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Riesz Potential and Its Commutators on Mixed Morrey Spaces

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Abstract. In this paper, we investigate necessary and sufficient conditions for the boundedness of the Riesz potential operator I_α and its commutator on mixed Morrey spaces $L_{\vec{p},\lambda}(\mathbb{R}^n)$. We characterize the Spanne-type and Adams-type boundedness of I_α on $L_{\vec{p},\lambda}(\mathbb{R}^n)$, respectively. Furthermore, we establish necessary and sufficient conditions for the boundedness of the commutator $[b, I_\alpha]$ on $L_{\vec{p},\lambda}(\mathbb{R}^n)$ when $b \in BMO(\mathbb{R}^n)$. As applications, we obtain estimates for the Marcinkiewicz operator and for fractional powers of certain analytic semigroups on mixed Morrey spaces.

Key Words and Phrases: Riesz potential operator, maximal operator, mixed Morrey space, commutator, BMO space, Marcinkiewicz operator, fractional powers of some analytic semigroups.

2010 Mathematics Subject Classifications: 42B20, 42B25, 42B35

1. Introduction

The theory of mixed-norm function spaces has undergone substantial development over the past few decades. Nevertheless, the standard literature still regards mixed Lebesgue spaces $L_{\vec{p}}(\mathbb{R}^n)$, $0 < \vec{p} \le \infty$, as a natural generalization of the classical Lebesgue spaces $L_p(\mathbb{R}^n)$, $0 < p \le \infty$, first introduced by Benedek and Panzone [7] in 1961. Owing to their finer structural properties compared with their classical counterparts, mixed-norm function spaces admit a richer analytical framework and consequently find broader applications in areas such as potential analysis, harmonic analysis, and partial differential equations.

Morrey spaces, denoted by $L_{p,\lambda}(\mathbb{R}^n)$ and named after Charles Morrey [21], generalize Lebesgue spaces by including a parameter λ that takes into account the local behavior of functions in a given domain, while Lebesgue spaces typically consider global properties of a function. By adding this new parameter, Morrey

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spaces provide a more detailed analysis of functions, which is especially useful in the study of partial differential equations (PDEs). Since then, many works have been devoted to various Morrey spaces. In 2019, Nogayama [22] generalized Morrey spaces and mixed Lebesgue spaces to introduce a new Morrey-type space called the mixed Morrey space, see also [23].

In this paper, we consider the mixed Morrey spaces $L_{\vec{p},\lambda}(\mathbb{R}^n)$. These spaces generalize the mixed Lebesgue spaces so that $L_{\vec{p},0}(\mathbb{R}^n) \equiv L_{\vec{p}}(\mathbb{R}^n)$ and the Morrey spaces so that $L_{\vec{p},\lambda}(\mathbb{R}^n) \equiv L_{p,\lambda}(\mathbb{R}^n)$, $\vec{p} = (p, \dots, p)$. We characterize the Spanne and Adams type boundedness of I_α on $L_{\vec{p},\lambda}(\mathbb{R}^n)$, respectively. We also give necessary and sufficient conditions for the boundedness of the commutator of the Riesz potential operator $[b, I_\alpha]$ on $L_{\vec{p},\lambda}(\mathbb{R}^n)$ when b belongs to the spaces $BMO(\mathbb{R}^n)$. As applications, we obtain some estimates for the Marcinkiewicz operator and fractional powers of some analytic semigroups on the mixed Morrey spaces, see [1, 5, 6, 10, 15, 16, 17, 18, 19, 20, 24, 25, 28].

The Hardy-Littlewood-Sobolev (HLS) inequality, which bounds the Riesz potential operator in Lebesgue spaces, has been extended to mixed Morrey spaces. These inequalities specify conditions under which the Riesz potential maps functions from a mixed Morrey space to another mixed Morrey space with potentially different parameters. In particular, the HLS inequality in mixed Morrey spaces investigates how applying the Riesz potential affects the integrability and decay of a function, as measured by the norms of the mixed Morrey space.

For $x \in \mathbb{R}^n$ and $t > 0$, let $B(x,t)$ denote the open ball centered at x of radius r and ${}^{\complement}B(x,t) = \mathbb{R}^n \setminus B(x,t)$.

One of the most important variants of the Hardy-Littlewood maximal function is the so-called fractional maximal function defined by the formula

$$M_\alpha f(x) = \sup_{t>0} |B(x,t)|^{-1+\alpha/n} \int_{B(x,t)} |f(y)| dy, \quad 0 \le \alpha < n,$$

where $|B(x,t)| = v_n t^n$ is the Lebesgue measure of the ball $B(x,t)$ and v_n is the volume of the unit ball in $\mathbb{R}^n$. It coincides with the Hardy-Littlewood maximal function $Mf \equiv M_0 f$ and is intimately related to the Riesz potential

$$I_\alpha f(x) = \int_{\mathbb{R}^n} \frac{f(y)}{|x-y|^{n-\alpha}}\, dy, \qquad 0 < \alpha < n$$

(see, for example, [3] and [4]). The operators M_α and I_α play important role in real and harmonic analysis (see, for example [12] and [30]).

The aim of this paper is to establish necessary and sufficient conditions for the boundedness of the Riesz potential operator I_α and its commutator $[b, I_\alpha]$ on $L_{\vec{p},\lambda}(\mathbb{R}^n)$, when b belongs to the spaces $BMO(\mathbb{R}^n)$.

The structure of the paper is as follows. Section 1 presents definitions, auxiliary results, and some embeddings into the mixed Morrey space $L_{\vec{p},\lambda}(\mathbb{R}^n)$. Section 2 characterizes the boundedness of the Spanne and Adams types for the Riesz potential operator I_α on the spaces $L_{\vec{p},\lambda}(\mathbb{R}^n)$, respectively. Section 3 provides necessary and sufficient conditions for the boundedness of the commutator of the Riesz potential operator $[b, I_\alpha]$ on the spaces $L_{\vec{p},\lambda}(\mathbb{R}^n)$. Section 4 provides sufficient conditions for the boundedness of the modified Riesz potential operator $\widetilde{I}_\alpha$ on the spaces $L_{\vec{p},\lambda}(\mathbb{R}^n)$. In Section 5, as an application, estimates are obtained for the Marcinkiewicz operator and fractional powers of some analytic semigroups in mixed Morrey spaces.

By $A \lesssim B$ we mean that $A \le CB$ with some positive constant C independent of appropriate quantities. If $A \lesssim B$ and $B \lesssim A$, we write $A \approx B$ and say that A and B are equivalent.

2. Definition and basic properties of mixed Morrey spaces

For any $r > 0$ and $x \in \mathbb{R}^n$, let $B(x,r) = \{y \ : \ |y - x| < r\}$ be the ball centered at x with radius r. Let $\mathrm{B} = \{B(x,r) \ : \ x \in \mathbb{R}^n, \ r > 0\}$ be the set of all such balls. We also use χ_E and $|E|$ to denote the characteristic function and the Lebesgue measure of a measurable set E.

The letter $\vec{p}$ denotes n-tuples of the numbers in $[0, \ \infty]$, $(n \ge 1)$, $\vec{p} = (p_1, \ \cdots \ , \ p_n)$. By definition, the inequality, for example, $0 < \vec{p} < \infty$ means $0 < p_i < \infty$ for all i. For $1 \le \vec{p} \le \infty$, we denote $\dfrac{1}{P} = \dfrac{1}{n}\displaystyle\sum_{i=1}^{n}\dfrac{1}{p_i}$, $\vec{p}' = (p_1', \ \cdots \ , p_n')$, where p_i', P' satisfies $\dfrac{1}{p_i} + \dfrac{1}{p_i'} = 1$, $\dfrac{1}{P} + \dfrac{1}{P'} = 1$.

