

Commutator of Parabolic Fractional Integral Operators with Parabolic Lipschitz Functions on Parabolic Total Morrey-Guliyev Spaces

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Abstract. In this paper, the main goal is to provide sufficient conditions for the boundedness of commutators of the parabolic fractional integral operator $[b, I_\alpha^P]$ in parabolic total Morrey-Guliyev spaces $L_{p,\lambda,\mu}^P(\mathbb{R}^n)$ with symbols b belonging to parabolic Lipschitz spaces $\dot{\Lambda}_{\beta,P}(\mathbb{R}^n)$.

Key Words and Phrases: parabolic total Morrey-Guliyev spaces, parabolic fractional integral, commutator, parabolic Lipschitz spaces.

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1. Introduction and main results

The aim of this paper is to study the commutators of the parabolic fractional integral operator $[b, I_\alpha^P]$ with parabolic Lipschitz functions in parabolic total Morrey-Guliyev spaces $L_{p,\lambda,\mu}^P(\mathbb{R}^n)$, in other words, to obtain the boundedness of the commutators of the parabolic fractional integral operator $[b, I_\alpha^P]$ with parabolic Lipschitz functions in parabolic total Morrey-Guliyev spaces $L_{p,\lambda,\mu}^P(\mathbb{R}^n)$, see for example, [4, 7, 11, 12, 17, 18, 20, 22, 25, 29, 30].

Let \mathbb{R}^n be the n -dimension Euclidean space with the norm $|x|$ for each $x \in \mathbb{R}^n$, S^{n-1} denotes the unit sphere on \mathbb{R}^n . For $x \in \mathbb{R}^n$ and $r > 0$, we denote by $B(x, r)$ the open ball centered at x of radius r . Let $|B(x, r)|$ be the Lebesgue measure of the ball $B(x, r)$.

Let P be a real $n \times n$ matrix, all of whose eigenvalues have positive real part. Let $A_t = t^P$ ($t > 0$), and set $\gamma = \text{tr}P$. Then, there exists a quasi-distance ρ associated with P such that (see [5, 6, 8])

$$(a) \quad \rho(A_t x) = t\rho(x), \quad t > 0, \quad \text{for every } x \in \mathbb{R}^n;$$

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- (b) $\rho(0) = 0$, $\rho(x - y) = \rho(y - x) \geq 0$ and $\rho(x - y) \leq k(\rho(x - z) + \rho(y - z))$;
 (c) $dx = \rho^{\gamma-1} d\sigma(w) d\rho$, where $\rho = \rho(x)$, $w = A_{\rho^{-1}}x$
 and $d\sigma(w)$ is a measure on the ellipsoid $\{w : \rho(w) = 1\}$.

Let $S_\rho = \{w \in \mathbb{R}^n : \rho(w) = 1\}$ be the unit ρ -sphere (ellipsoid) in \mathbb{R}^n ($n \geq 2$) equipped with the normalized Lebesgue surface measure $d\sigma$. Then, $\{\mathbb{R}^n, \rho, dx\}$ becomes a space of homogeneous type in the sense of Coifman-Weiss. Thus \mathbb{R}^n , endowed with the metric ρ , defines a homogeneous metric space ([5, 8]). The balls with respect to ρ , centered at x of radius r , are just the ellipsoids $\mathcal{E}(x, r) = \{y \in \mathbb{R}^n : \rho(x - y) < r\}$ with the Lebesgue measure $|\mathcal{E}(x, r)| = v_\rho r^\gamma$, where v_ρ is the volume of the unit ellipsoid in \mathbb{R}^n . If $P = I$, then clearly $\rho(x) = |x|$ and $\mathcal{E}(x, r) = B(x, r)$. Note that in the standard parabolic case $P = (1, \dots, 1, 2)$ we have

$$\rho(x) = \sqrt{\frac{|x'|^2 + \sqrt{|x'|^4 + x_n^2}}{2}}, \quad x = (x', x_n).$$

Let $f : \mathbb{R}^n \rightarrow \mathbb{R}$ be a locally integrable function. The parabolic maximal function $M^P f$ is given by

$$M^P f(x) = \sup_{\mathcal{E} \ni x} |\mathcal{E}|^{-1} \int_{\mathcal{E}} |f(y)| dy,$$

and the parabolic fractional integral $I_\alpha^P f$ is defined by

$$I_\alpha^P f(x) = \int_{\mathbb{R}^n} \frac{f(y)}{\rho(x - y)^{\gamma - \alpha}} dy, \quad 0 < \alpha < \gamma.$$

The commutators generated by the parabolic fractional integral operator I_α^P and the locally integrable function b are defined by the formula

$$[b, I_\alpha^P]f = b I_\alpha^P f - I_\alpha^P (bf).$$

Morrey spaces, introduced by C. B. Morrey [27], play important roles in the regularity theory of PDE, including heat equations and Navier-Stokes equations (see e.g. [3, 4, 9, 18, 31]). The total Morrey-Guliyev spaces $L_{p,\lambda,\mu}(\mathbb{R}^n)$, introduced by Guliyev [13], extend the classical Morrey space $L_{p,\lambda}(\mathbb{R}^n)$ by incorporating a second parameter μ . These spaces can be regarded as intermediate spaces between Lebesgue spaces and Morrey spaces. Their norms are defined through a combination of the norms of $L_{p,\lambda}(\mathbb{R}^n)$ and $L_{p,\mu}(\mathbb{R}^n)$, allowing for a broader range of function behaviors, see [15, 16]. In [1, 2] the authors considered the parabolic total Morrey-Guliyev spaces $L_{p,\lambda,\mu}^P(\mathbb{R}^n)$, give basic properties of the

spaces $L_{p,\lambda,\mu}^P(\mathbb{R}^n)$ and study some embeddings into the parabolic total Morrey-Guliyev space $L_{p,\lambda,\mu}^P(\mathbb{R}^n)$, see also [1, 2, 14, 19, 21, 26, 28, 29, 30, 32, 33, 34, 35].

