_1,\wp_4)+{}^{p_1,p_2}I^{\eta_1,\eta_2}_{\wp_2^-,\wp_4^-}\mathcal{F}(\wp_1,\wp_3)\bigg]\right|$$

$$\leq\frac{(\wp_2^{p_1}-\wp_1^{p_1})(\wp_4^{p_2}-\wp_3^{p_2})}{4p_1p_2}$$

$$\times\left[\Big(\psi_1(\wp_1,\wp_2,\wp_3,\wp_4;p_1;p_2)\Big)^{1-\frac{1}{\alpha}}\left(\left|\frac{\partial^2\mathcal{F}}{\partial t\partial s}(\wp_1,\wp_3)\right|^\alpha\psi_2(\wp_1,\wp_2,\wp_3,\wp_4;p_1;p_2)\right.\right.$$

$$+\left|\frac{\partial^2\mathcal{F}}{\partial t\partial s}(\wp_1,\wp_4)\right|^\alpha\psi_3(\wp_1,\wp_2,\wp_3,\wp_4;p_1;p_2)+\left|\frac{\partial^2\mathcal{F}}{\partial t\partial s}(\wp_2,\wp_3)\right|^\alpha\psi_4(\wp_1,\wp_2,\wp_3,\wp_4;p_1;p_2)$$

$$+\left.\left|\frac{\partial^2\mathcal{F}}{\partial t\partial s}(\wp_2,\wp_4)\right|^\alpha\psi_5(\wp_1,\wp_2,\wp_3,\wp_4;p_1;p_2)\right)^{1-\frac{1}{\alpha}}-\Big(\psi_6(\wp_1,\wp_2,\wp_3,\wp_4;p_1;p_2)\Big)^{1-\frac{1}{\alpha}}$$

$$\left(\left|\frac{\partial^2\mathcal{F}}{\partial t\partial s}(\wp_1,\wp_4)\right|^\alpha\psi_7(\wp_1,\wp_2,\wp_3,\wp_4;p_1;p_2)+\left|\frac{\partial^2\mathcal{F}}{\partial t\partial s}(\wp_1,\wp_3)\right|^\alpha\psi_8(\wp_1,\wp_2,\wp_3,\wp_4;p_1;p_2)\right.$$

$$+\left|\frac{\partial^2 \mathcal{F}}{\partial t \partial s}(\wp_2, \wp_4)\right|^\alpha \psi_9(\wp_1, \wp_2, \wp_3, \wp_4; p_1; p_2)$$

$$\left.+\left|\frac{\partial^2 \mathcal{F}}{\partial t \partial s}(\wp_2, \wp_3)\right|^\alpha \psi_{10}(\wp_1, \wp_2, \wp_3, \wp_4; p_1; p_2)\right)^{\frac{1}{\alpha}}$$

$$-\left(\psi_{11}(\wp_1, \wp_2, \wp_3, \wp_4; p_1; p_2)\right)^{1-\frac{1}{\alpha}}\left(\left|\frac{\partial^2 \mathcal{F}}{\partial t \partial s}(\wp_2, \wp_3)\right|^\alpha \psi_{12}(\wp_1, \wp_2, \wp_3, \wp_4; p_1; p_2)\right.$$

$$+\left|\frac{\partial^2 \mathcal{F}}{\partial t \partial s}(\wp_2, \wp_4)\right|^\alpha \psi_{13}(\wp_1, \wp_2, \wp_3, \wp_4; p_1; p_2)+\left|\frac{\partial^2 \mathcal{F}}{\partial t \partial s}(\wp_1, \wp_3)\right|^\alpha \psi_{14}(\wp_1, \wp_2, \wp_3, \wp_4; p_1; p_2)$$

$$\left.+\left|\frac{\partial^2 \mathcal{F}}{\partial t \partial s}(\wp_1, \wp_4)\right|^\alpha \psi_{15}(\wp_1, \wp_2, \wp_3, \wp_4; p_1; p_2)\right)^{\frac{1}{\alpha}}$$

$$+\left(\psi_{16}(\wp_1, \wp_2, \wp_3, \wp_4; p_1; p_2)\right)^{1-\frac{1}{\alpha}}$$

$$\times\left(\left|\frac{\partial^2 \mathcal{F}}{\partial t \partial s}(\wp_2, \wp_4)\right|^\alpha \psi_{17}(\wp_1, \wp_2, \wp_3, \wp_4; p_1; p_2)\right.$$

$$+\left|\frac{\partial^2 \mathcal{F}}{\partial t \partial s}(\wp_2, \wp_3)\right|^\alpha \psi_{18}(\wp_1, \wp_2, \wp_3, \wp_4; p_1; p_2)$$

$$+\left|\frac{\partial^2 \mathcal{F}}{\partial t \partial s}(\wp_1, \wp_4)\right|^\alpha \psi_{19}(\wp_1, \wp_2, \wp_3, \wp_4; p_1; p_2)$$

$$\left.\left.+\left|\frac{\partial^2 \mathcal{F}}{\partial t \partial s}(\wp_1, \wp_3)\right|^\alpha \psi_{20}(\wp_1, \wp_2, \wp_3, \wp_4; p_1; p_2)\right)^{\frac{1}{\alpha}}\right], \tag{6}$$

where

$$\psi_1(\wp_1, \wp_2, \wp_3, \wp_4; p_1; p_2)$$

$$=\int_0^1 \frac{t^{\eta_1}}{[t\wp_1^{p_1} + (1-t)\wp_2^{p_1}]}dt \int_0^1 \frac{s^{\eta_2}}{[s\wp_3^{p_2} + (1-s)\wp_4^{p_2}]}ds$$

$$=\left[\frac{\wp_2^{1-p_1}}{\eta_1 + 1}{}_2F_1\left(1 - \frac{1}{p_1}, \eta_1 + 1; \eta_1 + 2; 1 - \frac{\wp_1^{p_1}}{\wp_2^{p_1}}\right)\right]$$

$$\times\left[\frac{\wp_4^{1-p_2}}{\eta_2 + 1}{}_2F_1\left(1 - \frac{1}{p_2}, \eta_2 + 1; \eta_2 + 2; 1 - \frac{\wp_3^{p_2}}{\wp_4^{p_2}}\right)\right],$$

$$\psi_2(\wp_1, \wp_2, \wp_3, \wp_4; p_1; p_2)$$

$$= \frac{1}{n}\sum_{i=1}^{n}\int_0^1 \frac{t^{\eta_1}(1-(1-t)^i)}{[t\wp_1^{p_1}+(1-t)\wp_2^{p_1}]}dt \frac{1}{m}\sum_{j=1}^{m}\int_0^1 \frac{s^{\eta_2}(1-(1-s)^j)}{[s\wp_3^{p_2}+(1-s)\wp_4^{p_2}]}ds$$

$$= \frac{1}{n}\sum_{i=1}^{n}\left[\frac{\wp_2^{1-p_1}}{\eta_1+1}{}_2F_1\left(1-\frac{1}{p_1},\eta_1+1;\eta_1+2;1-\frac{\wp_1^{p_1}}{\wp_2^{p_1}}\right)\right.$$

$$\left. -\wp_2^{1-p_1}\beta(\eta_1+1,i+1){}_2F_1\left(1-\frac{1}{p_1},\eta_1+1;\eta_1+i+2;1-\frac{\wp_1^{p_1}}{\wp_2^{p_1}}\right)\right]$$