We first recall the definition of mixed Lebesgue space defined in [7].

Let $\vec{p} = (p_1, \dots, p_n) \in (0, \infty]^n$. Then the mixed Lebesgue norm $\| \cdot \|_{L_{\vec{p}}}$ or $\| \cdot \|_{L_{(p_1,\dots,p_n)}}$ is defined by

$$\|f\|_{L_{\vec{p}}} \equiv \|f\|_{L_{(p_1,\dots,p_n)}} = \Big(\int_{\mathbb{R}} \cdots \Big(\int_{\mathbb{R}} \Big(\int_{\mathbb{R}} |f(x_1, \ x_2, \ \dots, \ x_n)|^{p_1} dx_1 \Big)^{\frac{p_2}{p_1}} dx_2 \Big)^{\frac{p_3}{p_2}} \dots dx_n \Big)^{\frac{1}{p_n}},$$

where $f : \mathbb{R}^n \to \mathbb{R}$ is a measurable function. If $p_j = \infty$ for some $j = 1, \dots, n$, then we have to make appropriate modifications. We define the mixed Lebesgue space $L_{\vec{p}}(\mathbb{R}^n) = L_{(p_1,\dots,p_n)}(\mathbb{R}^n)$ to be the set of all $f \in L_0(\mathbb{R}^n)$ with $\|f\|_{L_{\vec{p}}} < \infty$, where $L_0(\mathbb{R}^n)$ denotes the set of measureable functions on $\mathbb{R}^n$.

The following analogue of the Hölder's inequality for $L_{\vec{p}}$ is well known (see, for example, [34]).

Theorem 1. *Let $\Omega \subset \mathbb{R}^n$ be a measurable set, $1 \le \vec{p} \le \infty$ and $\frac{1}{\vec{p}} + \frac{1}{\vec{p}'} = 1$. Then for any $f \in L_{\vec{p}}(\Omega)$ and $g \in L_{\vec{p}'}(\Omega)$, the following inequality is valid*

$$\int_{\Omega} |f(x)g(x)|dx \le \|f\|_{L_{\vec{p}}(\Omega)} \|g\|_{L_{\vec{p}'}(\Omega)}.$$

By elementary calculations we have the following property.

Lemma 1. *Let $0 < \vec{p} < \infty$ and B be a balls in $\mathbb{R}^n$. Then*

$$\|\chi_B\|_{L_{\vec{p}}} = \|\chi_B\|_{WL_{\vec{p}}} = |B|^{\frac{1}{P}}.$$

By Theorem 1 and Lemma 1 we get the following estimate.

Lemma 2. *For $1 \le \vec{p} < \infty$ and for the balls $B = B(x,r)$ the following inequality is valid:*

$$\int_B |f(y)|dy \le |B|^{\frac{1}{P'}} \|f\|_{L_{\vec{p}}(B)}.$$

The following lemma shows the Lebesgue differential theorem in mixed-norm Lebesgue spaces as follows.

Lemma 3. *[34, Lemma 2.4] Let $f \in L_1^{\rm loc}(\mathbb{R}^n)$ and $0 < \vec{p} < \infty$, then*

$$\lim_{r\to 0} \|\chi_{B(x,r)}\|_{L_{\vec{p}}}^{-1} \|f\|_{L_{\vec{p}}(B(x,r))} = |f(x)| \quad a.e.\ x \in \mathbb{R}^n.$$

Definition 1. *Let $0 < \vec{p} < \infty$, $\lambda \in \mathbb{R}$. We denote by $L_{\vec{p},\lambda}(\mathbb{R}^n)$ the mixed Morrey space [22] the set of all classes of locally integrable functions f with the finite norm*

$$\|f\|_{L_{\vec{p},\lambda}} = \sup_{x\in\mathbb{R}^n,\, t>0} t^{-\frac{\lambda}{P}} \|f\|_{L_{\vec{p}}(B(x,t))}.$$

Also we define the weak mixed Morrey space $WL_{\vec{p},\lambda}(\mathbb{R}^n)$ [22] as the set of all locally integrable functions f with finite norm

$$\|f\|_{WL_{\vec{p},\lambda}} = \sup_{x\in\mathbb{R}^n,\, t>0} t^{-\frac{\lambda}{P}} \|f\|_{WL_{\vec{p}}(B(x,t))}.$$

Note that

$$L_{\vec{p},0}(\mathbb{R}^n) = L_{\vec{p}}(\mathbb{R}^n) = L_{\vec{p}}(\mathbb{R}^n),\ WL_{\vec{p},0}(\mathbb{R}^n) = WL_{\vec{p}}(\mathbb{R}^n),$$
$$\|f\|_{WL_{\vec{p},\lambda}} \le \|f\|_{L_{\vec{p},\lambda}} \text{ and therefore } L_{\vec{p},\lambda}(\mathbb{R}^n) \subset WL_{\vec{p},\lambda}(\mathbb{R}^n)$$

and if $\lambda < 0$ or $\lambda > n$, then $L_{\vec{p},\lambda}(\mathbb{R}^n) = WL_{\vec{p},\lambda}(\mathbb{R}^n) = \Theta$, where $\Theta \equiv \Theta(\mathbb{R}^n)$ is the set of all functions equivalent to 0 on $\mathbb{R}^n$.

Lemma 4. *If $0 < \vec{p} < \infty$, $0 \le \lambda_2 \le \lambda_1 \le n$, then*

$$L_{\vec{p},\lambda_1}(\mathbb{R}^n) \subset\!\succ L_{\vec{p},\lambda_2}(\mathbb{R}^n)$$

and

$$\|f\|_{L_{\vec{p},\lambda_2}} \le \|f\|_{L_{\vec{p},\lambda_1}}.$$

Proof. Let $f \in L_{\vec{p},\lambda_1}$, $0 < \vec{p} < \infty$, $0 \le \lambda_2 \le \lambda_1 \le n$. Then

$$\|f\|_{L_{\vec{p},\lambda_2}} = \sup_{x\in\mathbb{R}^n,\, t>0} t^{-\frac{\lambda_2}{P}} \|f\|_{L_{\vec{p}}(B(x,t))} \le \sup_{x\in\mathbb{R}^n,\, t>0} t^{-\frac{\lambda_1}{P}} \|f\|_{L_{\vec{p}}(B(x,t))} \le \|f\|_{L_{\vec{p},\lambda_1}}.$$

◄

Lemma 5. *If $0 < \vec{p} < \infty$, $0 \le \lambda \le n$, then*

$$L_{\vec{p},n}(\mathbb{R}^n) = L_\infty(\mathbb{R}^n)$$

and

$$\|f\|_{L_{\vec{p},n}} = v_n^{\frac{n}{P}} \|f\|_{L_\infty}.$$

Proof. Let $f \in L_\infty(\mathbb{R}^n)$. Then for all $x \in \mathbb{R}^n$ and $t > 0$

$$t^{-\frac{n}{P}} \|f\|_{L_{\vec{p}}(B(x,t))} \le v_n^{\frac{n}{P}} \|f\|_{L_\infty}.$$

Therefore $f \in L_{\vec{p},n}(\mathbb{R}^n)$ and

$$\|f\|_{L_{\vec{p},n}} \le v_n^{\frac{n}{P}} \|f\|_{L_\infty}.$$

Let $f \in L_{\vec{p},n}(\mathbb{R}^n)$. By the Lebesgue's differentiation theorem we have (see Lemma 3)

$$\lim_{t\to 0} |B(x,t)|^{-\frac{1}{P}} \|f\|_{L_{\vec{p}}(B(x,t))} = |f(x)| \quad \text{for} \quad a.e.\ x \in \mathbb{R}^n.$$