In this paper we characterize the boundedness of the commutators of the parabolic fractional integral operator $[b, I_\alpha^P]$ with parabolic Lipschitz functions (Theorems 1 and 2) in parabolic total Morrey-Guliyev spaces.

Let us now give the notations useful in the sequel.

At first, we give the definition of the parabolic Lipschitz spaces on \mathbb{R}^n .

Definition 1. Let $0 < \beta < 1$, we say a function b belongs to the parabolic Lipschitz space $\dot{\Lambda}_{\beta,P}(\mathbb{R}^n)$ if there exists a constant C such that for all $x, y \in \mathbb{R}^n$,

$$|b(x) - b(y)| \leq C\rho(x - y)^\beta.$$

The smallest such constant C is called the $\dot{\Lambda}_{\beta,P}(\mathbb{R}^n)$ norm of b and is denoted by $\|b\|_{\dot{\Lambda}_{\beta,P}}$.

Definition 2. Let $0 < p < \infty$, $\lambda \in \mathbb{R}$, $\mu \in \mathbb{R}$, $[t]_1 = \min\{1, t\}$, $t > 0$. We denote by $L_{p,\lambda,\mu}^P(\mathbb{R}^n)$ the parabolic total Morrey space [1, 2, 13, 14] the set of all classes of locally integrable functions f with the finite quasi-norm

$$\|f\|_{L_{p,\lambda,\mu}^P} = \sup_{x \in \mathbb{R}^n, t > 0} [t]_1^{-\frac{\lambda}{p}} [1/t]_1^{\frac{\mu}{p}} \|f\|_{L_p(\mathcal{E}(x,t))}.$$

We denote by $WL_{p,\lambda,\mu}^P(\mathbb{R}^n)$ the weak parabolic total Morrey space [1, 2, 13, 14] the set of all classes of locally integrable functions f with the finite quasi-norm

$$\|f\|_{WL_{p,\lambda,\mu}^P} = \sup_{x \in \mathbb{R}^n, t > 0} [t]_1^{-\frac{\lambda}{p}} [1/t]_1^{\frac{\mu}{p}} \|f\|_{WL_p(\mathcal{E}(x,t))}.$$

Let us note that if $\lambda = \mu$, then $L_{p,\lambda}^P(\mathbb{R}^n) = L_{p,\lambda,\lambda}^P(\mathbb{R}^n)$ is the parabolic Morrey space, $WL_{p,\lambda}^P(\mathbb{R}^n) = WL_{p,\lambda,\lambda}^P(\mathbb{R}^n)$ is the weak parabolic Morrey space and if $\mu = 0$, then $\tilde{L}_{p,\lambda}^P(\mathbb{R}^n) = L_{p,\lambda,0}^P(\mathbb{R}^n)$ is the parabolic modified Morrey space, $W\tilde{L}_{p,\lambda}^P(\mathbb{R}^n) = WL_{p,\lambda,0}^P(\mathbb{R}^n)$ is the weak parabolic modified Morrey space [11, 12].

Our main results may be formulated as follows.

Theorem 1. (Spanne's type result) Let $b \in \dot{\Lambda}_{\beta,P}(\mathbb{R}^n)$, $1 \leq p < \infty$, $0 \leq \lambda, \mu < \gamma$, $0 < \beta < 1$, $0 < \alpha + \beta < \min\{\frac{\gamma-\lambda}{p}, \frac{\gamma-\mu}{p}\}$ and $\frac{1}{p} - \frac{1}{q} = \frac{\alpha+\beta}{\gamma}$.

1. If $p > 1$, $f \in L_{p,\lambda,\mu}^P(\mathbb{R}^n)$, then $[b, I_\alpha^P]f \in L_{q, \frac{\lambda q}{p}, \frac{\mu q}{p}}^P(\mathbb{R}^n)$ and

$$\|[b, I_\alpha^P]f\|_{L_{q, \frac{\lambda q}{p}, \frac{\mu q}{p}}^P} \leq C_{p,\lambda,\mu} \|f\|_{L_{p,\lambda,\mu}^P}, \quad (1)$$

where $C_{p,\lambda,\mu}$ depends only on p,λ,μ and n .

2. If $p = 1$, $f \in L_{1,\lambda,\mu}^P(\mathbb{R}^n)$, then $[b, I_\alpha^P]f \in WWL_{q,\lambda q,\mu q}^P(\mathbb{R}^n)$ and

$$\|[b, I_\alpha^P]f\|_{WL_{q,\lambda q,\mu q}^P} \leq C_{1,\lambda,\mu} \|f\|_{L_{1,\lambda,\mu}^P}, \quad (2)$$

where $C_{1,\lambda,\mu}$ is independent of f .