$$\times \frac{1}{m}\sum_{j=1}^{m}\left[\frac{\wp_4^{1-p_2}}{\eta_2+1}{}_2F_1\left(1-\frac{1}{p_2},\eta_2+1;\eta_2+2;1-\frac{\wp_3^{p_2}}{\wp_4^{p_2}}\right)\right.$$

$$\left. -\wp_4^{1-p_2}\beta(\eta_2+1,j+1){}_2F_1\left(1-\frac{1}{p_2},\eta_2+1;\eta_2+j+2;1-\frac{\wp_3^{p_2}}{\wp_4^{p_2}}\right)\right],$$

$$\psi_3(\wp_1, \wp_2, \wp_3, \wp_4; p_1; p_2)$$

$$= \frac{1}{n}\sum_{i=1}^{n}\int_0^1 \frac{t^{\eta_1}(1-(1-t)^i)}{[t\wp_1^{p_1}+(1-t)\wp_2^{p_1}]}dt \frac{1}{m}\sum_{j=1}^{m}\int_0^1 \frac{s^{\eta_2}(1-s^j)}{[s\wp_3^{p_2}+(1-s)\wp_4^{p_2}]}ds$$

$$= \frac{1}{n}\sum_{i=1}^{n}\left[\frac{\wp_2^{1-p_1}}{\eta_1+1}{}_2F_1\left(1-\frac{1}{p_1},\eta_1+1;\eta_1+2;1-\frac{\wp_1^{p_1}}{\wp_2^{p_1}}\right)\right.$$

$$\left. -\wp_2^{1-p_1}\beta(\eta_1+1,i+1){}_2F_1\left(1-\frac{1}{p_1},\eta_1+1;\eta_1+i+2;1-\frac{\wp_1^{p_1}}{\wp_2^{p_1}}\right)\right]$$

$$\times \frac{1}{m}\sum_{j=1}^{m}\left[\frac{\wp_4^{1-p_2}}{\eta_2+1}{}_2F_1\left(1-\frac{1}{p_2},\eta_2+1;\eta_2+2;1-\frac{\wp_3^{p_2}}{\wp_4^{p_2}}\right)\right.$$

$$\left. -\frac{\wp_4^{1-p_2}}{\eta_2+j+1}{}_2F_1\left(1-\frac{1}{p_2},\eta_2+j+1;\eta_2+j+2;1-\frac{\wp_3^{p_2}}{\wp_4^{p_2}}\right)\right],$$

$$\psi_4(\wp_1, \wp_2, \wp_3, \wp_4; p_1; p_2)$$

$$= \frac{1}{n}\sum_{i=1}^{n}\int_0^1 \frac{t^{\eta_1}(1-t^i)}{[t\wp_1^{p_1}+(1-t)\wp_2^{p_1}]}dt \cdot \frac{1}{m}\sum_{j=1}^{m}\int_0^1 \frac{s^{\eta_2}(1-(1-s)^j)}{[s\wp_3^{p_2}+(1-s)\wp_4^{p_2}]}ds$$

$$= \frac{1}{n}\sum_{i=1}^{n}\left[\frac{\wp_2^{1-p_1}}{\eta_1+1}{}_2F_1\left(1-\frac{1}{p_1},\eta_1+1;\eta_1+2;1-\frac{\wp_1^{p_1}}{\wp_2^{p_1}}\right)\right.$$

$$- \frac{\wp_2^{1-p_1}}{\eta_1 + i + 1} {}_2F_1\Big(1 - \tfrac{1}{p_1}, \eta_1 + 1; \eta_1 + i + 2; 1 - \tfrac{\wp_1^{p_1}}{\wp_2^{p_1}}\Big)\Bigg]$$

$$\times \frac{1}{m} \sum_{j=1}^{m} \Bigg[\frac{\wp_4^{1-p_2}}{\eta_2 + 1} {}_2F_1\Big(1 - \tfrac{1}{p_2}, \eta_2 + 1; \eta_2 + 2; 1 - \tfrac{\wp_3^{p_2}}{\wp_4^{p_2}}\Big)$$

$$- \wp_4^{1-p_2} \beta(\eta_2 + 1, j + 1) {}_2F_1\Big(1 - \tfrac{1}{p_2}, \eta_2 + 1; \eta_2 + j + 2; 1 - \tfrac{\wp_3^{p_2}}{\wp_4^{p_2}}\Big)\Bigg],$$

$$\psi_5(\wp_1, \wp_2, \wp_3, \wp_4; p_1; p_2)$$

$$= \frac{1}{n} \sum_{i=1}^{n} \int_0^1 \frac{t^{\eta_1}(1 - t^i)}{[t\wp_1^{p_1} + (1-t)\wp_2^{p_1}]} \, dt \cdot \frac{1}{m} \sum_{j=1}^{m} \int_0^1 \frac{s^{\eta_2}(1 - s^j)}{[s\wp_3^{p_2} + (1-s)\wp_4^{p_2}]} \, ds$$

$$= \frac{1}{n} \sum_{i=1}^{n} \Bigg[\frac{\wp_2^{1-p_1}}{\eta_1 + 1} {}_2F_1\Big(1 - \tfrac{1}{p_1}, \eta_1 + 1; \eta_1 + 2; 1 - \tfrac{\wp_1^{p_1}}{\wp_2^{p_1}}\Big)$$

$$- \frac{\wp_2^{1-p_1}}{\eta_1 + i + 1} {}_2F_1\Big(1 - \tfrac{1}{p_1}, \eta_1 + 1; \eta_1 + i + 2; 1 - \tfrac{\wp_1^{p_1}}{\wp_2^{p_1}}\Big)\Bigg]$$

$$\times \frac{1}{m} \sum_{j=1}^{m} \Bigg[\frac{\wp_4^{1-p_2}}{\eta_2 + 1} {}_2F_1\Big(1 - \tfrac{1}{p_2}, \eta_2 + 1; \eta_2 + 2; 1 - \tfrac{\wp_3^{p_2}}{\wp_4^{p_2}}\Big)$$

$$- \frac{\wp_4^{1-p_2}}{\eta_2 + j + 1} {}_2F_1\Big(1 - \tfrac{1}{p_2}, \eta_2 + j + 1; \eta_2 + j + 2; 1 - \tfrac{\wp_3^{p_2}}{\wp_4^{p_2}}\Big)\Bigg],$$

$$\psi_6(\wp_1, \wp_2, \wp_3, \wp_4; p_1; p_2)$$

$$= \int_0^1 \frac{t^{\eta_1}}{[t\wp_1^{p_1} + (1-t)\wp_2^{p_1}]} \, dt \int_0^1 \frac{s^{\eta_2}}{[s\wp_4^{p_2} + (1-s)\wp_3^{p_2}]} \, ds$$

$$= \Bigg[\frac{\wp_2^{1-p_1}}{\eta_1 + 1} {}_2F_1\Big(1 - \tfrac{1}{p_1}, \eta_1 + 1; \eta_1 + 2; 1 - \tfrac{\wp_1^{p_1}}{\wp_2^{p_1}}\Big)\Bigg]$$

$$\times \Bigg[\frac{\wp_3^{1-p_2}}{\eta_2 + 1} {}_2F_1\Big(1 - \tfrac{1}{p_2}, \eta_2 + 1; \eta_2 + 2; 1 - \tfrac{\wp_4^{p_2}}{\wp_3^{p_2}}\Big)\Bigg],$$