Then for a.e. $x \in \mathbb{R}^n$

$$\begin{aligned} |f(x)| &= \lim_{t\to 0} |B(x,t)|^{-\frac{1}{P}} \|f\|_{L_{\vec{p}}(B(x,t))} \\ &\le v_n^{-\frac{n}{P}} \sup_{x\in\mathbb{R}^n,\, 0<t\le 1} t^{-\frac{n}{P}} \|f\|_{L_{\vec{p}}(B(x,t))} \le v_n^{-\frac{n}{P}} \|f\|_{L_{\vec{p},n}}. \end{aligned}$$

Therefore $f \in L_\infty(\mathbb{R}^n)$ and

$$\|f\|_{L_\infty} \le v_n^{-\frac{n}{P}} \|f\|_{L_{\vec{p},n}}.$$

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Lemma 6. *If $0 \le \lambda < n$, $0 \le \alpha < n-\lambda$, then for $1 < \vec{p} < \infty$ and $P = \frac{n-\lambda}{\alpha}$*

$$L_{\vec{p},\lambda}(\mathbb{R}^n) \subset\!\succ L_{1,n-\alpha}(\mathbb{R}^n)$$

and for $f \in L_{\vec{p},\lambda}(\mathbb{R}^n)$ the following inequality

$$\|f\|_{L_{1,n-\alpha}} \le v_n^{\frac{n}{P'}} \|f\|_{L_{\vec{p},\lambda}}$$

is valid.

Proof. Let $0 < \alpha < n$, $0 \le \lambda < n$, $f \in L_{\vec{p},\lambda}(\mathbb{R}^n)$ and $P = \frac{n-\lambda}{\alpha}$. By the Hölder's inequality (see Theorem 1) we have

$$\|f\|_{L_{1,n-\alpha}} = \sup_{x\in\mathbb{R}^n,\, t>0} t^{\alpha-n} \|f\|_{L_1(B(x,t))} \le \sup_{x\in\mathbb{R}^n,\, t>0} t^{\alpha-n} \|f\|_{L_{\vec{p}}(B(x,t))} \|1\|_{L_{\vec{p}'}(B(x,t))}$$
$$\le v_n^{\frac{n}{P'}} \sup_{x\in\mathbb{R}^n,\, t>0} t^{\alpha-\frac{n-\lambda}{P}} t^{-\frac{\lambda}{P}} \|f\|_{L_{\vec{p}}(B(x,t))}$$
$$\le v_n^{\frac{n}{P'}} \|f\|_{L_{\vec{p},\lambda}} \sup_{t>0} t^{\alpha-\frac{n-\lambda}{P}} = v_n^{\frac{n}{P'}} \|f\|_{L_{\vec{p},\lambda}}.$$

Therefore $f \in L_{1,n-\alpha}(\mathbb{R}^n)$ and

$$\|f\|_{L_{1,n-\alpha}} \le v_n^{\frac{n}{P'}} \|f\|_{L_{\vec{p},\lambda}}.$$

◄

Theorem 2. [*32, Corollary* 5.1] *Let $1 < \vec{p} \le \vec{q} < \infty$, $\vec{p} \ne \vec{q}$, $0 \le \lambda < n$. If $f \in L_{\vec{p},\lambda}(\mathbb{R}^n)$, then $Mf \in L_{\vec{p},\lambda}(\mathbb{R}^n)$ and*

$$\|Mf\|_{L_{\vec{p},\lambda}} \le C_{\vec{p},\lambda} \|f\|_{L_{\vec{p},\lambda}},$$

where $C_{\vec{p},\lambda}$ depends only on $\vec{p}$, λ and n.

Theorem 3. [*32, Corollary* 5.6] *Let $1 < \vec{p} < \infty$, $0 \le \lambda < n$ and $b \in BMO(\mathbb{R}^n)$. If $f \in L_{\vec{p},\lambda}(\mathbb{R}^n)$, then $M_b f \in L_{\vec{p},\lambda}(\mathbb{R}^n)$ and*

$$\|M_b f\|_{L_{\vec{p},\lambda}} \le C_{p,\lambda} \|b\|_* \|f\|_{L_{\vec{p},\lambda}},$$

where $C_{\vec{p},\lambda}$ depends only on $\vec{p}$, λ and n.

3. Hardy-Littlewood-Sobolev inequality in mixed Morrey spaces

The classical Hardy-Littlewood-Sobolev theorem states that if $1 < p < q < \infty$, then I_α is bounded from $L_p(\mathbb{R}^n)$ to $L_q(\mathbb{R}^n)$ if and only if $\alpha = \frac{n}{p} - \frac{n}{q}$ and for $p = 1 < q < \infty$, I_α is bounded from $L_1(\mathbb{R}^n)$ to $WL_q(\mathbb{R}^n)$ if and only if $\alpha = n - \frac{n}{q}$. [2, 7] proved the boundedness of I_α in mixed-norm Lebesgue spaces. In 2020, Zhang and Zhou [34] gave necessary and sufficient conditions for I_α to be bounded in mixed Lebesgue spaces. Their result is stated that if $1 < \vec{p} \le \vec{q} < \infty$, $\vec{p} \ne \vec{q}$, then I_α is bounded from $L_{\vec{p}}(\mathbb{R}^n)$ to $L_{\vec{q}}(\mathbb{R}^n)$ if and only if $\frac{1}{P} - \frac{1}{Q} = \frac{\alpha}{n}$.

The following local estimate is valid (see also [13]).

Lemma 7. *[32, Lemma 3.2], [13, Theorem 5.1] Let $0 < \alpha < n$, $1 < \vec{p} \le \vec{q} < \infty$, $\vec{p} \ne \vec{q}$, and $\frac{1}{P} - \frac{1}{Q} = \frac{\alpha}{n}$. Then the following inequality*

$$\|I_\alpha f\|_{L_{\vec{q}}(B(x,r))} \lesssim r^{\frac{n}{Q}} \int_{2r}^{\infty} t^{-\frac{n}{Q}} \|f\|_{L_{\vec{p}}(B(x,t))} \frac{dt}{t} \tag{1}$$

holds for all $B(x,r)$ and for all $f \in L_{\vec{p}}^{\mathrm{loc}}(\mathbb{R}^n)$.

Below is a Spanne type result for the Riesz potential in mixed Morrey spaces (see, for example, [13]).

Theorem 4. *(Spanne type result) Let $1 < \vec{p} \le \vec{q} < \infty$, $\vec{p} \ne \vec{q}$, $0 \le \lambda < n$, $0 < \alpha < \frac{n-\lambda}{P}$, $\frac{1}{P} - \frac{1}{Q} = \frac{\alpha}{n}$. If $f \in L_{\vec{p},\lambda}(\mathbb{R}^n)$, then $I_\alpha f \in L_{\vec{q},\frac{\lambda P}{Q}}(\mathbb{R}^n)$ and*

$$\|I_\alpha f\|_{L_{\vec{q},\frac{\lambda P}{Q}}} \le C_{\vec{p},\lambda} \|f\|_{L_{\vec{p},\lambda}}, \tag{2}$$

where $C_{\vec{p},\lambda}$ depends only on $\vec{p}$,λ and n.