From Theorem 1 in the case $\lambda = \mu$ or $\mu = 0$ we get the following corollaries.

Corollary 1. Let $b \in \dot{\Lambda}_{\beta,P}(\mathbb{R}^n)$, $1 \leq p < \infty$, $0 < \lambda < \gamma$, $0 < \beta < 1$, $0 \leq \alpha + \beta < \frac{\gamma-\lambda}{p}$ and $\frac{1}{p} - \frac{1}{q} = \frac{\alpha+\beta}{\gamma}$.

1. If $p > 1$, $f \in L_{p,\lambda}^P(\mathbb{R}^n)$, then $[b, I_\alpha^P]f \in L_{q,\frac{\lambda q}{p}}^P(\mathbb{R}^n)$ and

$$\|[b, I_\alpha^P]f\|_{L_{q,\frac{\lambda q}{p}}^P} \leq C_{p,\lambda} \|f\|_{L_{p,\lambda}^P}, \quad (3)$$

where $C_{p,\lambda}$ depends only on p, λ .

2. If $p = 1$, $f \in L_{1,\lambda}^P(\mathbb{R}^n)$, then $[b, I_\alpha^P]f \in WL_{q,\lambda}^P(\mathbb{R}^n)$ and

$$\|[b, I_\alpha^P]f\|_{WL_{q,\lambda}^P} \leq C_{1,\lambda} \|f\|_{L_{1,\lambda}^P}, \quad (4)$$

where $C_{1,\lambda}$ is independent of f .

Corollary 2. Let $b \in \dot{\Lambda}_{\beta,P}(\mathbb{R}^n)$, $1 \leq p < \infty$, $0 < \lambda < \gamma$, $0 < \beta < 1$, $0 \leq \alpha + \beta < \frac{\gamma-\lambda}{p}$ and $\frac{1}{p} - \frac{1}{q} = \frac{\alpha+\beta}{\gamma}$.

1. If $p > 1$, $f \in \tilde{L}_{p,\lambda}^P(\mathbb{R}^n)$, then $[b, I_\alpha^P]f \in \tilde{L}_{q,\frac{\lambda q}{p}}^P(\mathbb{R}^n)$ and

$$\|[b, I_\alpha^P]f\|_{\tilde{L}_{q,\frac{\lambda q}{p}}^P} \leq C_{p,\lambda} \|f\|_{\tilde{L}_{p,\lambda}^P}, \quad (5)$$

where $C_{p,\lambda}$ depends only on p, λ and γ .

2. If $p = 1$, $f \in \tilde{L}_{1,\lambda}^P(\mathbb{R}^n)$, then $[b, I_\alpha^P]f \in W\tilde{L}_{q,\lambda}^P(\mathbb{R}^n)$ and

$$\|[b, I_\alpha^P]f\|_{W\tilde{L}_{q,\lambda}^P} \leq C_{1,\lambda} \|f\|_{\tilde{L}_{1,\lambda}^P}, \quad (6)$$

where $C_{1,\lambda}$ is independent of f .

Theorem 2. (Adams type result) Let $b \in \dot{\Lambda}_{\beta,P}(\mathbb{R}^n)$, $1 \leq p < \infty$, $0 \leq \mu \leq \lambda < \gamma$, $0 < \beta < 1$, $0 < \alpha + \beta < \frac{\gamma-\lambda}{p}$.

1) If $1 < p < \frac{\gamma-\lambda}{\alpha+\beta}$, then condition $\frac{\alpha+\beta}{\gamma-\mu} \leq \frac{1}{p} - \frac{1}{q} \leq \frac{\alpha+\beta}{\gamma-\lambda}$ is sufficient for the boundedness of the operator $[b, I_\alpha^P]$ from $L_{p,\lambda,\mu}^P(\mathbb{R}^n)$ to $L_{q,\lambda,\mu}^P(\mathbb{R}^n)$.

2) If $p = 1 < \frac{\gamma-\lambda}{\alpha+\beta}$, then condition $\frac{\alpha+\beta}{\gamma-\mu} \leq 1 - \frac{1}{q} \leq \frac{\alpha+\beta}{\gamma-\lambda}$ is sufficient for the boundedness of the operator $[b, I_\alpha^P]$ from $L_{1,\lambda,\mu}^P(\mathbb{R}^n)$ to $WL_{q,\lambda,\mu}^P(\mathbb{R}^n)$.

From Theorem 2 in the case $\lambda = \mu$ or $\mu = 0$ we get the following corollaries.

Corollary 3. Let $b \in \dot{\Lambda}_{\beta,P}(\mathbb{R}^n)$, $0 < \beta < 1$, $0 < \alpha + \beta < \gamma$, $0 \leq \lambda < \gamma - \alpha - \beta$ and $1 \leq p < \frac{\gamma-\lambda}{\alpha+\beta}$.

1) If $1 < p < \frac{\gamma-\lambda}{\alpha+\beta}$, then condition $\frac{1}{p} - \frac{1}{q} = \frac{\alpha+\beta}{\gamma-\lambda}$ is sufficient for the boundedness of the operator $[b, I_\alpha^P]$ from $L_{p,\lambda}^P(\mathbb{R}^n)$ to $L_{q,\lambda}^P(\mathbb{R}^n)$.