$$\psi_7(\wp_1, \wp_2, \wp_3, \wp_4; p_1; p_2)$$

$$= \frac{1}{n} \sum_{i=1}^{n} \int_0^1 \frac{t^{\eta_1}(1 - (1-t)^i)}{[t\wp_1^{p_1} + (1-t)\wp_2^{p_1}]} \, dt \cdot \frac{1}{m} \sum_{j=1}^{m} \int_0^1 \frac{s^{\eta_2}(1 - (1-s)^j)}{[s\wp_4^{p_1} + (1-s)\wp_3^{p_2}]} \, ds$$

$$= \frac{1}{n} \sum_{i=1}^{n} \left[\frac{\wp_2^{1-p_1}}{\eta_1 + 1} {}_2F_1\left(1 - \frac{1}{p_1}, \eta_1 + 1; \eta_1 + 2; 1 - \frac{\wp_1^{p_1}}{\wp_2^{p_1}}\right) \right.$$

$$\left. - \wp_2^{1-p_1} \beta(\eta_1 + 1, i + 1) {}_2F_1\left(1 - \frac{1}{p_1}, \eta_1 + 1; \eta_1 + i + 2; 1 - \frac{\wp_1^{p_1}}{\wp_2^{p_1}}\right) \right]$$

$$\times \frac{1}{m} \sum_{j=1}^{m} \left[\frac{\wp_3^{1-p_2}}{\eta_2 + 1} {}_2F_1\left(1 - \frac{1}{p_2}, \eta_2 + 1; \eta_2 + 2; 1 - \frac{\wp_4^{p_2}}{\wp_3^{p_2}}\right) \right.$$

$$\left. - \wp_3^{1-p_2} \beta(\eta_2 + 1, j + 1) {}_2F_1\left(1 - \frac{1}{p_2}, \eta_2 + 1; \eta_2 + j + 2; 1 - \frac{\wp_4^{p_2}}{\wp_3^{p_2}}\right) \right],$$

$$\psi_8(\wp_1, \wp_2, \wp_3, \wp_4; p_1; p_2)$$

$$= \frac{1}{n} \sum_{i=1}^{n} \int_0^1 \frac{t^{\eta_1}(1 - (1-t)^i)}{[t\wp_1^{p_1} + (1-t)\wp_2^{p_1}]} \, dt \cdot \frac{1}{m} \sum_{j=1}^{m} \int_0^1 \frac{s^{\eta_2}(1 - s^j)}{[s\wp_4^{p_1} + (1-s)\wp_3^{p_2}]} \, ds$$

$$= \frac{1}{n} \sum_{i=1}^{n} \left[\frac{\wp_2^{1-p_1}}{\eta_1 + 1} {}_2F_1\left(1 - \frac{1}{p_1}, \eta_1 + 1; \eta_1 + 2; 1 - \frac{\wp_1^{p_1}}{\wp_2^{p_1}}\right) \right.$$

$$\left. - \wp_2^{1-p_1} \beta(\eta_1 + 1, i + 1) {}_2F_1\left(1 - \frac{1}{p_1}, \eta_1 + 1; \eta_1 + i + 2; 1 - \frac{\wp_1^{p_1}}{\wp_2^{p_1}}\right) \right]$$

$$\times \frac{1}{m} \sum_{j=1}^{m} \left[\frac{\wp_3^{1-p_2}}{\eta_2 + 1} {}_2F_1\left(1 - \frac{1}{p_2}, \eta_2 + 1; \eta_2 + 2; 1 - \frac{\wp_4^{p_2}}{\wp_3^{p_2}}\right) \right.$$

$$\left. - \frac{\wp_3^{1-p_2}}{\eta_2 + j + 1} {}_2F_1\left(1 - \frac{1}{p_2}, \eta_2 + j + 1; \eta_2 + j + 2; 1 - \frac{\wp_4^{p_2}}{\wp_3^{p_2}}\right) \right],$$

$$\psi_9(\wp_1, \wp_2, \wp_3, \wp_4; p_1; p_2)$$

$$= \frac{1}{n} \sum_{i=1}^{n} \int_0^1 \frac{t^{\eta_1}(1 - t^i)}{[t\wp_1^{p_1} + (1-t)\wp_2^{p_1}]} \, dt \cdot \frac{1}{m} \sum_{j=1}^{m} \int_0^1 \frac{s^{\eta_2}(1 - (1-s)^j)}{[s\wp_4^{p_1} + (1-s)\wp_3^{p_2}]} \, ds$$

$$= \frac{1}{n} \sum_{i=1}^{n} \left[\frac{\wp_2^{1-p_1}}{\eta_1 + 1} {}_2F_1\left(1 - \frac{1}{p_1}, \eta_1 + 1; \eta_1 + 2; 1 - \frac{\wp_1^{p_1}}{\wp_2^{p_1}}\right) \right.$$

$$\left. - \frac{\wp_2^{1-p_1}}{\eta_1 + i + 1} {}_2F_1\left(1 - \frac{1}{p_1}, \eta_1 + 1; \eta_1 + i + 2; 1 - \frac{\wp_1^{p_1}}{\wp_2^{p_1}}\right) \right]$$

$$\times \frac{1}{m} \sum_{j=1}^{m} \left[\frac{\wp_3^{1-p_2}}{\eta_2 + 1} {}_2F_1\left(1 - \frac{1}{p_2}, \eta_2 + 1; \eta_2 + 2; 1 - \frac{\wp_4^{p_2}}{\wp_3^{p_2}}\right) \right.$$

$$- \wp_3^{1-p_2} \beta(\eta_2 + 1, j + 1) {}_2F_1\left(1 - \tfrac{1}{p_2}, \eta_2 + 1; \eta_2 + j + 2; 1 - \tfrac{\wp_4^{p_2}}{\wp_3^{p_2}}\right)\Bigg],$$

$$\psi_{10}(\wp_1, \wp_2, \wp_3, \wp_4; p_1; p_2)$$

$$= \frac{1}{n} \sum_{i=1}^{n} \int_0^1 \frac{t^{\eta_1}(1 - t^i)}{[t\wp_1^{p_1} + (1-t)\wp_2^{p_1}]} \, dt \cdot \frac{1}{m} \sum_{j=1}^{m} \int_0^1 \frac{s^{\eta_2}(1 - s^j)}{[s\wp_4^{p_1} + (1-s)\wp_3^{p_2}]} \, ds$$

$$= \frac{1}{n} \sum_{i=1}^{n} \left[\frac{\wp_2^{1-p_1}}{\eta_1 + 1} {}_2F_1\left(1 - \tfrac{1}{p_1}, \eta_1 + 1; \eta_1 + 2; 1 - \tfrac{\wp_1^{p_1}}{\wp_2^{p_1}}\right) \right.$$

$$\left. - \frac{\wp_2^{1-p_1}}{\eta_1 + i + 1} {}_2F_1\left(1 - \tfrac{1}{p_1}, \eta_1 + 1; \eta_1 + i + 2; 1 - \tfrac{\wp_1^{p_1}}{\wp_2^{p_1}}\right) \right]$$

$$\times \frac{1}{m} \sum_{j=1}^{m} \left[\frac{\wp_3^{1-p_2}}{\eta_2 + 1} {}_2F_1\left(1 - \tfrac{1}{p_2}, \eta_2 + 1; \eta_2 + 2; 1 - \tfrac{\wp_4^{p_2}}{\wp_3^{p_2}}\right) \right.$$

