Azerbaijan Journal of Mathematics
V. 16, No 1, 2026, January
ISSN 2218-6816
https://doi.org/10.59849/2218-6816.2026.1.16

Weyl-Titcmarsh Theory for a Discrete Schrödinger Operator With Bounded Coefficients

A.Kh. Khanmamedov*, A.H. Yagubov

Abstract. A discrete Schrödinger equation with an absolutely continuous spectrum of multiplicity two is considered on the line L. The basic properties of the Weyl solutions of this equation are investigated. It is proved that the system consisting of the Weyl solutions, is a Bessel system in some weight space.

Key Words and Phrases: Schrödinger operator, transformation operators, Weyl-Titcmarsh theory, expansion formula, Bessel system, Hilbert system.

2010 Mathematics Subject Classifications: 34A55, 34B24

---

1. Introduction

Consider a discrete Schrödinger equation on the line

$$a_{n-1}y_{n-1} + b_n y_n + a_n y_{n+1} = \lambda y_n,\ n \in Z, \tag{1}$$

in the important special case where the coefficients $a_n > 0, b_n$ are real and the difference expression $(ly)_n = a_{n-1}y_{n-1} + b_n y_n + a_n y_{n+1}$ generates in the space $\ell_2(-\infty, +\infty)$ a bounded operator L, whose spectrum is absolutely continuous of multiplicity two. This condition is assumed to hold throughout the work. For the equation (1), direct and inverse spectral problems have been investigated in the case where the coefficients a_n, b_n rapidly decrease (see [1]) and in the periodic case (see [2,4]). In [5], a new type of spectral data, which are similar to scattering data, is introduced for the operator L. Moreover, an integral equation that allows us to reconstruct the operator L from this spectral data, is obtained. On the other hand, in the work of [6,7], by means of the method of transformation operators the inverse spectral problems were studied for the equation (1) with

*Corresponding author.

asymptotically periodic coefficients. At the same time, as follows from [6,7], when constructing the transformation operator, a special role is played by the basic properties of explicit solutions of the corresponding unperturbed equation. In the case of equation (1) with bounded coefficients, the establishment of such properties is facing significant difficulties.

In this work, the basic properties of the Weyl solutions of the equation (1) are investigated. It is proved that the system consisting of the Weyl solutions, is a Bessel system in some weight space.

2. The basic properties of the Weyl solutions

In the spaces $\ell_2[\pm 1, \pm\infty)$, consider the operators $L_\pm$ generated by the difference expressions $(l_\pm y)_n = a_{n-1}y_{n-1} + b_n y_n + a_n y_{n+1}, \pm n \geq \pm 1$ and the boundary condition $y_0 = 0$.

For definiteness, we assume that the spectrum of the operator L lies in the segment $[-2, 2]$. We denote by $\varphi_n(\lambda)$ and $\theta_n(\lambda)$ the solutions of the equation (1) with initial data $\varphi_0(\lambda) = \theta_1(\lambda) = 0$, $\varphi_1(\lambda) = 1$, $\theta_0(\lambda) = 1$. It is evident that every solution of this equation is their linear combination.

It is known (see, for example, [6,7,9,10]) that for $\lambda \in C\backslash R$ the equation (1) has two solutions (the Weyl solutions of the operator L)

$$\Psi_n^+(\lambda) = -\frac{\theta_n(\lambda)}{a_0} + m_+(\lambda)\varphi_n(\lambda), \, n \in Z, \tag{2}$$

$$\Psi_n^-(\lambda) = \frac{\theta_n(\lambda)}{a_0} + m_-(\lambda)\varphi_n(\lambda), \, n \in Z, \tag{3}$$

such that $\sum\limits_{k=N}^{\pm\infty} \left|\Psi_k^\pm(\lambda)\right|^2 < \infty$ for any finite number N.