2) If $p = 1 < \frac{\gamma-\lambda}{\alpha+\beta}$, then condition $1 - \frac{1}{q} = \frac{\alpha+\beta}{\gamma-\lambda}$ is sufficient for the boundedness of the operator $[b, I_\alpha^P]$ from $L_{1,\lambda}^P(\mathbb{R}^n)$ to $WL_{q,\lambda}^P(\mathbb{R}^n)$.

Corollary 4. Let $b \in \dot{\Lambda}_{\beta,P}(\mathbb{R}^n)$, $0 < \alpha < \gamma$, $0 \leq \lambda < \gamma - \alpha$ and $1 \leq p < \frac{\gamma-\lambda}{\alpha}$.

1) If $1 < p < \frac{\gamma-\lambda}{\alpha}$, then condition $\frac{\alpha}{\gamma} \leq \frac{1}{p} - \frac{1}{q} \leq \frac{\alpha}{\gamma-\lambda}$ is sufficient for the boundedness of the operator $[b, I_\alpha^P]$ from $\tilde{L}_{p,\lambda}^P(\mathbb{R}^n)$ to $\tilde{L}_{q,\lambda}^P(\mathbb{R}^n)$.

2) If $p = 1 < \frac{\gamma-\lambda}{\alpha}$, then condition $\frac{\alpha}{\gamma} \leq 1 - \frac{1}{q} \leq \frac{\alpha}{\gamma-\lambda}$ is sufficient for the boundedness of the operator $[b, I_\alpha^P]$ from $\tilde{L}_{1,\lambda}^P(\mathbb{R}^n)$ to $W\tilde{L}_{q,\lambda}^P(\mathbb{R}^n)$.

The structure of the paper is as follows. In Section 2, we present some definitions and auxiliary results, and discuss certain embeddings into the parabolic total Morrey-Guliyev space $L_{p,\lambda,\mu}^P(\mathbb{R}^n)$. In Section 3, we establish sufficient conditions for the Spanne- and Adams- type boundedness of the commutators of the parabolic fractional integral operator $[b, I_\alpha^P]$ with parabolic Lipschitz functions from the spaces $L_{p,\lambda,\mu}^P(\mathbb{R}^n)$ to the spaces $L_{q,\lambda,\mu}^P(\mathbb{R}^n)$, $1 \leq p < q < \infty$.

By $A \lesssim B$ we mean that $A \leq CB$ with some positive constant C independent of appropriate quantities. If $A \lesssim B$ and $B \lesssim A$, we write $A \approx B$ and say that A and B are equivalent.

2. Preliminaries

In this section, we provide essential lemmas and definitions that are crucial for establishing our main results.

Lemma 1. [2, Lemma 0.1] If $0 < p < \infty$, $0 \leq \mu \leq \lambda \leq \gamma$, then

$$L_{p,\lambda,\mu}^P(\mathbb{R}^n) = L_{p,\lambda}^P(\mathbb{R}^n) \cap L_{p,\mu}^P(\mathbb{R}^n)$$

and

$$\|f\|_{L_{p,\lambda,\mu}^P(\mathbb{R}^n)} = \max \left\{ \|f\|_{L_{p,\lambda}^P}, \|f\|_{L_{p,\mu}^P} \right\}.$$

Lemma 2. [2, Lemma 0.2] If $0 < p < \infty$, $0 \leq \mu \leq \lambda \leq \gamma$, then

$$WL_{p,\lambda,\mu}^P(\mathbb{R}^n) = WL_{p,\lambda}^P(\mathbb{R}^n) \cap WL_{p,\mu}^P(\mathbb{R}^n)$$

and

$$\|f\|_{WL_{p,\lambda,\mu}^P} = \max \left\{ \|f\|_{WL_{p,\lambda}^P}, \|f\|_{WL_{p,\mu}^P} \right\}.$$

Lemma 3. [1, Lemma 2.4] *If $0 < p < \infty$, $0 \leq \lambda \leq \gamma$ and $0 \leq \mu \leq \gamma$, then*

$$L_{p,\gamma,\mu}^P(\mathbb{R}^n) \subset_{\gamma} L_{\infty}(\mathbb{R}^n) \subset_{\gamma} L_{p,\lambda,\gamma}^P(\mathbb{R}^n)$$

and

$$\|f\|_{L_{p,\lambda,\gamma}^P} \leq v_{\rho}^{1/p} \|f\|_{L_{\infty}} \leq \|f\|_{L_{p,\gamma,\mu}^P}.$$

Corollary 5. *If $0 < p < \infty$, then*

$$L_{p,\gamma}^P(\mathbb{R}^n) = \tilde{L}_{p,\gamma}^P(\mathbb{R}^n) = L_{\infty}(\mathbb{R}^n)$$

and

$$\|f\|_{L_{p,\gamma}^P} = \|f\|_{\tilde{L}_{p,\gamma}^P} = v_{\rho}^{1/p} \|f\|_{L_{\infty}}.$$

Remark 1. *Let $0 < p < \infty$. If $\mu < 0$ or $\lambda > \gamma$, then*

$$L_{p,\lambda,\mu}^P(\mathbb{R}^n) = WL_{p,\lambda,\mu}^P(\mathbb{R}^n) = \Theta(\mathbb{R}^n),$$

where $\Theta = \Theta(\mathbb{R}^n)$ is the set of all functions equivalent to 0 on \mathbb{R}^n .