$$\left. - \frac{\wp_3^{1-p_2}}{\eta_2 + j + 1} {}_2F_1\left(1 - \tfrac{1}{p_2}, \eta_2 + j + 1; \eta_2 + j + 2; 1 - \tfrac{\wp_4^{p_2}}{\wp_3^{p_2}}\right) \right],$$

$$\psi_{11}(\wp_1, \wp_2, \wp_3, \wp_4; p_1; p_2) = \int_0^1 \frac{t^{\eta_1}}{[t\wp_2^{p_1} + (1-t)\wp_1^{p_1}]} \, dt \int_0^1 \frac{s^{\eta_2}}{[s\wp_3^{p_1} + (1-s)\wp_4^{p_2}]} \, ds$$

$$= \left[\frac{\wp_1^{1-p_1}}{\eta_1 + 1} {}_2F_1\left(1 - \tfrac{1}{p_1}, \eta_1 + 1; \eta_1 + 2; 1 - \tfrac{\wp_2^{p_1}}{\wp_1^{p_1}}\right) \right]$$

$$\times \left[\frac{\wp_4^{1-p_2}}{\eta_2 + 1} {}_2F_1\left(1 - \tfrac{1}{p_2}, \eta_2 + 1; \eta_2 + 2; 1 - \tfrac{\wp_3^{p_2}}{\wp_4^{p_2}}\right) \right],$$

$$\psi_{12}(\wp_1, \wp_2, \wp_3, \wp_4; p_1; p_2)$$

$$= \frac{1}{n} \sum_{i=1}^{n} \int_0^1 \frac{t^{\eta_1}(1 - (1-t)^i)}{[t\wp_2^{p_1} + (1-t)\wp_1^{p_1}]} \, dt \cdot \frac{1}{m} \sum_{j=1}^{m} \int_0^1 \frac{s^{\eta_2}(1 - (1-s)^j)}{[s\wp_3^{p_1} + (1-s)\wp_4^{p_2}]} \, ds$$

$$= \frac{1}{n} \sum_{i=1}^{n} \left[\frac{\wp_1^{1-p_1}}{\eta_1 + 1} {}_2F_1\left(1 - \tfrac{1}{p_1}, \eta_1 + 1; \eta_1 + 2; 1 - \tfrac{\wp_2^{p_1}}{\wp_1^{p_1}}\right) \right.$$

$$\left. - \wp_1^{1-p_1} \beta(\eta_1 + 1, i + 1) {}_2F_1\left(1 - \tfrac{1}{p_1}, \eta_1 + 1; \eta_1 + i + 2; 1 - \tfrac{\wp_2^{p_1}}{\wp_1^{p_1}}\right) \right]$$

$$\times \frac{1}{m} \sum_{j=1}^{m} \left[\frac{\wp_4^{1-p_2}}{\eta_2+1} {}_2F_1\left(1-\frac{1}{p_2}, \eta_2+1; \eta_2+2; 1-\frac{\wp_3^{p_2}}{\wp_4^{p_2}}\right)\right.$$

$$\left. - \wp_4^{1-p_2}\beta(\eta_2+1, j+1){}_2F_1\left(1-\frac{1}{p_2}, \eta_2+1; \eta_2+j+2; 1-\frac{\wp_3^{p_2}}{\wp_4^{p_2}}\right)\right],$$

$$\psi_{13}(\wp_1, \wp_2, \wp_3, \wp_4; p_1; p_2)$$

$$= \frac{1}{n} \sum_{i=1}^{n} \int_0^1 \frac{t^{\eta_1}(1-(1-t)^i)}{[t\wp_2^{p_1}+(1-t)\wp_1^{p_1}]} \, dt \cdot \frac{1}{m} \sum_{j=1}^{m} \int_0^1 \frac{s^{\eta_2}(1-s^j)}{[s\wp_3^{p_1}+(1-s)\wp_4^{p_2}]} \, ds$$

$$= \frac{1}{n} \sum_{i=1}^{n} \left[\frac{\wp_1^{1-p_1}}{\eta_1+1} {}_2F_1\left(1-\frac{1}{p_1}, \eta_1+1; \eta_1+2; 1-\frac{\wp_2^{p_1}}{\wp_1^{p_1}}\right)\right.$$

$$\left. - \wp_1^{1-p_1}\beta(\eta_1+1, i+1){}_2F_1\left(1-\frac{1}{p_1}, \eta_1+1; \eta_1+i+2; 1-\frac{\wp_2^{p_1}}{\wp_1^{p_1}}\right)\right]$$

$$\times \frac{1}{m} \sum_{j=1}^{m} \left[\frac{\wp_4^{1-p_2}}{\eta_2+1} {}_2F_1\left(1-\frac{1}{p_2}, \eta_2+1; \eta_2+2; 1-\frac{\wp_3^{p_2}}{\wp_4^{p_2}}\right)\right.$$

$$\left. - \frac{\wp_4^{1-p_2}}{\eta_2+j+1} {}_2F_1\left(1-\frac{1}{p_2}, \eta_2+j+1; \eta_2+j+2; 1-\frac{\wp_3^{p_2}}{\wp_4^{p_2}}\right)\right],$$

$$\psi_{14}(\wp_1, \wp_2, \wp_3, \wp_4; p_1; p_2)$$

$$= \frac{1}{n} \sum_{i=1}^{n} \int_0^1 \frac{t^{\eta_1}(1-t^i)}{[t\wp_2^{p_1}+(1-t)\wp_1^{p_1}]} \, dt \cdot \frac{1}{m} \sum_{j=1}^{m} \int_0^1 \frac{s^{\eta_2}(1-(1-s)^j)}{[s\wp_3^{p_1}+(1-s)\wp_4^{p_2}]} \, ds$$

$$= \frac{1}{n} \sum_{i=1}^{n} \left[\frac{\wp_1^{1-p_1}}{\eta_1+1} {}_2F_1\left(1-\frac{1}{p_1}, \eta_1+1; \eta_1+2; 1-\frac{\wp_2^{p_1}}{\wp_1^{p_1}}\right)\right.$$

$$\left. - \frac{\wp_1^{1-p_1}}{\eta_1+i+1} {}_2F_1\left(1-\frac{1}{p_1}, \eta_1+1; \eta_1+i+2; 1-\frac{\wp_2^{p_1}}{\wp_1^{p_1}}\right)\right]$$

$$\times \frac{1}{m} \sum_{j=1}^{m} \left[\frac{\wp_4^{1-p_2}}{\eta_2+1} {}_2F_1\left(1-\frac{1}{p_2}, \eta_2+1; \eta_2+2; 1-\frac{\wp_3^{p_2}}{\wp_4^{p_2}}\right)\right.$$

$$\left. - \wp_4^{1-p_2}\beta(\eta_2+1, j+1){}_2F_1\left(1-\frac{1}{p_2}, \eta_2+1; \eta_2+j+2; 1-\frac{\wp_3^{p_2}}{\wp_4^{p_2}}\right)\right],$$

$$\psi_{15}(\wp_1, \wp_2, \wp_3, \wp_4; p_1; p_2)$$

$$= \frac{1}{n} \sum_{i=1}^{n} \int_0^1 \frac{t^{\eta_1}(1-t^i)}{[t\wp_2^{p_1} + (1-t)\wp_1^{p_1}]} \, dt \cdot \frac{1}{m} \sum_{j=1}^{m} \int_0^1 \frac{s^{\eta_2}(1-s^j)}{[s\wp_3^{p_2} + (1-s)\wp_4^{p_2}]} \, ds$$