We denote by $s_n(\lambda)$ and $c_n(\lambda)$ the solutions of the equation (1) with initial data $s_0(\lambda) = c_{-1}(\lambda) = 0$, $s_{-1}(\lambda) = 1$, $c_0(\lambda) = 1$. Then for $\lambda \in C\backslash R$ the equation (1) has a Weyl solution

$$\Psi_n^L(\lambda) = -\frac{c_n(\lambda)}{a_{-1}} + m_-^L(\lambda)s_n(\lambda), \, n \in Z, \tag{4}$$

such that $\sum\limits_{k=-\infty}^{N} \left|\Psi_k^L(\lambda)\right|^2 < \infty$ for any finite number N. Let $\delta_n = (\delta_{nm})_{m\in Z}$ be the canonical basis of $\ell_2(-\infty, +\infty)$, where δ_{nm} is the Kronecker delta. In this notation, $m_+(\lambda)$ and $m_-^L(\lambda)$ are the Weyl functions of the operators $L_\pm$ represented in the form of scalar product

$$m_\pm(\lambda) = \left\langle (L_\pm - zI)^{-1}\delta_{\pm 1}, \delta_{\pm 1}\right\rangle. \tag{5}$$

Moreover,

$$m_+(\lambda)=\int_{-2}^{2}\frac{d\rho_+(t)}{t-\lambda},\ m_-^L(\lambda)=\int_{-2}^{2}\frac{d\rho_-(t)}{t-\lambda},\tag{6}$$

Here $d\rho_\pm(t)$ is the spectral measure of the operator $L_\pm$. The connection between functions $m_-(\lambda)$ and $m_-^L(\lambda)$ is given by the formula

$$a_0^2 m_-(\lambda)=\lambda-b_0+a_{-1}^2 m_-^L(\lambda).\tag{7}$$

In this case, the function $m_-^L(\lambda)$ can be represented as

$$m_-(\lambda)=\frac{\lambda-b_0}{a_0^2}+\frac{a_{-1}^2}{a_0^2}\int_{-2}^{2}\frac{d\rho_-(t)}{t-\lambda}.\tag{8}$$

It should be noted that for each $\lambda\notin[-2,2]$ the equation (1) has a unique solution $u_n(\lambda)$ from the space $\ell_2(-\infty,0)$ up to a constant factor. It follows that

$$\Psi_n^L(\lambda)=-\frac{a_0}{a_{-1}}\Psi_n^-(\lambda).\tag{9}$$

Let Γ be the complex λ-plane with cuts along the segment $[-2,2]$.According to (2), (3), and (6), (8) the functions $\Psi_n^+(\lambda)$ and $\Psi_n^-(\lambda)$ are continuous for $\lambda\in\partial\Gamma$. Furthermore, the relations $m_\pm(\lambda+i0)=\overline{m_\pm(\lambda-i0)}$, $\lambda\in[-2,2]$ and (2), (3) imply that $\Psi_n^\pm(\lambda+i0)=\overline{\Psi_n^\pm(\lambda-i0)}$, $\lambda\in[-2,2]$, where i is the imaginary unit. On the other hand, the expressions (2), (3) imply that for $\lambda\in\partial\Gamma,\lambda\neq\pm2$, the two solutions $\Psi_n^\pm(\lambda)$ and $\overline{\Psi_n^\pm(\lambda)}$ of Eq. (1) are linearly independent, and their Wronskian is given by the formula

$$W\left[\Psi^\pm,\overline{\Psi^\pm}\right]=a_0\left\{\Psi_0^\pm(\lambda)\overline{\Psi_1^\pm(\lambda)}-\Psi_1^\pm(\lambda)\overline{\Psi_0^\pm(\lambda)}\right\}=$$

$$=\pm\left[m_\pm(\lambda)-\overline{m_\pm(\lambda)}\right].\tag{10}$$

Therefore, for $\lambda\in\partial\Gamma,\lambda\neq\pm2$, we have

$$\Psi_n^-(\lambda)=a_+(\lambda)\overline{\Psi_n^+(\lambda)}+b_+(\lambda)\Psi_n^+(\lambda),\tag{11}$$

$$\Psi_n^+(\lambda)=a_-(\lambda)\overline{\Psi_n^-(\lambda)}+b_-(\lambda)\Psi_n^-(\lambda).\tag{12}$$