The following result is valid.

Theorem 3. [14, Corollary 2.1] *Let $1 \leq p < \infty$, $0 \leq \lambda, \mu < \gamma$.*

1. *If $p > 1$, $f \in L_{p,\lambda,\mu}^P(\mathbb{R}^n)$, then $M^P f \in L_{p,\lambda,\mu}^P(\mathbb{R}^n)$ and*

$$\|M^P f\|_{L_{p,\lambda,\mu}^P} \lesssim \|f\|_{L_{p,\lambda,\mu}^P}.$$

2. *If $f \in L_{1,\lambda,\mu}^P(\mathbb{R}^n)$, then $M^P f \in WL_{1,\lambda,\mu}^P(\mathbb{R}^n)$ and*

$$\|M^P f\|_{WL_{1,\lambda,\mu}^P} \lesssim \|f\|_{L_{1,\lambda,\mu}^P}.$$

Since $\{\mathbb{R}^n, \rho, dx\}$ (see [5, 6, 8]) becomes a space of homogeneous type in the sense of Coifman-Weiss, hence, the following results can be derived from [23, 24].

Lemma 4. *Let $0 < \beta < 1$, we have*

$$\|b\|_{\dot{\Lambda}_{\beta,p}} \approx \sup_{\mathcal{E}} \frac{1}{|\mathcal{E}|^{1+\beta/\gamma}} \int_{\mathcal{E}} |b(x) - b_{\mathcal{E}}| dx,$$

where $b_{\mathcal{E}} = \frac{1}{|\mathcal{E}|} \int_{\mathcal{E}} b(y) dy$.

The following local estimate is valid.

Lemma 5. *Let $0 < \beta < 1$, $0 < \alpha < \alpha + \beta < \gamma$ and $b \in \dot{\Lambda}_{\beta,P}(\mathbb{R}^n)$, then the following pointwise estimate holds:*

$$|[b, I_\alpha^P]f(x)| \lesssim \|b\|_{\dot{\Lambda}_{\beta,P}} I_{\alpha+\beta}^P |f|(x).$$

Proof. Let $0 < \beta < 1$, $0 < \alpha < \alpha + \beta < \gamma$ and $b \in \dot{\Lambda}_{\beta,P}(\mathbb{R}^n)$, then

$$\begin{aligned} |[b, I_\alpha^P]f(x)| &\lesssim \int_{\mathbb{R}^n} \frac{|b(x) - b(y)||f(y)|}{\rho(x-y)^{\gamma-\alpha}} dy \\ &\lesssim \|b\|_{\dot{\Lambda}_{\beta,P}} \int_{\mathbb{R}^n} \frac{|f(y)|}{\rho(x-y)^{\gamma-\alpha-\beta}} dy \\ &= \|b\|_{\dot{\Lambda}_{\beta,P}} I_{\alpha+\beta}^P |f|(x). \end{aligned}$$

◀

3. Commutator of parabolic fractional integral operators with parabolic Lipschitz functions

In this section, we establish the boundedness of the commutator of the parabolic fractional integral operator I_α^P with parabolic Lipschitz functions in parabolic total Morrey-Guliyev spaces $L_{p,\lambda,\mu}^P(\mathbb{R}^n)$.

The following local estimate is valid.

Lemma 6. [10] *Let $0 < \alpha < \gamma$, $1 \leq p < \frac{\gamma}{\alpha}$, and $\frac{1}{p} - \frac{1}{q} = \frac{\alpha}{\gamma}$. Then, for $p > 1$ the inequality*

$$\|I_\alpha^P f\|_{L_q(\mathcal{E}(x,r))} \lesssim r^{\frac{\gamma}{q}} \int_{2r}^\infty t^{-\frac{\gamma}{q}} \|f\|_{L_p(\mathcal{E}(x,t))} \frac{dt}{t} \quad (7)$$

holds for all $\mathcal{E}(x,r)$ and for all $f \in L_p^{\text{loc}}(\mathbb{R}^n)$.

Moreover if $p = 1$, then the inequality

$$\|I_\alpha^P f\|_{W L_q(\mathcal{E}(x,r))} \lesssim r^{\frac{\gamma}{q}} \int_{2r}^\infty t^{-\frac{\gamma}{q}} \|f\|_{L_1(\mathcal{E}(x,t))} \frac{dt}{t} \quad (8)$$

holds for all $\mathcal{E}(x,r)$ and for all $f \in L_1^{\text{loc}}(\mathbb{R}^n)$.

Proof of Theorem 1. Let $1 \leq p < \infty$, $0 \leq \lambda, \mu < \gamma$, $0 < \beta < 1$, $0 < \alpha + \beta < \min\{\frac{\gamma-\lambda}{p}, \frac{\gamma-\mu}{p}\}$ and $\frac{1}{p} - \frac{1}{q} = \frac{\alpha+\beta}{\gamma}$.