$$= \frac{1}{n} \sum_{i=1}^{n} \left[\frac{\wp_1^{1-p_1}}{\eta_1 + 1} {}_2F_1 \left(1 - \frac{1}{p_1}, \eta_1 + 1; \eta_1 + 2; 1 - \frac{\wp_2^{p_1}}{\wp_1^{p_1}} \right) \right.$$

$$\left. - \frac{\wp_1^{1-p_1}}{\eta_1 + i + 1} {}_2F_1 \left(1 - \frac{1}{p_1}, \eta_1 + 1; \eta_1 + i + 2; 1 - \frac{\wp_2^{p_1}}{\wp_1^{p_1}} \right) \right]$$

$$\times \frac{1}{m} \sum_{j=1}^{m} \left[\frac{\wp_4^{1-p_2}}{\eta_2 + 1} {}_2F_1 \left(1 - \frac{1}{p_2}, \eta_2 + 1; \eta_2 + 2; 1 - \frac{\wp_3^{p_2}}{\wp_4^{p_2}} \right) \right.$$

$$\left. - \frac{\wp_4^{1-p_2}}{\eta_2 + j + 1} {}_2F_1 \left(1 - \frac{1}{p_2}, \eta_2 + j + 1; \eta_2 + j + 2; 1 - \frac{\wp_3^{p_2}}{\wp_4^{p_2}} \right) \right],$$

$$\psi_{16}(\wp_1, \wp_2, \wp_3, \wp_4; p_1; p_2)$$

$$= \int_0^1 \frac{t^{\eta_1}}{[t\wp_2^{p_1} + (1-t)\wp_1^{p_1}]} \, dt \int_0^1 \frac{s^{\eta_2}}{[s\wp_4^{p_2} + (1-s)\wp_3^{p_2}]} \, ds$$

$$= \left[\frac{\wp_1^{1-p_1}}{\eta_1 + 1} {}_2F_1 \left(1 - \frac{1}{p_1}, \eta_1 + 1; \eta_1 + 2; 1 - \frac{\wp_2^{p_1}}{\wp_1^{p_1}} \right) \right]$$

$$\times \left[\frac{\wp_3^{1-p_2}}{\eta_2 + 1} {}_2F_1 \left(1 - \frac{1}{p_2}, \eta_2 + 1; \eta_2 + 2; 1 - \frac{\wp_4^{p_2}}{\wp_3^{p_2}} \right) \right],$$

$$\psi_{17}(\wp_1, \wp_2, \wp_3, \wp_4; p_1; p_2)$$

$$= \frac{1}{n} \sum_{i=1}^{n} \int_0^1 \frac{t^{\eta_1}(1-(1-t)^i)}{[t\wp_2^{p_1} + (1-t)\wp_1^{p_1}]} \, dt \cdot \frac{1}{m} \sum_{j=1}^{m} \int_0^1 \frac{s^{\eta_2}(1-(1-s)^j)}{[s\wp_4^{p_2} + (1-s)\wp_3^{p_2}]} \, ds$$

$$= \frac{1}{n} \sum_{i=1}^{n} \left[\frac{\wp_1^{1-p_1}}{\eta_1 + 1} {}_2F_1 \left(1 - \frac{1}{p_1}, \eta_1 + 1; \eta_1 + 2; 1 - \frac{\wp_2^{p_1}}{\wp_1^{p_1}} \right) \right.$$

$$\left. - \wp_1^{1-p_1} \beta(\eta_1 + 1, i + 1) {}_2F_1 \left(1 - \frac{1}{p_1}, \eta_1 + 1; \eta_1 + i + 2; 1 - \frac{\wp_2^{p_1}}{\wp_1^{p_1}} \right) \right]$$

$$\times \frac{1}{m} \sum_{j=1}^{m} \left[\frac{\wp_3^{1-p_2}}{\eta_2 + 1} {}_2F_1 \left(1 - \frac{1}{p_2}, \eta_2 + 1; \eta_2 + 2; 1 - \frac{\wp_4^{p_2}}{\wp_3^{p_2}} \right) \right.$$

$$- \wp_3^{1-p_2} \beta(\eta_2 + 1, j + 1) {}_2F_1\left(1 - \frac{1}{p_2}, \eta_2 + 1; \eta_2 + j + 2; 1 - \frac{\wp_4^{p_2}}{\wp_3^{p_2}}\right)\Bigg],$$

$$\psi_{18}(\wp_1, \wp_2, \wp_3, \wp_4; p_1; p_2)$$

$$= \frac{1}{n} \sum_{i=1}^{n} \int_0^1 \frac{t^{\eta_1}(1 - (1 - t)^i)}{[t\wp_2^{p_1} + (1 - t)\wp_1^{p_1}]}\, dt \cdot \frac{1}{m} \sum_{j=1}^{m} \int_0^1 \frac{s^{\eta_2}(1 - s^j)}{[s\wp_4^{p_2} + (1 - s)\wp_3^{p_2}]}\, ds$$

$$= \frac{1}{n} \sum_{i=1}^{n} \Bigg[\frac{\wp_1^{1-p_1}}{\eta_1 + 1} {}_2F_1\left(1 - \frac{1}{p_1}, \eta_1 + 1; \eta_1 + 2; 1 - \frac{\wp_2^{p_1}}{\wp_1^{p_1}}\right)$$

$$- \wp_1^{1-p_1} \beta(\eta_1 + 1, i + 1) {}_2F_1\left(1 - \frac{1}{p_1}, \eta_1 + 1; \eta_1 + i + 2; 1 - \frac{\wp_2^{p_1}}{\wp_1^{p_1}}\right)\Bigg]$$

$$\times \frac{1}{m} \sum_{j=1}^{m} \Bigg[\frac{\wp_3^{1-p_2}}{\eta_2 + 1} {}_2F_1\left(1 - \frac{1}{p_2}, \eta_2 + 1; \eta_2 + 2; 1 - \frac{\wp_4^{p_2}}{\wp_3^{p_2}}\right)$$

$$- \frac{\wp_3^{1-p_2}}{\eta_2 + j + 1} {}_2F_1\left(1 - \frac{1}{p_2}, \eta_2 + j + 1; \eta_2 + j + 2; 1 - \frac{\wp_4^{p_2}}{\wp_3^{p_2}}\right)\Bigg],$$

$$\psi_{19}(\wp_1, \wp_2, \wp_3, \wp_4; p_1; p_2)$$

$$= \frac{1}{n} \sum_{i=1}^{n} \int_0^1 \frac{t^{\eta_1}(1 - t^i)}{[t\wp_2^{p_1} + (1 - t)\wp_1^{p_1}]}\, dt \cdot \frac{1}{m} \sum_{j=1}^{m} \int_0^1 \frac{s^{\eta_2}(1 - (1 - s)^j)}{[s\wp_4^{p_2} + (1 - s)\wp_3^{p_2}]}\, ds$$

$$= \frac{1}{n} \sum_{i=1}^{n} \Bigg[\frac{\wp_1^{1-p_1}}{\eta_1 + 1} {}_2F_1\left(1 - \frac{1}{p_1}, \eta_1 + 1; \eta_1 + 2; 1 - \frac{\wp_2^{p_1}}{\wp_1^{p_1}}\right)$$