It follows from (10), (11), (12) that

$$|a_\pm(\lambda)|^2-|b_\pm(\lambda)|^2=\left(\frac{\overline{m_-(\lambda)}-m_-(\lambda)}{\overline{m_+(\lambda)}-m_+(\lambda)}\right)^{\pm1},\tag{13}$$

$$\left[\overline{m_+(\lambda)} - m_+(\lambda)\right] a_+(\lambda) = \left[\overline{m_-(\lambda)} - m_-(\lambda)\right] a_-(\lambda) = m_+(\lambda) + m_-(\lambda), \quad (14)$$

$$\left[\overline{m_+(\lambda)} - m_+(\lambda)\right] b_+(\lambda) = \left[m_-(\lambda) - \overline{m_-(\lambda)}\right] \overline{b_-(\lambda)}. \quad (15)$$

It follows from (14) that $\left[\overline{m_\pm(\lambda)} - m_\pm(\lambda)\right] a_\pm(\lambda)$ admits an analytic continuation to the plane Γ and is continuous for $\lambda \in \partial\Gamma, \lambda \neq \pm 2$.

We introduce the function

$$w(\lambda) = m_+(\lambda) + m_-(\lambda). \quad (16)$$

Since the spectrum of the operator $\hat{L}_\pm$ is absolutely continuous, we have $\pm Imm_+(\lambda \pm i0) > 0$ and $\pm Imm_-(\lambda \pm i0) > 0$ for $\lambda \in (-2, 2)$. Consequently, $\pm\left(\overline{m_+(\lambda \pm i0)} - m_+(\lambda \pm i0)\right) > 0$ and $\pm\left(\overline{m_-(\lambda \pm i0)} - m_-(\lambda \pm i0)\right) > 0$ for $\lambda \in (-2, 2)$. Moreover, $a_\pm(\lambda)$ does not vanish at $\lambda \in \partial\Gamma, \lambda \neq \pm 2$ by normalization condition (19). At the points $\lambda = \pm 2$, the functions

$$\omega_+^{-1}(\lambda) = \overline{m_+(\lambda)} - m_+(\lambda),$$

$$\omega_-^{-1}(\lambda) = \overline{m_-(\lambda)} - m_-(\lambda)$$

may turn to zero. However, we assume that the function $\omega_\pm(\lambda)$ has an integrated feature at the points $\lambda = \pm 2$. Moreover, the functions $\left(\frac{\overline{m_-(\lambda)} - m_-(\lambda)}{\overline{m_+(\lambda)} - m_+(\lambda)}\right)^{\pm 1}$ have the finite limits at the points $\lambda = \pm 2$.

We now assume that $w(\lambda) = 0, \lambda \notin \partial\Gamma$. It then follows from (7) that the solutions $\Psi_n^+(\lambda)$ and $\Psi_n^-(\lambda)$ are linearly dependent and the self-adjoint operator $\hat{L}$ has an eigenvalue, which, according to condition II, is impossible.