Suppose $p > 1$. From the inequality (7) (see Lemma 6) we get

$$\|[b, I_\alpha^P]f\|_{L_{q, \frac{\lambda q}{p}, \frac{\mu q}{p}}^P} \lesssim \|I_{\alpha+\beta}^P f\|_{L_{q, \frac{\lambda q}{p}, \frac{\mu q}{p}}^P} = \sup_{x \in \mathbb{R}^n, r > 0} [r]_1^{-\frac{\lambda}{p}} [1/r]_1^{\frac{\mu}{p}} \|I_{\alpha+\beta}^P f\|_{L_q(\mathcal{E}(x,r))}$$

$$\begin{aligned}
 &\lesssim \sup_{x \in \mathbb{R}^n, r > 0} [r]_1^{-\frac{\lambda}{p}} [1/r]_1^{\frac{\mu}{p}} r^{\frac{\gamma}{q}} \int_{2r}^{\infty} t^{-\frac{\gamma}{q}} \|f\|_{L_p(\mathcal{E}(x,t))} \frac{dt}{t} \\
 &\lesssim \|f\|_{L_{p,\lambda,\mu}^P} \sup_{r > 0} [r]_1^{-\frac{\lambda}{p}} [1/r]_1^{\frac{\mu}{p}} r^{\frac{\gamma}{q}} \int_r^{\infty} t^{-\frac{\gamma}{q}} [t]_1^{\frac{\lambda}{p}} [1/t]_1^{-\frac{\mu}{p}} \frac{dt}{t} \\
 &= \|f\|_{L_{p,\lambda,\mu}^P} \sup_{r > 0} [r]_1^{-\frac{\lambda}{p}} [1/r]_1^{\frac{\mu}{p}} \int_1^{\infty} t^{-\frac{\gamma}{q}} [tr]_1^{\frac{\lambda}{p}} [1/(tr)]_1^{-\frac{\mu}{p}} \frac{dt}{t} \\
 &= \|f\|_{L_{p,\lambda,\mu}^P} \max \left\{ \sup_{0 < r \leq 1} r^{-\frac{\lambda}{p}} \int_1^{\infty} t^{-\frac{\gamma}{q}} [tr]_1^{\frac{\lambda}{p}} [1/(tr)]_1^{-\frac{\mu}{p}} \frac{dt}{t}, \right. \\
 &\quad \left. \sup_{r > 1} r^{-\frac{\mu}{p}} \int_1^{\infty} t^{-\frac{\gamma}{q}} (tr)^{\frac{\mu}{p}} \frac{dt}{t} \right\} = \|f\|_{L_{p,\lambda,\mu}^P} \\
 &\times \max \left\{ \sup_{0 < r \leq 1} r^{-\frac{\lambda}{p}} \left(\int_1^{1/r} t^{-\frac{\gamma}{q}} (tr)^{\frac{\lambda}{p}} \frac{dt}{t} \right. \right. \\
 &\quad \left. \left. + \int_{1/r}^{\infty} t^{-\frac{\gamma}{q}} (tr)^{\frac{\mu}{p}} \frac{dt}{t} \right), \int_1^{\infty} t^{-\frac{\gamma}{q} + \frac{\mu}{p}} \frac{dt}{t} \right\} \\
 &= \|f\|_{L_{p,\lambda,\mu}^P} \max \left\{ \sup_{0 < r \leq 1} \left(\int_1^{1/r} t^{-\frac{\gamma}{q} + \frac{\lambda}{p}} \frac{dt}{t} \right. \right. \\
 &\quad \left. \left. + r^{-\frac{\lambda}{p} + \frac{\mu}{p}} \int_{1/r}^{\infty} t^{-\frac{\gamma}{q} + \frac{\mu}{p}} \frac{dt}{t} \right), \int_1^{\infty} t^{-\frac{\gamma}{q} + \frac{\mu}{p}} \frac{dt}{t} \right\} \\
 &= \|f\|_{L_{p,\lambda,\mu}^P} \max \left\{ \sup_{0 < r \leq 1} \left(\int_1^{1/r} t^{\alpha + \beta - \frac{\gamma - \lambda}{p}} \frac{dt}{t} \right. \right. \\
 &\quad \left. \left. + r^{-\frac{\lambda - \mu}{p}} \int_{1/r}^{\infty} t^{\alpha + \beta - \frac{\gamma - \mu}{p}} \frac{dt}{t} \right), \int_1^{\infty} t^{\alpha + \beta - \frac{\gamma - \mu}{p}} \frac{dt}{t} \right\} \\
 &\leq \|f\|_{L_{p,\lambda,\mu}^P} \max \left\{ \sup_{0 < r \leq 1} \left(\int_1^{1/r} t^{\alpha + \beta - \frac{\gamma - \lambda}{p}} \frac{dt}{t} \right. \right. \\
 &\quad \left. \left. + \int_{1/r}^{\infty} t^{\alpha + \beta - \frac{\gamma - \lambda}{p}} \frac{dt}{t} \right), \int_1^{\infty} t^{\alpha + \beta - \frac{\gamma - \lambda}{p}} \frac{dt}{t} \right\} \\
 &= \|f\|_{L_{p,\lambda,\mu}^P} \max \left\{ \int_1^{\infty} t^{\alpha + \beta - \frac{\gamma - \lambda}{p}} \frac{dt}{t} \approx \|f\|_{L_{p,\lambda,\mu}^P}, \right.
 \end{aligned}$$

which implies that the operator $[b, I_{\alpha}^P]$ is bounded from $L_{p,\lambda,\mu}^P(\mathbb{R}^n)$ to $L_{q,\frac{\lambda q}{p},\frac{\mu q}{p}}(\mathbb{R}^n)$.