$$- \frac{\wp_1^{1-p_1}}{\eta_1 + i + 1} {}_2F_1\left(1 - \frac{1}{p_1}, \eta_1 + 1; \eta_1 + i + 2; 1 - \frac{\wp_2^{p_1}}{\wp_1^{p_1}}\right)\Bigg]$$

$$\times \frac{1}{m} \sum_{j=1}^{m} \Bigg[\frac{\wp_3^{1-p_2}}{\eta_2 + 1} {}_2F_1\left(1 - \frac{1}{p_2}, \eta_2 + 1; \eta_2 + 2; 1 - \frac{\wp_4^{p_2}}{\wp_3^{p_2}}\right)$$

$$- \wp_3^{1-p_2} \beta(\eta_2 + 1, j + 1) {}_2F_1\left(1 - \frac{1}{p_2}, \eta_2 + 1; \eta_2 + j + 2; 1 - \frac{\wp_4^{p_2}}{\wp_3^{p_2}}\right)\Bigg],$$

$$\psi_{20}(\wp_1, \wp_2, \wp_3, \wp_4; p_1; p_2)$$

$$= \frac{1}{n} \sum_{i=1}^{n} \int_0^1 \frac{t^{\eta_1}(1-t^i)}{[t\wp_2^{p_1} + (1-t)\wp_1^{p_1}]} \, dt \cdot \frac{1}{m} \sum_{j=1}^{m} \int_0^1 \frac{s^{\eta_2}(1-s^j)}{[s\wp_4^{p_2} + (1-s)\wp_3^{p_2}]} \, ds$$

$$= \frac{1}{n} \sum_{i=1}^{n} \left[\frac{\wp_1^{1-p_1}}{\eta_1 + 1} {}_2F_1\left(1 - \frac{1}{p_1}, \eta_1 + 1; \eta_1 + 2; 1 - \frac{\wp_2^{p_1}}{\wp_1^{p_1}}\right) \right.$$

$$\left. - \frac{\wp_1^{1-p_1}}{\eta_1 + i + 1} {}_2F_1\left(1 - \frac{1}{p_1}, \eta_1 + 1; \eta_1 + i + 2; 1 - \frac{\wp_2^{p_1}}{\wp_1^{p_1}}\right) \right]$$

$$\times \frac{1}{m} \sum_{j=1}^{m} \left[\frac{\wp_3^{1-p_2}}{\eta_2 + 1} {}_2F_1\left(1 - \frac{1}{p_2}, \eta_2 + 1; \eta_2 + 2; 1 - \frac{\wp_4^{p_2}}{\wp_3^{p_2}}\right) \right.$$

$$\left. - \frac{\wp_3^{1-p_2}}{\eta_2 + j + 1} {}_2F_1\left(1 - \frac{1}{p_2}, \eta_2 + j + 1; \eta_2 + j + 2; 1 - \frac{\wp_4^{p_2}}{\wp_3^{p_2}}\right) \right].$$

Proof. It follows from Lemma 1 and the Hölder inequality

$$\frac{\mathcal{F}(\wp_1, \wp_3) + \mathcal{F}(\wp_1, \wp_4) + \mathcal{F}(\wp_2, \wp_3) + \mathcal{F}(\wp_2, \wp_4)}{4} - A$$

$$+ \frac{p_1^{\eta_1} p_2^{\eta_2} \Gamma(\eta_1 + 1)\Gamma(\eta_2 + 1)}{4(\wp_2^{p_1} - \wp_1^{p_1})^{\eta_1}(\wp_4^{p_2} - \wp_3^{p_2})^{\eta_2}} \left[{}^{p_1, p_2} I_{\wp_1^+, \wp_3^+}^{\eta_1, \eta_2} \mathcal{F}(\wp_2, \wp_4) + {}^{p_1, p_2} I_{\wp_1^+, \wp_4^-}^{\eta_1, \eta_2} \mathcal{F}(\wp_2, \wp_3) \right.$$

$$\left. + {}^{p_1, p_2} I_{\wp_2^-, \wp_3^+}^{\eta_1, \eta_2} \mathcal{F}(\wp_1, \wp_4) + {}^{p_1, p_2} I_{\wp_2^-, \wp_4^-}^{\eta_1, \eta_2} \mathcal{F}(\wp_1, \wp_3) \right]$$

$$= \frac{(\wp_2^{p_1} - \wp_1^{p_1})(\wp_4^{p_2} - \wp_3^{p_2})}{4p_1 p_2} \left[\left| \int_0^1 \int_0^1 \frac{t^{\eta_1} s^{\eta_2}}{[t\wp_1^{p_1} + (1-t)\wp_2^{p_1}]^{1-\frac{1}{p_1}} [s\wp_3^{p_2} + (1-s)\wp_4^{p_2}]^{1-\frac{1}{p_2}}} \right. \right.$$

$$\left. \times \frac{\partial^2 \mathcal{F}}{\partial t \partial s} \left(\left[t\wp_1^{p_1} + (1-t)\wp_2^{p_1}\right]^{\frac{1}{p_1}}, \left[s\wp_3^{p_2} + (1-s)\wp_4^{p_2}\right]^{\frac{1}{p_2}} \right) \right| dt ds$$

$$- \left| \int_0^1 \int_0^1 \frac{t^{\eta_1} s^{\eta_2}}{[t\wp_1^{p_1} + (1-t)\wp_2^{p_1}]^{1-\frac{1}{p_1}} [s\wp_4^{p_2} + (1-s)\wp_3^{p_2}]^{1-\frac{1}{p_2}}} \right.$$

$$\left. \times \frac{\partial^2 \mathcal{F}}{\partial t \partial s} \left(\left[t\wp_1^{p_1} + (1-t)\wp_2^{p_1}\right]^{\frac{1}{p_1}}, \left[s\wp_4^{p_2} + (1-s)\wp_3^{p_2}\right]^{\frac{1}{p_2}} \right) \right| dt ds$$

$$- \left| \int_0^1 \int_0^1 \frac{t^{\eta_1} s^{\eta_2}}{[t\wp_2^{p_1} + (1-t)\wp_1^{p_1}]^{1-\frac{1}{p_1}} [s\wp_3^{p_2} + (1-s)\wp_4^{p_2}]^{1-\frac{1}{p_2}}} \right.$$

$$\left. \times \frac{\partial^2 \mathcal{F}}{\partial t \partial s} \left(\left[t\wp_2^{p_1} + (1-t)\wp_1^{p_1}\right]^{\frac{1}{p_1}}, \left[s\wp_3^{p_2} + (1-s)\wp_4^{p_2}\right]^{\frac{1}{p_2}} \right) \right| dt ds$$

$$
\left. -\int_0^1\int_0^1 \frac{t^{\eta_1}s^{\eta_2}}{[t\wp_2^{p_1}+(1-t)\wp_1^{p_1}]^{1-\frac{1}{p_1}}[s\wp_4^{p_2}+(1-s)\wp_3^{p_2}]^{1-\frac{1}{p_2}}}\right|
$$

$$
\left.\times \frac{\partial^2 \mathcal{F}}{\partial t \partial s}\left(\left[t\wp_2^{p_1}+(1-t)\wp_1^{p_1}\right]^{\frac{1}{p_1}},\left[s\wp_4^{p_2}+(1-s)\wp_3^{p_2}\right]^{\frac{1}{p_2}}\right)\right| dt ds \right]
$$