Let

$$R_{nm}(\lambda) = \begin{cases} \frac{\Psi_n^+(\lambda)\Psi_m^-(\lambda)}{w(\lambda)} & for\ n \geq m \\ \frac{\Psi_m^+(\lambda)\Psi_n^-(\lambda)}{w(\lambda)} & for\ n < m. \end{cases}$$

Consider the operator R_λ on $\ell_2(-\infty, +\infty)$ with kernel $R_{nm}(\lambda)$, i.e.

$$(R_\lambda h)_n = \sum_{m=-\infty}^{\infty} R_{nm}(\lambda) h_m, h = \{h_m\}_{m=-\infty}^{\infty} \in l^2(-\infty, \infty). \quad (17)$$

It is easy to verify that the operator R_λ, $\lambda \notin [-2, 2]$, is the resolvent of L. In other words, $R_\lambda = (L - \lambda I)^{-1}$. The explicit formula (17) for the resolvent R_λ leads immediately to the eigenfunction expansion theorem for L. Namely, when there is no point spectrum, we have the formula

$$h_n = \frac{1}{2\pi i}\int_{-2}^{2}\left\{(R_{\lambda+i0}h)_n - (R_{\lambda-i0}h)_n\right\} d\lambda = \sum_{m=-\infty}^{+\infty}\left(\frac{1}{2\pi i}\int_{\partial\Gamma} R_{nm}(\lambda) h_m d\lambda\right) =$$

$$= \frac{1}{2\pi i}\int_{\partial\Gamma} w^{-1}(\lambda)\,\Psi_n^+(\lambda)\left\{\sum_{m=-\infty}^{n}\Psi_m^-(\lambda)\,h_m\right\}d\lambda +$$

$$+\frac{1}{2\pi i}\int_{\partial\Gamma} w^{-1}(\lambda)\,\Psi_n^-(\lambda)\left\{\sum_{m=n+1}^{\infty}\Psi_m^+(\lambda)\,h_m\right\}d\lambda.$$

(see [3], Ch. XII), which is sometimes referred to as Stone's formula. It is this formula that serves as a basis for the derivation of the eigenfunction expansion theorem.

Theorem 1. *We have the expansion formula*

$$\frac{1}{2\pi i}\int_{\partial\Gamma} w^{-1}(\lambda)\,\Psi_n^+(\lambda)\,\Psi_m^-(\lambda)\,d\lambda = \delta_{nm}. \tag{18}$$

Using equality (16), we rewrite the formula (18) as follows:

$$\frac{1}{2\pi i}\int_{\partial\Gamma} \omega_\pm(\lambda)\,\Psi_n^\pm(\lambda)\,\frac{\Psi_m^\mp(\lambda)}{a_\pm(\lambda)}d\lambda = \delta_{nm}, \tag{19}$$

where $\omega_\pm(\lambda) = \frac{1}{m_\pm(\lambda) - m_\pm(\lambda)}$. We introduce the space $L_2(\partial\Gamma;\omega_\pm(\lambda))$ of functions $f(\lambda)$ defined almost everywhere on $\partial\Gamma$ and satisfying the conditions $\overline{f(\lambda+i0)} = \overline{f(\lambda-i0)}$ for almost every $\lambda\in[-2,2]$,

$$\left|\int_{\partial\Gamma}\omega_\pm(\lambda)\,|f(\lambda)|^2\,d\lambda\right| < \infty.$$

It is clear that $L_2(\partial\Gamma;\omega_\pm(\lambda))$ is a separable Hilbert space with the scalar product

$$\langle f, g\rangle = \frac{1}{2\pi i}\int_{\partial\Gamma}\omega_\pm(\lambda)\,f(\lambda)\,\overline{g(\lambda)}d\lambda.$$

Let $\{\varphi_n(\lambda)\}_{n\in Z}$ and $\{\psi_n(\lambda)\}_{n\in Z}$ be two complete systems of functions in $L_2(\partial\Gamma;\omega_\pm(\lambda))$, forming a biorthogonal system of functions, i.e. $\langle\varphi_n(\lambda),\psi_m(\lambda)\rangle = \delta_n^m$. Following S.S. Levin [11], we will call the system $\{\varphi_n(\lambda)\}_{n\in Z}$ a B-system, if there is such $\{\psi_n(\lambda)\}_{n\in Z}$, which together with it forms a biorthogonal system. B-system of functions $\{\varphi_n(\lambda)\}_{n\in Z}$ is said to be a Bessel system if, for any function $f(\lambda)\in L_2(\partial\Gamma;\omega_\pm(\lambda))$, the series