Let $p = 1$. From the inequality (8) (see Lemma 6) we get

$$\begin{aligned}
 &\| [b, I_{\alpha}^P] f \|_{WL_{q,\lambda q,\mu q}^P} \lesssim \| I_{\alpha + \beta}^P f \|_{WL_{q,\lambda q,\mu q}^P} = \sup_{x \in \mathbb{R}^n, r > 0} [r]_1^{-\lambda} [1/r]_1^{\mu} \| I_{\alpha + \beta}^P f \|_{WL_q(\mathcal{E}(x,r))} \\
 &\lesssim \sup_{x \in \mathbb{R}^n, r > 0} [r]_1^{-\lambda} [1/r]_1^{\mu} r^{\frac{\gamma}{q}} \int_r^{\infty} t^{-\frac{\gamma}{q}} \|f\|_{L_1(\mathcal{E}(x,t))} \frac{dt}{t}
 \end{aligned}$$

$$\begin{aligned}
&= \|f\|_{L_{1,\lambda,\mu}^P} \sup_{r>0} [r]_1^{-\lambda} [1/r]_1^\mu \int_1^\infty t^{-\frac{\gamma}{q}} [tr]_1^\lambda [1/(tr)]_1^{-\mu} \frac{dt}{t} = \|f\|_{L_{1,\lambda,\mu}^P} \\
&\times \max \left\{ \sup_{0<r\leq 1} r^{-\lambda} \int_1^\infty t^{-\frac{\gamma}{q}} [tr]_1^\lambda [1/(tr)]_1^{-\mu} \frac{dt}{t}, \sup_{r>1} r^{-\mu} \int_1^\infty t^{-\frac{\gamma}{q}} (tr)^\mu \frac{dt}{t} \right\} \\
&= \|f\|_{L_{1,\lambda,\mu}^P} \max \left\{ \sup_{0<r\leq 1} r^{-\lambda} \left(\int_1^{1/r} t^{-\frac{\gamma}{q}} (tr)^\lambda \frac{dt}{t} \right. \right. \\
&\left. \left. + \int_{1/r}^\infty t^{-\frac{\gamma}{q}} (tr)^\mu \frac{dt}{t} \right), \int_1^\infty t^{-\frac{\gamma}{q}+\mu} \frac{dt}{t} \right\} \\
&= \|f\|_{L_{1,\lambda,\mu}^P} \max \left\{ \sup_{0<r\leq 1} \left(\int_1^{1/r} t^{-\frac{\gamma}{q}+\lambda} \frac{dt}{t} \right. \right. \\
&\left. \left. + r^{-\lambda+\mu} \int_{1/r}^\infty t^{-\frac{\gamma}{q}+\mu} \frac{dt}{t} \right), \int_1^\infty t^{-\frac{\gamma}{q}+\mu} \frac{dt}{t} \right\} \\
&= \|f\|_{L_{1,\lambda,\mu}^P} \max \left\{ \sup_{0<r\leq 1} \left(\int_1^{1/r} t^{\alpha+\beta-(\gamma-\lambda)} \frac{dt}{t} \right. \right. \\
&\left. \left. + r^{-(\lambda-\mu)} \int_{1/r}^\infty t^{\alpha+\beta-(\gamma-\mu)} \frac{dt}{t} \right), \int_1^\infty t^{\alpha+\beta-(\gamma-\mu)} \frac{dt}{t} \right\} \\
&= \|f\|_{L_{1,\lambda,\mu}^P} \max \left\{ \sup_{0<r\leq 1} \left(\int_1^{1/r} t^{\alpha+\beta-(\gamma-\lambda)} \frac{dt}{t} \right. \right. \\
&\left. \left. + \int_{1/r}^\infty t^{\alpha+\beta-(\gamma-\lambda)} \frac{dt}{t} \right), \int_1^\infty t^{\alpha+\beta-(\gamma-\lambda)} \frac{dt}{t} \right\} \\
&\approx \|f\|_{L_{1,\lambda,\mu}^P} \int_1^\infty t^{\alpha+\beta-(\gamma-\lambda)} \frac{dt}{t} \approx \|f\|_{L_{1,\lambda,\mu}^P},
\end{aligned}$$

which implies that the operator $[b, I_\alpha^P]$ is bounded from $L_{1,\lambda,\mu}^P(\mathbb{R}^n)$ to $WL_{q,\lambda q,\mu q}^P(\mathbb{R}^n)$.