Proof. Let $1 < \vec{p} < \infty$. From the inequality (1) (see Lemma 7) we get

$$\begin{aligned}
\|I_\alpha f\|_{L_{\vec{q},\frac{\lambda P}{Q}}} &= \sup_{x\in\mathbb{R}^n,\, r>0} r^{-\frac{\lambda}{P}} \|I_\alpha f\|_{L_{\vec{q}}(B(x,r))} \\
&\lesssim \sup_{x\in\mathbb{R}^n,\, r>0} r^{-\frac{\lambda}{P}+\frac{n}{Q}} \int_{2r}^{\infty} t^{-\frac{n}{Q}} \|f\|_{L_{\vec{p}}(B(x,t))} \frac{dt}{t} \\
&\lesssim \|f\|_{L_{\vec{p},\lambda}} \sup_{r>0} r^{-\alpha+\frac{n-\lambda}{P}} \int_{r}^{\infty} t^{\alpha-\frac{n-\lambda}{P}} \frac{dt}{t} \\
&\approx \|f\|_{L_{\vec{p},\lambda}},
\end{aligned}$$

which implies that the operator $I_\alpha f$ is bounded from $L_{\vec{p},\lambda}(\mathbb{R}^n)$ to $L_{\vec{q},\frac{\lambda P}{Q}}(\mathbb{R}^n)$. ◀

Below is a Adams type result for the Riesz potential in mixed Morrey spaces, see, for example, [13, 14].

Theorem 5. *(Adams type result) Let* $1 < \vec{p} \le \vec{q} < \infty$, $\vec{p} \ne \vec{q}$, $0 < \alpha < \frac{n-\lambda}{P}$ *and* $\vec{q} = \frac{Q}{P}\,\vec{p}$. *Then condition* $\frac{1}{P} - \frac{1}{Q} = \frac{\alpha}{n-\lambda}$ *is necessary and sufficient for the boundedness of the operator* I_α *from* $L_{\vec{p},\lambda}(\mathbb{R}^n)$ *to* $L_{\vec{q},\lambda}(\mathbb{R}^n)$.

Proof. Sufficiency. Let $1 < \vec{p} \le \vec{q} < \infty$, $\vec{p} \ne \vec{q}$, $0 < \alpha < \frac{n-\lambda}{P}$, $\frac{1}{P} - \frac{1}{Q} = \frac{\alpha}{n-\lambda}$, $f \in L_{\vec{p},\lambda}(\mathbb{R}^n)$ and r arbitrary positive number.

$$\begin{aligned}
I_\alpha|f|(x) &= \int_{\mathbb{R}^n} \frac{|f(y)|}{|x-y|^{n-\alpha}}\,dy \approx \int_{\mathbb{R}^n} \Big(\int_{|x-y|}^{\infty} t^{\alpha-n-1}\,dt\Big)\,|f(y)|\,dy \\
&= \int_{\mathbb{R}^n}\int_0^{\infty} \chi_{(|x-y|,\infty)}(t)\,t^{\alpha-n-1}\,|f(y)|dtdy \\
&= \int_0^{\infty}\Big(\int_{\mathbb{R}^n} \chi_{(|x-y|,\infty)}(t)\,|f(y)|dy\Big)\,t^{\alpha-n-1}dt = \int_0^{\infty} t^{\alpha-n}\,\|f\|_{L_1(B(x,t))}\,\frac{dt}{t} \\
&= \int_0^{r} t^{\alpha-n}\,\|f\|_{L_1(B(x,t))}\,\frac{dt}{t} + \int_r^{\infty} t^{\alpha-n}\,\|f\|_{L_1(B(x,t))}\,\frac{dt}{t} \\
&\lesssim r^\alpha\,Mf(x) + r^{\alpha-\frac{n}{P}}\,\|f\|_{L_{\vec{p}}(B(x,r))} \\
&\le r^\alpha\,Mf(x) + r^{\alpha-\frac{n-\lambda}{P}}\,\|f\|_{L_{\vec{p},\lambda}}.
\end{aligned}$$

Minimizing with respect to r, at

$$r = \Big(\frac{\|f\|_{L_{\vec{p},\lambda}}}{Mf(x)}\Big)^{\frac{P}{n-\lambda}}$$

we have

$$I_\alpha|f|(x) \le \big(Mf(x)\big)^{1-\frac{\alpha P}{n-\lambda}}\,\|f\|_{L_{\vec{p},\lambda}}^{\frac{\alpha P}{n-\lambda}}, \tag{3}$$

where we have used that the supremum is achieved when the minimum parts are balanced. From Theorem 2, the inequality (3) and the condition $\vec{q} = \frac{Q}{P}\,\vec{p}$, we get

$$\begin{aligned}
\|I_\alpha f\|_{L_{\vec{q},\lambda}} &\lesssim \|f\|_{L_{\vec{p},\lambda}}^{1-\frac{P}{Q}}\,\|(Mf)^{\frac{P}{Q}}\|_{L_{\vec{q},\lambda}} = \|f\|_{L_{\vec{p},\lambda}}^{1-\frac{P}{Q}}\,\|Mf\|_{L_{\frac{P}{Q}\vec{q},\lambda}}^{\frac{P}{Q}} \\
&= \|f\|_{L_{\vec{p},\lambda}}^{1-\frac{P}{Q}}\,\|Mf\|_{L_{\vec{p},\lambda}}^{\frac{P}{Q}} \lesssim \|f\|_{L_{\vec{p},\lambda}}.
\end{aligned}$$

Necessity. Let $1 < \vec{p} \le \vec{q} < \infty$, $\vec{p} \ne \vec{q}$, $\frac{1}{P} - \frac{1}{Q} = \frac{\alpha}{n-\lambda}$, $f \in L_{\vec{p},\lambda}(\mathbb{R}^n)$ and assume that I_α is bounded from $L_{\vec{p},\lambda}(\mathbb{R}^n)$ to $L_{\vec{q},\lambda}(\mathbb{R}^n)$.

Define $f_t(x) =: f(tx)$, $[t]_{1,+} = \max\{1, t\}$. Then

$$\big\|f_t\big\|_{L_{\vec{p},\lambda}} = \sup_{x\in\mathbb{R}^n, r>0} r^{-\frac{\lambda}{P}}\,\|f_t\|_{L_{\vec{p}}(B(x,r))} = t^{-\frac{n}{P}} \sup_{x\in\mathbb{R}^n, r>0} r^{-\frac{\lambda}{P}}\,\|f\|_{L_{\vec{p}}(B(x,tr))}$$

$$= t^{-\frac{n-\lambda}{P}} \sup_{x\in\mathbb{R}^n,\, r>0} (tr)^{-\frac{\lambda}{P}} \|f\|_{L_{\vec{p}}(B(x,tr))} = t^{-\frac{n-\lambda}{P}} \|f\|_{L_{\vec{p},\lambda}},$$

and

$$I_\alpha f_t(x) = t^{-\alpha} I_\alpha f(tx),$$

$$\left\| I_\alpha f_t \right\|_{L_{\vec{q},\lambda}} = t^{-\alpha} \sup_{x\in\mathbb{R}^n,\, r>0} r^{-\frac{\lambda}{Q}} \|I_\alpha f(t\,\cdot)\|_{L_{\vec{q}}(B(x,r))}$$
$$= t^{-\alpha-\frac{n-\lambda}{Q}} \sup_{x\in\mathbb{R}^n,\, r>0} (tr)^{-\frac{\lambda}{Q}} \|I_\alpha f\|_{L_{\vec{q}}(B(tx,tr))} = t^{-\alpha-\frac{n-\lambda}{Q}} \|I_\alpha f\|_{L_{\vec{q},\lambda}}.$$

By the boundedness of I_α from $L_{\vec{p},\lambda}(\mathbb{R}^n)$ to $L_{\vec{q},\lambda}(\mathbb{R}^n)$ we have

$$\|I_\alpha f\|_{L_{\vec{q},\lambda}} = t^{\alpha+\frac{n-\lambda}{Q}} \left\|I_\alpha f_t\right\|_{L_{\vec{q},\lambda}} \lesssim t^{\alpha+\frac{n-\lambda}{Q}} \left\|f_t\right\|_{L_{\vec{p},\lambda}} = t^{\alpha+\frac{n-\lambda}{Q}-\frac{n-\lambda}{P}} \|f\|_{L_{\vec{p},\lambda}}.$$

If $\frac{1}{P} < \frac{1}{Q} + \frac{\alpha}{n-\lambda}$, then by letting $t \to 0$ we have $\|I_\alpha f\|_{L_{\vec{q},\lambda}} = 0$ for all $f \in L_{\vec{p},\lambda}(\mathbb{R}^n)$.