$$\sum_{n\in Z}|\langle f(\lambda),\psi_n(\lambda)\rangle|^2$$

is convergent(see [8]). B-system of functions$\{\varphi_n(\lambda)\}_{n\in Z}$ is said to be a Hilbert system if, for any sequence of numbers c_n, such that

$$\sum_{n\in Z}|c_n|^2<\infty,$$

there is one and only one $f(\lambda)$, for which $c_n=\langle f(\lambda),\psi_n(\lambda)\rangle$.

Theorem 2. *The* $\{\Psi_n^{\pm}(\lambda)\}_{n\in Z}$ *is a Bessel system in* $L_2(\partial\Gamma;\omega_{\pm}(\lambda))$. *Moreover,* $\frac{\overline{\Psi_n^{\mp}(\lambda)}}{a_{\pm}(\lambda)}$ *is a Hilbert system in* $L_2(\partial\Gamma;\omega_{\pm}(\lambda))$.

Proof. From the formula (19) it follows that the systems of functions $\Psi_n^{\pm}(\lambda)$ and $\left\{\overline{\left(\frac{\Psi_n^{\mp}(\lambda)}{a_{\pm}(\lambda)}\right)}\right\}$ are biorthonormal in the space $L_2(\partial\Gamma;\omega_{\pm}(\lambda))$. Consequently, the system $\{\Psi_n^{\pm}(\lambda)\}_{n\in Z}$ is minimal in $L_2(\partial\Gamma;\omega_{\pm}(\lambda))$. Let us prove that the system $\{\Psi_n^{\pm}(\lambda)\}_{n\in Z}$ is complete in $L_2(\partial\Gamma;\omega_{\pm}(\lambda))$. It is sufficient to show that $L_2(\partial\Gamma;\omega_{\pm}(\lambda))$ contains no non-zero element biorthogonal to all the elements of the system $\{\Psi_n^{\pm}(\lambda)\}_{n\in Z}$. Let $f(\lambda)\in L_2(\partial\Gamma;\omega_{\pm}(\lambda))$ and suppose that

$$\int_{\partial\Gamma}\omega_{\pm}(\lambda)\,\Psi_n^{\pm}(\lambda)\,\overline{f(\lambda)}d\lambda=0,n\in Z.$$

By (1) we have the equality

$$\int_{\partial\Gamma}\omega(\lambda)\,\lambda^k\Psi_n^{\pm}(\lambda)\,\overline{f(\lambda)}d\lambda=0,n=0,\pm1,\pm2,...,k=0,1,2,....$$

Consequently, $f(\lambda)=0$ for $\lambda\in\partial\Gamma$. Further, in the space $L_2(\partial\Gamma;\omega_{\pm}(\lambda))$ we consider the operator $B_{\pm}$ defined by the formula

$$(B_{\pm}f)(\lambda)=f(\lambda)+\overline{r_{\pm}(\lambda)f(\lambda)}.$$

Since $r_{\pm}(\lambda-i0)=\overline{r_{\pm}(\lambda+i0)}$, $|r_{+}(\lambda)|<1$, $B_{\pm}$ is a bounded positive Hermite operator. On the other hand, by virtue of (11) we have

$$B_{\pm}\Psi_n^{\pm}(\lambda)=\frac{\overline{\Psi_n^{\mp}(\lambda)}}{a_{\pm}(\lambda)}.$$

Hence, from Theorem 4 of [6] it follows that $\{\Psi_n^{\pm}(\lambda)\}_{n\in Z}$ is a Bessel system in $L_2(\partial\Gamma;\omega_{\pm}(\lambda))$. Moreover, $\left\{\frac{\overline{\Psi_n^{\mp}(\lambda)}}{a_{\pm}(\lambda)}\right\}_{n\in Z}$ is a Hilbert system in $L_2(\partial\Gamma;\omega_{\pm}(\lambda))$ (see Theorem 3 of [1]). ◄