Proof of Theorem 2. Let $1 \leq p < \infty$, $0 \leq \mu \leq \lambda < \gamma$, $0 < \beta < 1$, $0 < \alpha + \beta < \frac{\gamma-\lambda}{p}$, $\frac{\alpha+\beta}{\gamma-\mu} \leq \frac{1}{p} - \frac{1}{q} \leq \frac{\alpha+\beta}{\gamma-\lambda}$, $f \in L_{p,\lambda,\mu}^P(\mathbb{R}^n)$ and r arbitrary positive number.

$$\begin{aligned}
|[b, I_\alpha^P]f(x)| &\lesssim I_{\alpha+\beta}^P |f|(x) = \int_{\mathbb{R}^n} \frac{|f(y)|}{\rho(x-y)^{\gamma-\alpha+\beta-\beta}} dy \\
&\approx \int_{\mathbb{R}^n} \left(\int_{\rho(x-y)}^\infty t^{\alpha+\beta-\gamma-1} dt \right) |f(y)| dy \\
&= \int_{\mathbb{R}^n} \int_0^\infty \chi_{(\rho(x-y), \infty)}(t) t^{\alpha+\beta-\gamma-1} |f(y)| dt dy \\
&= \int_0^\infty \left(\int_{\mathbb{R}^n} \chi_{(\rho(x-y), \infty)}(t) |f(y)| dy \right) t^{\alpha+\beta-\gamma-1} dt = \int_0^\infty t^{\alpha+\beta-\gamma} \|f\|_{L_1(\mathcal{E}(x,t))} \frac{dt}{t}
\end{aligned}$$

$$\begin{aligned}
 &= \int_0^r t^{\alpha+\beta-\gamma} \|f\|_{L_1(\mathcal{E}(x,t))} \frac{dt}{t} + \int_r^\infty t^{\alpha+\beta-\gamma} \|f\|_{L_1(\mathcal{E}(x,t))} \frac{dt}{t} \\
 &\lesssim r^{\alpha+\beta} M^P f(x) + r^{\alpha+\beta-\frac{\gamma}{p}} \|f\|_{L_p(\mathcal{E}(x,r))} \\
 &\leq r^{\alpha+\beta} M^P f(x) + r^{\alpha-\frac{\gamma}{p}} [r]_1^{\frac{\lambda}{p}} [1/r]_1^{-\frac{\mu}{p}} \|f\|_{L_{p,\lambda,\mu}^P} \\
 &= r^{\alpha+\beta} M^P f(x) + [r]_1^{\alpha-\frac{\gamma-\lambda}{p}} [1/r]_1^{-\alpha-\beta+\frac{\gamma-\mu}{p}} \|f\|_{L_{p,\lambda,\mu}^P} \\
 &\leq \min \left\{ r^{\alpha+\beta} M^P f(x) + r^{\alpha+\beta-\frac{\gamma-\lambda}{p}} \|f\|_{L_{p,\lambda,\mu}^P}, r^{\alpha+\beta} M^P f(x) + r^{\alpha+\beta-\frac{\gamma-\mu}{p}} \|f\|_{L_{p,\lambda,\mu}^P} \right\}.
 \end{aligned}$$

Minimizing with respect to r , at

$$r = \left(\frac{\|f\|_{L_{p,\lambda,\mu}^P}}{M^P f(x)} \right)^{\frac{p}{\gamma-\lambda}} \quad \text{and} \quad r = \left(\frac{\|f\|_{L_{p,\lambda,\mu}^P}}{M^P f(x)} \right)^{\frac{p}{\gamma-\mu}}$$

we have

$$|[b, I_\alpha^P]f(x)| \leq \min \left\{ (M^P f(x))^{1-\frac{(\alpha+\beta)p}{\gamma-\lambda}} \|f\|_{L_{p,\lambda,\mu}^P}^{\frac{(\alpha+\beta)p}{\gamma-\lambda}}, (M^P f(x))^{1-\frac{(\alpha+\beta)p}{\gamma-\mu}} \|f\|_{L_{p,\lambda,\mu}^P}^{\frac{(\alpha+\beta)p}{\gamma-\mu}} \right\}, \tag{9}$$

where we used the fact that the supremum is achieved when the minimal parts are balanced. From theorem 3 and inequality (9) we obtain

$$\begin{aligned}
 \|[b, I_\alpha^P]f\|_{L_{q,\lambda,\mu}^P} &\lesssim \|I_{\alpha+\beta}^P f\|_{L_{q,\lambda,\mu}^P} \lesssim \|f\|_{L_{p,\lambda,\mu}^P}^{1-\frac{p}{q}} \|(M^P f)^{\frac{p}{q}}\|_{L_{q,\lambda,\mu}^P} \\
 &= \|f\|_{L_{p,\lambda,\mu}^P}^{1-\frac{p}{q}} \|M^P f\|_{L_{p,\lambda,\mu}^P}^{\frac{p}{q}} \lesssim \|f\|_{L_{p,\lambda,\mu}^P},
 \end{aligned}$$

if $1 < p < q < \infty$ and

$$\|[b, I_\alpha^P]f\|_{WL_{q,\lambda,\mu}^P} \lesssim \|I_{\alpha+\beta}^P f\|_{WL_{q,\lambda,\mu}^P} \lesssim \|f\|_{L_{1,\lambda,\mu}}^{1-\frac{1}{q}} \|M^P f\|_{WL_{1,\lambda,\mu}}^{\frac{1}{q}} \lesssim \|f\|_{L_{1,\lambda,\mu}},$$

if $p = 1 < q < \infty$.

4. Conclusion

In this paper, we present sufficient conditions for the boundedness of commutators of the parabolic fractional integral operator $[b, I_\alpha^P]$ in parabolic total Morrey-Guliyev spaces $L_{p,\lambda,\mu}^P(\mathbb{R}^n)$ with symbols b belonging to parabolic Lipschitz spaces $\dot{\Lambda}_{\beta,P}(\mathbb{R}^n)$.

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