As well as if $\frac{1}{P} > \frac{1}{Q} + \frac{\alpha}{n-\lambda}$, then at $t \to \infty$ we obtain $\|I_\alpha f\|_{L_{\vec{q},\lambda}} = 0$ for all $f \in L_{\vec{p},\lambda}(\mathbb{R}^n)$.

Therefore $\frac{1}{P} - \frac{1}{Q} = \frac{\alpha}{n-\lambda}$. ◄

Remark 1. *Note that, in the case of $\vec{p} = (p, \ldots, p)$ Theorems 4 and 5 were given in [27] and [3], respectively.*

4. Commutator of the Riesz potential in mixed Morrey spaces

In this section we find necessary and sufficient conditions for the boundedness of the commutator of Riesz potential $[b, I_\alpha]$ in the $L_{\vec{p},\lambda,\mu}(\mathbb{R}^n)$ spaces.

Definition 2. *We define the space $BMO(\mathbb{R}^n)$ as the set of all locally integrable functions f with finite norm*

$$\|f\|_* = \sup_{x\in\mathbb{R}^n, t>0} |B(x,t)|^{-1} \int_{B(x,t)} |f(y) - f_{B(x,t)}| dy < \infty,$$

where $f_{B(x,t)} = |B(x,t)|^{-1} \int_{B(x,t)} f(y) dy$.

The following local estimate is valid (see also [14]).

Lemma 8. *[32, Lemma 4.3], [14, Lemma 7.4] Let* $0 < \alpha < n$, $1 < \vec{p} \le \vec{q} < \infty$, $\vec{p} \ne \vec{q}$, $\frac{1}{P} - \frac{1}{Q} = \frac{\alpha}{n}$ *and* $b \in BMO(\mathbb{R}^n)$. *Then the inequality*

$$\|[b, I_\alpha]f\|_{L_{\vec{q}}(B(x,r))} \lesssim \|b\|_* \, r^{\frac{n}{Q}} \int_{2r}^{\infty} \log\left(e + \frac{t}{r}\right) t^{-\frac{n}{Q}} \, \|f\|_{L_{\vec{p}}(B(x,t))} \, \frac{dt}{t} \tag{4}$$

holds for all $B(x,r)$ *and for all* $f \in L^{\rm loc}_{\vec{p}}(\mathbb{R}^n)$.

The following is Spanne type result for commutator of Riesz potential in mixed Morrey spaces.

Theorem 6. *(Spanne's type result) Let* $0 < \alpha < n$, $1 < \vec{p} \le \vec{q} < \infty$, $\vec{p} \ne \vec{q}$, $0 < \alpha < \frac{n-\lambda}{P}$, $\frac{1}{P} - \frac{1}{Q} = \frac{\alpha}{n}$, $0 \le \lambda \le n$ *and* $b \in BMO(\mathbb{R}^n)$. *If* $f \in L_{\vec{p},\lambda}(\mathbb{R}^n)$, *then* $[b, I_\alpha]f \in L_{\vec{q},\frac{\lambda P}{Q}}(\mathbb{R}^n)$ *and*

$$\|[b, I_\alpha]f\|_{L_{\vec{q},\frac{\lambda P}{Q}}} \le C_{\vec{p},\lambda} \, \|b\|_* \, \|f\|_{L_{\vec{p},\lambda}}, \tag{5}$$

where $C_{\vec{p},\lambda}$ *depends only on* $\vec{p}$,λ *and* n.

Proof. Let $1 < \vec{p} \le \vec{q} < \infty$, $\vec{p} \ne \vec{q}$. From the inequality (4) we get

$$\begin{aligned}
\|[b, I_\alpha]f\|_{L_{\vec{q},\frac{\lambda P}{Q}}} &= \sup_{x\in\mathbb{R}^n,\, r>0} r^{-\frac{\lambda}{P}} \, \|[b, I_\alpha]f\|_{L_{\vec{q}}(B(x,r))} \\
&\lesssim \|b\|_* \sup_{x\in\mathbb{R}^n,\, r>0} r^{-\frac{\lambda}{P}+\frac{n}{Q}} \int_{2r}^{\infty} \log\left(e + \frac{t}{r}\right) t^{-\frac{n}{Q}} \, \|f\|_{L_{\vec{p}}(B(x,t))} \, \frac{dt}{t} \\
&\lesssim \|f\|_{L_{\vec{p},\lambda}} \sup_{r>0} r^{-\alpha+\frac{n-\lambda}{P}} \int_{r}^{\infty} \log\left(e + \frac{t}{r}\right) t^{\alpha-\frac{n-\lambda}{P}} \, \frac{dt}{t} \\
&\approx \|f\|_{L_{\vec{p},\lambda}} \int_{1}^{\infty} \log(e+t) \, t^{\alpha-\frac{n-\lambda}{P}} \, \frac{dt}{t} \approx \|f\|_{L_{\vec{p},\lambda}},
\end{aligned}$$

which implies that the operator $[b, I_\alpha]$ is bounded from $L_{\vec{p},\lambda}(\mathbb{R}^n)$ to $L_{\vec{q},\frac{\lambda P}{Q}}(\mathbb{R}^n)$. ◀

The following theorem is one of our main s, in which we obtain conditions that guarantee the boundedness of the commutator of Riesz potential $[b, I_\alpha]$ from the space $L_{\vec{p},\lambda}(\mathbb{R}^n)$ to $L_{\vec{p},\lambda}(\mathbb{R}^n)$ for $b \in BMO(\mathbb{R}^n)$.

Theorem 7. *Let* $1 < \vec{p} \le \vec{q} < \infty$, $\vec{p} \ne \vec{q}$, $0 \le \lambda < n$, $0 < \alpha < \frac{n-\lambda}{P}$, $\vec{q} = \frac{Q}{P}\vec{p}$ *and* $b \in BMO(\mathbb{R}^n)$. *Then the condition* $\frac{1}{P} - \frac{1}{Q} = \frac{\alpha}{n-\lambda}$ *is sufficient for the boundedness of the operator* $[b, I_\alpha]$ *from* $L_{\vec{p},\lambda}(\mathbb{R}^n)$ *to* $L_{\vec{q},\lambda}(\mathbb{R}^n)$.