We introduce the function

$$F(n,k,m,r)=$$

$$=\frac{1}{2\pi i}\int_{\partial\Gamma} w^{-2}(\lambda)\left\{\Psi_n^+(\lambda)\Psi_k^-(\lambda)-\Psi_n^-(\lambda)\Psi_k^+(\lambda)\right\}\Psi_m^+(\lambda)\Psi_r^-(\lambda)\,d\lambda. \qquad (20)$$

Theorem 3. *The following equality holds:* $F(n,k,m,r)=0$ *for* $\pm r\geq\pm(m\pm(k-n))$.

Proof. Let $r\leq m-(k-n)$. It follows from the definition of $F(n,k,m,r)$ and the residue theorem that

$$F(n,k,m,r)=\underset{\lambda=\infty}{res}\, w^{-2}(\lambda)\Psi_n^+(\lambda)\Psi_k^-(\lambda)\Psi_m^+(\lambda)\Psi_r^-(\lambda)-$$

$$\underset{\lambda=\infty}{res}\, w^{-2}(\lambda)\Psi_k^+(\lambda)\Psi_n^-(\lambda)\Psi_m^+(\lambda)\Psi_r^-(\lambda).$$

We need to know the behavior of the function $\Psi_n^{\pm}(\lambda)$ as $\lambda\to\infty$. From [5,6,9] we have

$$\Psi_n^+(\lambda)\lambda^n\to h_n^+,\ \ \lambda\to\infty,$$

where

$$h_n^+=\begin{cases}-a_1...a_{n-1},\ n>1,\\ -1,n=1,\\ -\frac{1}{a_n...a_0},\ n\leq 0.\end{cases}$$

Similarly,

$$\Psi_n^-(\lambda)\lambda^{-n}\to h_n^-,\ \ \lambda\to\infty,$$

where

$$h_n^-=\begin{cases}a_0^{-2}(a_1...a_{n-1})^{-1},\ n>1,\\ a_0^{-2},n=1,\\ a_0^{-1}a_n...a_{-1},\ n\leq 0.\end{cases}$$

According to the last relations and (8), (16), for $r\leq m-(k-n)$ we find up to a constant factor that

$$w^{-2}(\lambda)\Psi_n^+(\lambda)\Psi_k^-(\lambda)\Psi_m^+(\lambda)\Psi_r^-(\lambda)\sim\lambda^{-2}\lambda^{r-m+(k-n)},\lambda\to\infty,$$

$$w^{-2}(\lambda)\Psi_k^+(\lambda)\Psi_n^-(\lambda)\Psi_m^+(\lambda)\Psi_r^-(\lambda)\sim\lambda^{-2}\lambda^{r-m-(k-n)},\lambda\to\infty.$$

Recalling that $r-m+(k-n)\leq 0$, from the last two relations we conclude

$$\underset{\lambda=\infty}{res}\, w^{-2}(\lambda)\Psi_n^+(\lambda)\Psi_k^-(\lambda)\Psi_m^+(\lambda)\Psi_r^-(\lambda)=0.$$

Since the condition $r-m+(k-n)\leq 0$ implies $r-m-(k-n)\leq 0$, we similarly find that

$$\underset{\lambda=\infty}{res}\, w^{-2}(\lambda)\,\Psi_k^+(\lambda)\,\Psi_n^-(\lambda)\,\Psi_m^+(\lambda)\,\Psi_r^-(\lambda)=0.$$

Thus, $F(n,k,m,r)=0$ for $r\leq m-(k-n)$.