$$
\leq \frac{(\wp_2^{p_1}-\wp_1^{p_1})(\wp_4^{p_2}-\wp_3^{p_2})}{4p_1p_2}\left[\left(\int_0^1\int_0^1 \frac{t^{\eta_1}s^{\eta_2}}{[t\wp_1^{p_1}+(1-t)\wp_2^{p_1}]^{1-\frac{1}{p_1}}[s\wp_3^{p_2}+(1-s)\wp_4^{p_2}]^{1-\frac{1}{p_2}}}\right)^{1-\frac{1}{\alpha}}\right.
$$

$$
\times \left(\int_0^1\int_0^1 \frac{t^{\eta_1}s^{\eta_2}}{[t\wp_1^{p_1}+(1-t)\wp_2^{p_1}]^{1-\frac{1}{p_1}}[s\wp_3^{p_2}+(1-s)\wp_4^{p_2}]^{1-\frac{1}{p_2}}}\right.
$$

$$
\times \left.\left.\frac{\partial^2 \mathcal{F}}{\partial t \partial s}\left([t\wp_1^{p_1}+(1-t)\wp_2^{p_1}]^{\frac{1}{p_1}},[s\wp_3^{p_2}+(1-s)\wp_4^{p_2}]^{\frac{1}{p_2}}\right)\right|^\alpha dt ds\right)^{\frac{1}{\alpha}} \tag{7}
$$

$$
-\left(\int_0^1\int_0^1 \frac{t^{\eta_1}s^{\eta_2}}{[t\wp_1^{p_1}+(1-t)\wp_2^{p_1}]^{1-\frac{1}{p_1}}[s\wp_4^{p_2}+(1-s)\wp_3^{p_2}]^{1-\frac{1}{p_2}}}\right)^{1-\frac{1}{\alpha}}
$$

$$
\times \left(\int_0^1\int_0^1 \frac{t^{\eta_1}s^{\eta_2}}{[t\wp_1^{p_1}+(1-t)\wp_2^{p_1}]^{1-\frac{1}{p_1}}[s\wp_4^{p_2}+(1-s)\wp_3^{p_2}]^{1-\frac{1}{p_2}}}\right.
$$

$$
\times \left.\left.\frac{\partial^2 \mathcal{F}}{\partial t \partial s}\left([t\wp_1^{p_1}+(1-t)\wp_2^{p_1}]^{\frac{1}{p_1}},[s\wp_4^{p_2}+(1-s)\wp_3^{p_2}]^{\frac{1}{p_2}}\right)\right|^\alpha dt ds\right)^{\frac{1}{\alpha}} \tag{8}
$$

$$
-\left(\int_0^1\int_0^1 \frac{t^{\eta_1}s^{\eta_2}}{[t\wp_2^{p_1}+(1-t)\wp_1^{p_1}]^{1-\frac{1}{p_1}}[s\wp_3^{p_2}+(1-s)\wp_4^{p_2}]^{1-\frac{1}{p_2}}}\right)^{1-\frac{1}{\alpha}}
$$

$$
\times \left(\int_0^1\int_0^1 \frac{t^{\eta_1}s^{\eta_2}}{[t\wp_2^{p_1}+(1-t)\wp_1^{p_1}]^{1-\frac{1}{p_1}}[s\wp_3^{p_2}+(1-s)\wp_4^{p_2}]^{1-\frac{1}{p_2}}}\right.
$$

$$
\times \left.\left.\frac{\partial^2 \mathcal{F}}{\partial t \partial s}\left([t\wp_2^{p_1}+(1-t)\wp_1^{p_1}]^{\frac{1}{p_1}},[s\wp_3^{p_2}+(1-s)\wp_4^{p_2}]^{\frac{1}{p_2}}\right)\right|^\alpha dt ds\right)^{\frac{1}{\alpha}} \tag{9}
$$

$$
+\left(\int_0^1\int_0^1 \frac{t^{\eta_1}s^{\eta_2}}{[t\wp_2^{p_1}+(1-t)\wp_1^{p_1}]^{1-\frac{1}{p_1}}[s\wp_4^{p_2}+(1-s)\wp_3^{p_2}]^{1-\frac{1}{p_2}}}\right)^{1-\frac{1}{\alpha}}
$$

$$
\times \left(\int_0^1\int_0^1 \frac{t^{\eta_1}s^{\eta_2}}{[t\wp_2^{p_1}+(1-t)\wp_1^{p_1}]^{1-\frac{1}{p_1}}[s\wp_4^{p_2}+(1-s)\wp_3^{p_2}]^{1-\frac{1}{p_2}}}\right|
$$

$$\times \frac{\partial^2 \mathcal{F}}{\partial t \partial s}\left([t\wp_2^{p_1} + (1-t)\wp_1^{p_1}]^{\frac{1}{p_1}}, [s\wp_4^{p_2} + (1-s)\wp_3^{p_2}]^{\frac{1}{p_2}}\right)\bigg|^{\alpha} dtds\bigg)^{\frac{1}{\alpha}}. \tag{10}$$

From the (m,n)-polynomial (p_1, p_2)-convexity of function $\left|\frac{\partial^2 \mathcal{F}}{\partial t \partial s}\right|^{\alpha}$ on Δ, we get

$$\int_0^1 \int_0^1 \frac{t^{\eta_1} s^{\eta_2}}{[t\wp_1^{p_1} + (1-t)\wp_2^{p_1}]^{1-\frac{1}{p_1}} [s\wp_3^{p_2} + (1-s)\wp_4^{p_2}]^{1-\frac{1}{p_2}}}$$

$$\times \left|\frac{\partial^2 \mathcal{F}}{\partial t \partial s}\left(\left[t\wp_1^{p_1} + (1-t)\wp_2^{p_1}\right]^{\frac{1}{p_1}}, \left[s\wp_3^{p_2} + (1-s)\wp_4^{p_2}\right]^{\frac{1}{p_2}}\right)\right|^{\alpha} dtds$$

$$\leq \int_0^1 \int_0^1 \frac{t^{\eta_1} s^{\eta_2}}{[t\wp_1^{p_1} + (1-t)\wp_2^{p_1}]^{1-\frac{1}{p_1}} [s\wp_3^{p_2} + (1-s)\wp_4^{p_2}]^{1-\frac{1}{p_2}}}$$

$$\times \left[\frac{1}{n}\sum_{i=1}^{n}[1 - (1-t)^i]\frac{1}{m}\sum_{j=1}^{m}[1 - (1-s)^j]\left|\frac{\partial^2 \mathcal{F}}{\partial t \partial s}(\wp_1, \wp_3)\right|^{\alpha}\right.$$

$$+ \frac{1}{n}\sum_{i=1}^{n}[1 - t^i]\frac{1}{m}\sum_{j=1}^{m}[1 - s^j]\left|\frac{\partial^2 \mathcal{F}}{\partial t \partial s}(\wp_1, \wp_4)\right|^{\alpha}$$

$$+ \frac{1}{n}\sum_{i=1}^{n}[1 - (1-t)^i]\frac{1}{m}\sum_{j=1}^{m}[1 - (1-s)^j]\left|\frac{\partial^2 \mathcal{F}}{\partial t \partial s}(\wp_2, \wp_3)\right|^{\alpha}$$

$$+ \left.\frac{1}{n}\sum_{i=1}^{n}[1 - t^i]\frac{1}{m}\sum_{j=1}^{m}[1 - s^j]\left|\frac{\partial^2 \mathcal{F}}{\partial t \partial s}(\wp_2, \wp_4)\right|^{\alpha}\right] dtds$$

$$= \left|\frac{\partial^2 \mathcal{F}}{\partial t \partial s}(\wp_1, \wp_3)\right|^{\alpha} \psi_2(\wp_1, \wp_2, \wp_3, \wp_4; p_1; p_2)$$

$$+ \left|\frac{\partial^2 \mathcal{F}}{\partial t \partial s}(\wp_1, \wp_4)\right|^{\alpha} \psi_3(\wp_1, \wp_2, \wp_3, \wp_4; p_1; p_2)$$

$$+ \left|\frac{\partial^2 \mathcal{F}}{\partial t \partial s}(\wp_2, \wp_3)\right|^{\alpha} \psi_4(\wp_1, \wp_2, \wp_3, \wp_4; p_1; p_2)$$

$$+ \left|\frac{\partial^2 \mathcal{F}}{\partial t \partial s}(\wp_2, \wp_4)\right|^{\alpha} \psi_5(\wp_1, \wp_2, \wp_3, \wp_4; p_1; p_2). \tag{11}$$