Proof. Let $0 < \alpha < n$, $0 \le \lambda < n$, $1 < P < \frac{n-\lambda}{\alpha}$ and $b \in BMO(\mathbb{R}^n)$. Let also $\frac{1}{P} - \frac{1}{Q} = \frac{\alpha}{n-\lambda}$ and $f \in L_{\vec{p},\lambda}(\mathbb{R}^n)$. For arbitrary $x_0 \in \mathbb{R}^n$, set $B = B(x_0, r)$. Write $f = f_1 + f_2$ with $f_1 = f\chi_{2B}$ and $f_2 = f\chi_{\complement_{2B}}$.

$$\big|[b, I_\alpha] f_1(x)\big| \le \int_{2B} \frac{|b(x) - b(y)|}{|x-y|^{n-\alpha}} |f(y)|\, dy \lesssim r^\alpha\, M_b f(x). \tag{6}$$

For $x \in B$ we have

$$\begin{aligned}\big|[b, I_\alpha] f_2(x)\big| &\le \int_{\complement_{2B}} \frac{|b(x) - b(y)|}{|x-y|^{n-\alpha}} |f(y)|\, dy \\ &\approx \int_{\complement_{2B}} \frac{|b(x) - b(y)|}{|x_0-y|^{n-\alpha}} |f(y)|\, dy.\end{aligned}$$

Analogously to [14, Section 7.1], for all $\vec{p} \in (1, \infty)$ and $x \in B$ we get

$$\big|[b, I_\alpha] f_2(x)\big| \lesssim \|b\|_* \int_{2r}^{\infty} \log\big(e + \frac{t}{r}\big)\, t^{\alpha - \frac{n}{P}}\, \|f\|_{L_{\vec{p}}(B(x,t))} \frac{dt}{t}. \tag{7}$$

Then from inequalities (6) and (7) we get

$$\begin{aligned}\big|[b, I_\alpha] f(x)\big| &\lesssim r^\alpha\, M_b f(x) + \|b\|_* \int_{2r}^{\infty} \log\big(e + \frac{t}{r}\big)\, t^{\alpha - \frac{n}{P}}\, \|f\|_{L_{\vec{p}}(B(x,t))} \frac{dt}{t} \\ &\le r^\alpha\, M_b f(x) + \|b\|_*\, \|f\|_{L_{\vec{p},\lambda}} \int_{r}^{\infty} \log\big(e + \frac{t}{r}\big)\, t^{\alpha - \frac{n-\lambda}{P}} \frac{dt}{t} \\ &\le r^\alpha\, M_b f(x) + \|b\|_*\, \|f\|_{L_{\vec{p},\lambda}}\, r^{\alpha - \frac{n-\lambda}{P}} \int_{1}^{\infty} \log(e + t)\, t^{\alpha - \frac{n-\lambda}{P}} \frac{dt}{t} \\ &\approx r^\alpha\, M_b f(x) + \|b\|_*\, \|f\|_{L_{\vec{p},\lambda}}\, r^{\alpha - \frac{n-\lambda}{P}}.\end{aligned} \tag{8}$$

Minimizing with respect to r, at

$$r = \Big(\frac{\|f\|_{L_{\vec{p},\lambda}}}{M_b f(x)}\Big)^{\frac{P}{n-\lambda}}$$

we have

$$\big|[b, I_\alpha] f(x)\big| \le \big(M_b f(x)\big)^{1 - \frac{\alpha P}{n-\lambda}}\, \|f\|_{L_{\vec{p},\lambda}}^{\frac{\alpha P}{n-\lambda}}, \tag{9}$$

where we have used that the supremum is achieved when the minimum parts are balanced. From Theorem 3, the inequality (9) and the condition $\vec{q} = \frac{Q}{P}\,\vec{p}$, we get

$$\begin{aligned}\|[b, I_\alpha] f\|_{L_{\vec{q},\lambda}} &\lesssim \|f\|_{L_{\vec{p},\lambda}}^{1-\frac{P}{Q}}\, \|(M_b f)^{\frac{P}{Q}}\|_{L_{\vec{q},\lambda}} = \|f\|_{L_{\vec{p},\lambda}}^{1-\frac{P}{Q}}\, \|M_b f\|_{L_{\frac{P}{Q}\vec{p},\lambda}}^{\frac{P}{Q}} \\ &= \|f\|_{L_{\vec{p},\lambda}}^{1-\frac{P}{Q}}\, \|M_b f\|_{L_{\vec{p},\lambda}}^{\frac{P}{Q}} \lesssim \|f\|_{L_{\vec{p},\lambda}}.\end{aligned}$$

◀

5. Modified Riesz potential in mixed Morrey spaces

We consider the modified Riesz potential

$$\widetilde{I}_\alpha f(x) = \int_{\mathbb{R}^n} \left(|x-y|^{\alpha-n} - |y|^{\alpha-n}\chi_{\complement_{B(0,1)}}(y)\right) f(y)dy.$$

Note that in the limiting case $P = \frac{n-\lambda}{\alpha}$ the statement of Theorem 5 is not true. Moreover, there exists $f \in L_{\vec{p},\lambda}(\mathbb{R}^n)$ such that $I_\alpha f(x) = \infty$ for all $x \in \mathbb{R}^n$. In [15, Theorem 3.2] we showed that if $P = \frac{n-\lambda}{\alpha}$, then the operator M_α is bounded from $L_{\vec{p},\lambda}(\mathbb{R}^n)$ to $L_\infty(\mathbb{R}^n)$. However, as will be shown, statement of Theorem 5 is valid for the modified Riesz potential $\widetilde{I}_\alpha$ if the space $L_\infty(\mathbb{R}^n)$ is replaced by the wider space $BMO(\mathbb{R}^n)$.

The following theorem is one of our main results, in which we obtain conditions that guarantee that the modified Riesz potential operator $\widetilde{I}_\alpha$ is bounded from the space $L_{\vec{p},\lambda}(\mathbb{R}^n)$ to $BMO(\mathbb{R}^n)$.

Theorem 8. *Let $0 < \alpha < n$, $1 < \vec{p} < \infty$, $0 \le \mu \le \lambda < n$ and $P = \frac{n-\lambda}{\alpha}$, then the operator $\widetilde{I}_\alpha$ is bounded from $L_{\vec{p},\lambda}(\mathbb{R}^n)$ to $BMO(\mathbb{R}^n)$. Moreover, if the integral $I_\alpha f$ exists almost everywhere for $f \in L_{\vec{p},\lambda}(\mathbb{R}^n)$, $P = \frac{n-\lambda}{\alpha}$, then $I_\alpha f \in BMO(\mathbb{R}^n)$ and the following inequality is valid*

$$\|I_\alpha f\|_* \le C\|f\|_{L_{\vec{p},\lambda}},$$

where $C > 0$ is independent of f.

Proof. For given $x \in \mathbb{R}^n$, $y \in B(x,t)$ and $t > 0$ we denote

$$f_1(y) = f(y)\chi_{B(x,2t)}(y), \quad f_2(y) = f(y) - f_1(y), \tag{10}$$

where $\chi_{B(x,2t)}$ is the characteristic function of the set $B(x,2t)$. Then

$$\widetilde{I}_\alpha f(y) = \widetilde{I}_\alpha f_1(y) + \widetilde{I}_\alpha f_2(y) = F_1(y) + F_2(y), \tag{11}$$

where

$$F_1(y) = \int_{B(x,2t)} \left(|y-z|^{\alpha-n} - |x-z|^{\alpha-n}\chi_{\complement_{B(x,1)}}(z)\right) f(z)dz,$$

$$F_2(y) = \int_{\complement_{B(x,2t)}} \left(|y-z|^{\alpha-n} - |x-z|^{\alpha-n}\chi_{\complement_{B(x,1)}}(z)\right) f(z)dz.$$

Note that the function f_1 has compact (bounded) support and thus

$$a_1 = -\int_{B(x,2t)\setminus B(x,\min\{1,2t\})} |x-z|^{\alpha-n} f(z)dz$$

is finite.