Similarly, we can write

$$\int_0^1 \int_0^1 \frac{t^{\eta_1} s^{\eta_2}}{[t\wp_1^{p_1} + (1-t)\wp_2^{p_1}]^{1-\frac{1}{p_1}} [s\wp_4^{p_2} + (1-s)\wp_3^{p_2}]^{1-\frac{1}{p_2}}}$$

$$\times \left|\frac{\partial^2 \mathcal{F}}{\partial t \partial s}\left(\left[t\wp_1^{p_1} + (1-t)\wp_2^{p_1}\right]^{\frac{1}{p_1}}, \left[s\wp_4^{p_2} + (1-s)\wp_3^{p_2}\right]^{\frac{1}{p_2}}\right)\right|^{\alpha} dtds$$

$$\leq \left| \frac{\partial^2 \mathcal{F}}{\partial t \partial s}(\wp_1, \wp_4) \right|^\alpha \psi_7(\wp_1, \wp_2, \wp_3, \wp_4; p_1; p_2)$$

$$+ \left| \frac{\partial^2 \mathcal{F}}{\partial t \partial s}(\wp_1, \wp_3) \right|^\alpha \psi_8(\wp_1, \wp_2, \wp_3, \wp_4; p_1; p_2)$$

$$+ \left| \frac{\partial^2 \mathcal{F}}{\partial t \partial s}(\wp_2, \wp_4) \right|^\alpha \psi_9(\wp_1, \wp_2, \wp_3, \wp_4; p_1; p_2)$$

$$+ \left| \frac{\partial^2 \mathcal{F}}{\partial t \partial s}(\wp_2, \wp_3) \right|^\alpha \psi_{10}(\wp_1, \wp_2, \wp_3, \wp_4; p_1; p_2), \tag{12}$$

$$\int_0^1 \int_0^1 \frac{t^{\eta_1} s^{\eta_2}}{[t\wp_2^{p_1} + (1-t)\wp_1^{p_1}]^{1-\frac{1}{p_1}} [s\wp_3^{p_2} + (1-s)\wp_4^{p_2}]^{1-\frac{1}{p_2}}}$$

$$\times \left| \frac{\partial^2 \mathcal{F}}{\partial t \partial s}\left(\left[t\wp_2^{p_1} + (1-t)\wp_1^{p_1} \right]^{\frac{1}{p_1}}, \left[s\wp_3^{p_2} + (1-s)\wp_4^{p_2} \right]^{\frac{1}{p_2}} \right) \right|^\alpha dtds$$

$$\leq \left| \frac{\partial^2 \mathcal{F}}{\partial t \partial s}(\wp_2, \wp_3) \right|^\alpha \psi_{12}(\wp_1, \wp_2, \wp_3, \wp_4; p_1; p_2)$$

$$+ \left| \frac{\partial^2 \mathcal{F}}{\partial t \partial s}(\wp_2, \wp_4) \right|^\alpha \psi_{13}(\wp_1, \wp_2, \wp_3, \wp_4; p_1; p_2)$$

$$+ \left| \frac{\partial^2 \mathcal{F}}{\partial t \partial s}(\wp_1, \wp_3) \right|^\alpha \psi_{14}(\wp_1, \wp_2, \wp_3, \wp_4; p_1; p_2)$$

$$+ \left| \frac{\partial^2 \mathcal{F}}{\partial t \partial s}(\wp_1, \wp_4) \right|^\alpha \psi_{15}(\wp_1, \wp_2, \wp_3, \wp_4; p_1; p_2), \tag{13}$$

$$\int_0^1 \int_0^1 \frac{t^{\eta_1} s^{\eta_2}}{[t\wp_2^{p_1} + (1-t)\wp_1^{p_1}]^{1-\frac{1}{p_1}} [s\wp_4^{p_2} + (1-s)\wp_3^{p_2}]^{1-\frac{1}{p_2}}}$$

$$\times \left| \frac{\partial^2 \mathcal{F}}{\partial t \partial s}\left(\left[t\wp_2^{p_1} + (1-t)\wp_1^{p_1} \right]^{\frac{1}{p_1}}, \left[s\wp_4^{p_2} + (1-s)\wp_3^{p_2} \right]^{\frac{1}{p_2}} \right) \right|^\alpha dtds$$

$$\leq \left| \frac{\partial^2 \mathcal{F}}{\partial t \partial s}(\wp_2, \wp_4) \right|^\alpha \psi_{17}(\wp_1, \wp_2, \wp_3, \wp_4; p_1; p_2)$$

$$+ \left| \frac{\partial^2 \mathcal{F}}{\partial t \partial s}(\wp_2, \wp_3) \right|^\alpha \psi_{18}(\wp_1, \wp_2, \wp_3, \wp_4; p_1; p_2)$$

$$+ \left| \frac{\partial^2 \mathcal{F}}{\partial t \partial s}(\wp_1, \wp_4) \right|^\alpha \psi_{19}(\wp_1, \wp_2, \wp_3, \wp_4; p_1; p_2)$$

$$+ \left| \frac{\partial^2 \mathcal{F}}{\partial t \partial s}(\wp_1, \wp_3) \right|^\alpha \psi_{20}(\wp_1, \wp_2, \wp_3, \wp_4; p_1; p_2). \tag{14}$$

Inequalities (7)–(14) give the desired inequality (6).

5. Conclusions

Fractional calculus has a bigger impact and yields more accurate results when looking at computer models. The fields of applied mathematics, mathematical biology, engineering, simulation, and inequality theory all make extensive use of fractional calculus. Fractional calculus has attracted the attention of numerous researchers from a variety of scientific fields. In this work, we have introduced the new definition-(m, n)-polynomial (p_1, p_2)-convexity on coordinates. On the basis of this newly introduced concept, we were able to formulate new Hermite-Hadamard-type inequalities via generalized fractional operators. Further, we derived the refinements of Hermite-Hadamard type inequalities on the basis of newly introduced lemma. The novel idea proposed in this paper can be applied to a wide range of inequalities that use the fractional Hermite-Hadamard. Some novel concepts, like interval-valued R-L convexities, center-radius order convexities, and fuzzy interval convexities, can be used to make additional extensions. Researchers who are enthusiastic and motivated are able to employ quantum calculus and interval-valued functions to see how this inequality behaves. It will be fascinating to observe how different new convex function types are used to investigate fractional versions of inequalities.

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