Note also that

$$F_1(y)-a_1=\int_{B(x,2t)}|y-z|^{\alpha-n}f(z)dz$$

$$-\int_{B(x,2t)\setminus B(x,\min\{1,2t\})}|x-z|^{\alpha-n}f(z)dz$$

$$+\int_{B(x,2t)\setminus B(x,\min\{1,2t\})}|x-z|^{\alpha-n}f(z)dz$$

$$=\int_{\mathbb{R}^n}|y-z|^{\alpha-n}f_1(z)dz=I_\alpha f_1(y).$$

Therefore

$$|F_1(y)-a_1|=|I_\alpha f_1(y)|\le\int_{B(0,3t)}|z|^{\alpha-n}|f(y-z)|dz.$$

Then

$$\begin{aligned}
&|B(x,t)|^{-1}\int_{B(x,t)}\big|F_1(y)-a_1\big|dy\\
&\le|B(x,t)|^{-1}\int_{B(x,t)}\Big(\int_{B(0,3t)}|z|^{\alpha-n}|f(y-z)|dz\Big)dy\\
&=|B(x,t)|^{-1}\int_{B(0,3t)}\Big(\int_{B(x,t)}|f(y-z)|dy\Big)|z|^{\alpha-n}dz\\
&\lesssim t^{-n}t^{n-\alpha}\|f\|_{L_{1,n-\alpha}}\int_{B(0,3t)}|z|^{\alpha-n}dz\approx\|f\|_{L_{1,n-\alpha}}.
\end{aligned}\tag{12}$$

Denote

$$a_2=\int_{B(x,\max\{1,2t\})\setminus B(x,2t)}|x-z|^{\alpha-n}f(z)dz.$$

If $2|x-y|\le|x-z|$, then

$$\big||y-z|^{\alpha-n}-|x-z|^{\alpha-n}\big|\le C|x-y||x-z|^{\alpha-n-1}.$$

By the Hölder's inequality we have

$$\Big|F_2(y)-a_2\Big|\lesssim|x-y|\int_{\complement_{B(x,2t)}}|f(z)|\,|x-z|^{\alpha-n-1}dz$$

$$=|x-y|\sum_{j=0}^{\infty}\int_{B(x,2^{j+2}t)\setminus B(x,2^{j+1}t)}|f(z)|\,|x-z|^{\alpha-n-1}\,dz$$

$$\lesssim |x-y| \sum_{j=0}^{\infty} (2^{j+1}t)^{\alpha-n-1} \int_{B(x,2^{j+2}t)} |f(z)|\,dz$$

$$\lesssim |x-y|\, \|f\|_{L_{1,n-\alpha}} \sum_{j=0}^{\infty} (2^{j+1}t)^{\alpha-n-1}\,(2^{j+2}t)^{n-\alpha}$$

$$\approx |x-y|\, t^{-1}\, \|f\|_{L_{1,n-\alpha}} \sum_{j=0}^{\infty} 2^{-j}$$

$$\approx |x-y|\, t^{-1}\, \|f\|_{L_{1,n-\alpha}} = \|f\|_{L_{1,n-\alpha}}. \tag{13}$$

Therefore, from (12) and (13) we have

$$\sup_{x,t} \frac{1}{|B(x,t)|} \int_{B(x,t)} \left| \widetilde{I}^{\alpha} f(y) - a_f \right| dy \lesssim \|f\|_{L_{1,n-\alpha}}. \tag{14}$$

Finally for $P = \frac{n-\lambda}{\alpha}$ and $f \in L_{\vec{p},\lambda}(\mathbb{R}^n)$ from (14) and Lemma 6 we get

$$\left\|\widetilde{I}_{\alpha} f\right\|_{*} \leq 2 \sup_{x,t} \frac{1}{|B(x,t)|} \int_{B(x,t)} \left|\widetilde{I}_{\alpha} f(y) - a_f\right| dy \leq C \|f\|_{L_{\vec{p},\lambda}}.$$

The Theorem 8 is proved. ◄

6. Some applications

In this section, we shall apply Theorems 4, 5, 6 and 7 to several particular operators such as the Marcinkiewicz operator and fractional powers of the some analytic semigroups.

6.1. Marcinkiewicz operator

Let $S^{n-1} = \{x \in \mathbb{R}^n : |x| = 1\}$ be the unit sphere in $\mathbb{R}^n$, equipped with Lebesgue measure $d\sigma$. Suppose that Ω is a homogeneous function of degree zero on $\mathbb{R}^n$, has zero mean on S^{n-1}, and satisfies the condition $\Omega \in L_\infty(S^{n-1})$.

The Marcinkiewicz integral operator μ_Ω is defined by

$$\mu_\Omega(f)(x) = \left(\int_0^\infty \Big| \int_{|x-y|\leq t} \frac{\Omega(x-y)}{|x-y|^{n-1}} f(y) dy \Big|^2 \frac{dt}{t^3} \right)^{1/2}.$$

As is known, the Marcinkiewicz integral is one of the classical operators of harmonic analysis, belonging to a wide class of Littlewood-Paley g-functions and playing an important role in harmonic analysis and the theory of partial differential equations. Research into the mapping properties of the Marcinkiewicz integral and its commutators in various functional spaces is a topical issue. In 1958, Stein [29] first introduced the operator μ_Ω, which is the higher dimensional generalization of Marcinkiewicz integral in one-dimension, and showed that μ_Ω is bounded on $L_p(\mathbb{R}^n)$ for $1 < p \le 2$ and weak type (1.1), provided $\Omega \in \mathrm{Lip}_\gamma(S^{n-1})$, $0 < \gamma \le 1$.

For $0 \le \alpha < n$ the fractional Marcinkiewicz operator $\mu_{\Omega,\alpha}$ is defined by (see [31])

$$\mu_{\Omega,\alpha}(f)(x) = \left(\int_0^\infty \Big| \int_{|x-y|\le t} \frac{\Omega(x-y)}{|x-y|^{n-1-\alpha}} f(y)dy \Big|^2 \frac{dt}{t^3} \right)^{1/2}.$$

Note that $\mu_\Omega f = \mu_{\Omega,0} f$.

The sublinear commutator of the operator $\mu_{\Omega,\alpha}$ is defined by

$$[b, \mu_{\Omega,\alpha}](f)(x) = \left(\int_0^\infty \Big| \int_{|x-y|\le t} \frac{\Omega(x-y)}{|x-y|^{n-1-\alpha}} [b(x) - b(y)] f(y)dy \Big|^2 \frac{dt}{t^3} \right)^{1/2}.$$

By Minkowski inequality and the conditions on Ω, we get

$$\mu_{\Omega,\alpha}(f)(x) \le \int_{\mathbb{R}^n} \frac{|\Omega(x-y)|}{|x-y|^{n-1-\alpha}} |f(y)| \left(\int_{|x-y|}^\infty \frac{dt}{t^3} \right)^{1/2} dy \le C \int_{\mathbb{R}^n} \frac{|f(y)|}{|x-y|^{n-\alpha}} dy.$$

It is known that for $b \in BMO(\mathbb{R}^n)$ the operators $\mu_{\Omega,\alpha}$ and $[b, \mu_{\Omega,\alpha}]$ are bounded from $L_p(\mathbb{R}^n)$ to $L_q(\mathbb{R}^n)$ for $p > 1$, and bounded from $L_1(\mathbb{R}^n)$ to $WL_q(\mathbb{R}^n)$ (see [26, 31, 33]), then from Theorems 4, 5, 6 and 7 we get

Corollary 1. *Let* $1 < \vec{p} \le \vec{q} < \infty$, $\vec{p} \ne \vec{q}$, $0 \le \lambda < n$, $0 < \alpha < \frac{n-\lambda}{P}$, *and* $\frac{1}{P} - \frac{1}{Q} = \frac{\alpha}{n}$. *Then* $\mu_{\Omega,\alpha}$ *is bounded from* $L_{\vec{p},\lambda}$ *to* $L_{\vec{q},\frac{\lambda P}{Q}}$.

Corollary 2. *Let* $1 < \vec{p} \le \vec{q} < \infty$, $\vec{p} \ne \vec{q}$, $0 \le \mu \le \lambda < n$, $0 < \alpha < \frac{n-\lambda}{P}$, *and* $\frac{1}{P} - \frac{1}{Q} = \frac{\alpha}{n-\lambda}$. *Then* $\mu_{\Omega,\alpha}$ *is bounded from* $L_{\vec{p},\lambda}$ *to* $L_{\vec{q},\lambda}$.

Corollary 3. *Let* $1 < \vec{p} \le \vec{q} < \infty$, $\vec{p} \ne \vec{q}$, $0 \le \lambda < n$, $0 < \alpha < \frac{n-\lambda}{P}$, $\frac{1}{P} - \frac{1}{Q} = \frac{\alpha}{n}$ *and* $b \in BMO(\mathbb{R}^n)$. *Then* $[b, \mu_{\Omega,\alpha}]$ *is bounded from* $L_{\vec{p},\lambda}$ *to* $L_{\vec{q},\frac{\lambda P}{Q}}$

Corollary 4. *Let* $1 < \vec{p} \le \vec{q} < \infty$, $\vec{p} \ne \vec{q}$, $0 \le \lambda < n$, $0 < \alpha < \frac{n-\lambda}{P}$, $\frac{1}{P} - \frac{1}{Q} = \frac{\alpha}{n-\lambda}$ *and* $b \in BMO(\mathbb{R}^n)$. *Then* $[b, \mu_{\Omega,\alpha}]$ *is bounded from* $L_{\vec{p},\lambda}$ *to* $L_{\vec{q},\lambda}$.

6.2. Fractional powers of the some analytic semigroups

The theorems of the previous sections can be applied to various operators which are estimated from above by Riesz potentials. We give some examples.

Suppose that L is a linear operator on L_2 which generates an analytic semigroup e^{-tL} with the kernel $p_t(x,y)$ satisfying a Gaussian upper bound, that is,

$$|p_t(x,y)| \leq \frac{c_1}{t^{n/2}} e^{-c_2 \frac{|x-y|^2}{t}} \tag{15}$$

for $x, y \in \mathbb{R}^n$ and all $t > 0$, where c_1, $c_2 > 0$ are independent of x, y and t.

For $0 < \alpha < n$, the fractional powers $L^{-\alpha/2}$ of the operator L are defined by

$$L^{-\alpha/2} f(x) = \frac{1}{\Gamma(\alpha/2)} \int_0^\infty e^{-tL} f(x) \frac{dt}{t^{-\alpha/2+1}}.$$

Note that if $L = -\triangle$ is the Laplacian on $\mathbb{R}^n$, then $L^{-\alpha/2}$ is the Riesz potential I_α. See, for example, Chapter 5 in [30].

Property (15) is satisfied for large classes of differential operators (see, for example [8]). In [8] also other examples of operators which are estimates from above by Riesz potentials are given. In these cases Theorems 4 and 5 are also applicable for proving boundedness of those operators and commutators from $L_{\vec{p},\lambda}$ to $L_{q,\frac{\lambda P}{Q}}$ and from $L_{\vec{p},\lambda}$ to $L_{\vec{q},\lambda}$.

Corollary 5. *Let condition* (15) *be satisfied. Moreover, let* $1 < \vec{p} \leq \vec{q} < \infty$, $\vec{p} \neq \vec{q}$, $0 \leq \lambda < n$, $0 < \alpha < \frac{n-\lambda}{P}$, *and* $\frac{1}{P} - \frac{1}{Q} = \frac{\alpha}{n}$. *Then* $L^{-\alpha/2}$ *is bounded from* $L_{\vec{p},\lambda}$ *to* $L_{\vec{q},\frac{\lambda P}{Q}}$.

Proof. Since the semigroup e^{-tL} has the kernel $p_t(x,y)$ which satisfies condition (15), it follows that

$$|L^{-\alpha/2} f(x)| \lesssim I_\alpha(|f|)(x)$$

(see [11]). Hence by the aforementioned theorems we have

$$\|L^{-\alpha/2} f\|_{L_{\vec{q},\frac{\lambda P}{Q}}} \lesssim \|I_\alpha(|f|)\|_{L_{\vec{q},\frac{\lambda P}{Q}}} \lesssim \|f\|_{L_{\vec{p},\lambda}}.$$

◀

Corollary 6. *Let condition* (15) *be satisfied. Moreover, let* $1 < \vec{p} \leq \vec{q} < \infty$, $\vec{p} \neq \vec{q}$, $0 \leq \lambda < n$, $0 < \alpha < \frac{n-\lambda}{P}$, *and* $\frac{1}{P} - \frac{1}{Q} = \frac{\alpha}{n-\lambda}$. *Then* $L^{-\alpha/2}$ *is bounded from* $L_{\vec{p},\lambda}$ *to* $L_{\vec{q},\lambda}$.

Let b be a locally integrable function on $\mathbb{R}^n$, the commutator of b and $L^{-\alpha/2}$ is defined as follows

$$[b, L^{-\alpha/2}]f(x) = b(x)L^{-\alpha/2}f(x) - L^{-\alpha/2}(bf)(x).$$

In [11], Chanillo's result was generalized from $(-\Delta)$ to the more general operator L defined above. More precisely, see [9], they showed that if $b \in BMO(\mathbb{R}^n)$, then the commutator operator $[b, L^{-\alpha/2}]$ is bounded from $L_p(\mathbb{R}^n)$ to $L_q(\mathbb{R}^n)$ for $1 < p < \infty$ and $\frac{1}{q} = \frac{1}{p} - \frac{\alpha}{n}$. Then, from theorems 6 and 7 it follows

Corollary 7. *Let condition* (15) *be satisfied. Moreover, let* $1 < \vec{p} \le \vec{q} < \infty$, $\vec{p} \neq \vec{q}$, $0 \le \lambda < n$, $0 < \alpha < \frac{n-\lambda}{P}$, $\frac{1}{P} - \frac{1}{Q} = \frac{\alpha}{n}$ *and* $b \in BMO(\mathbb{R}^n)$. *Then* $[b, L^{-\alpha/2}]$ *is bounded from* $L_{\vec{p},\lambda}$ *to* $L_{\vec{q},\frac{\lambda P}{Q}}$.

Corollary 8. *Let condition* (15) *be satisfied. Moreover, let* $1 < \vec{p} \le \vec{q} < \infty$, $\vec{p} \neq \vec{q}$, $0 \le \lambda < n$, $0 < \alpha < \frac{n-\lambda}{P}$, $\frac{1}{P} - \frac{1}{Q} = \frac{\alpha}{n-\lambda}$ *and* $b \in BMO(\mathbb{R}^n)$. *Then* $[b, L^{-\alpha/2}]$ *is bounded from* $L_{\vec{p},\lambda}$ *to* $L_{\vec{q},\lambda}$.

7. Conclusion

In this paper, we present necessary and sufficient conditions for the boundedness of the Riesz potential I_α and the commutator of the Riesz potential $[b, I_\alpha]$ in the mixed Morrey spaces $L_{\vec{p},\lambda}(\mathbb{R}^n)$, when b belongs to the BMO spaces $BMO(\mathbb{R}^n)$. As an application, we obtain estimates for the Marcinkiewicz operator and fractional powers of some analytic semigroups in the mixed Morrey spaces.

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