Let us now consider the difference

$$F(n,k,m,r)-F(n,k,r,m)=\frac{1}{2\pi i}\int_{\partial\Gamma} w^{-2}(\lambda)\left\{\Psi_n^+(\lambda)\,\Psi_k^-(\lambda)-\Psi_k^+(\lambda)\,\Psi_n^-(\lambda)\right\}\times$$
$$\times\left\{\Psi_m^+(\lambda)\,\Psi_r^-(\lambda)-\Psi_r^+(\lambda)\,\Psi_m^-(\lambda)\right\}d\lambda.$$

Note that

$$w^{-1}(\lambda)\left\{\Psi_n^+(\lambda)\,\Psi_k^-(\lambda)-\Psi_k^+(\lambda)\,\Psi_n^-(\lambda)\right\}=$$
$$=\frac{1}{\left[\overline{m_+(\lambda)}-m_+(\lambda)\right]}\left\{\Psi_n^+(\lambda)\left(\overline{\Psi_k^+(\lambda)}+r_+(\lambda)\,\Psi_k^+(\lambda)\right)-\right.$$
$$\left.-\Psi_k^+(\lambda)\left(\overline{\Psi_n^+(\lambda)}+r_+(\lambda)\,\Psi_n^+(\lambda)\right)\right\}=$$
$$=\frac{1}{\overline{m_+(\lambda)}-m_+(\lambda)}\left\{\Psi_n^+(\lambda)\,\overline{\Psi_k^+(\lambda)}-\Psi_k^+(\lambda)\,\overline{\Psi_n^+(\lambda)}\right\}=$$
$$=\frac{1}{\overline{m_+(\lambda)}-m_+(\lambda)}\left\{\left(-\frac{\theta_n(\lambda)}{a_0}+m_+(\lambda)\,\varphi_n(\lambda)\right)\left(-\frac{\theta_k(\lambda)}{a_0}+\overline{m_+(\lambda)}\varphi_k(\lambda)\right)\right\}-$$
$$-\frac{1}{\overline{m_+(\lambda)}-m_+(\lambda)}\left\{\left(-\frac{\theta_k(\lambda)}{a_0}+m_+(\lambda)\,\varphi_k(\lambda)\right)\left(-\frac{\theta_n(\lambda)}{a_0}+\overline{m_+(\lambda)}\varphi_n(\lambda)\right)\right\}=$$
$$=\frac{1}{\overline{m_+(\lambda)}-m_+(\lambda)}\left\{\overline{m_+(\lambda)}\left(\frac{\theta_k(\lambda)}{a_0}\varphi_n(\lambda)-\frac{\theta_n(\lambda)}{a_0}\varphi_k(\lambda)\right)-\right.$$
$$\left.-m_+(\lambda)\left(\frac{\theta_k(\lambda)}{a_0}\varphi_n(\lambda)-\frac{\theta_n(\lambda)}{a_0}\varphi_k(\lambda)\right)\right\}=\frac{\theta_k(\lambda)}{a_0}\varphi_n(\lambda)-\frac{\theta_n(\lambda)}{a_0}\varphi_k(\lambda).$$

It follows that

$$F(n,k,m,r)-F(n,k,r,m)=$$
$$=\frac{1}{2\pi i}\int_{\partial\Gamma}\left(\frac{\theta_k(\lambda)}{a_0}\varphi_n(\lambda)-\frac{\theta_n(\lambda)}{a_0}\varphi_k(\lambda)\right)\left(\frac{\theta_r(\lambda)}{a_0}\varphi_m(\lambda)-\frac{\theta_m(\lambda)}{a_0}\varphi_r(\lambda)\right)d\lambda=0,$$

since for all j,s, the function $\frac{\theta_j(\lambda)}{a_0}\varphi_s(\lambda)-\frac{\theta_s(\lambda)}{a_0}\varphi_j(\lambda)$ takes real values. Thus, we found that $F(n,k,m,r)=F(n,k,r,m)$. It follows from the last equality that $F(n,k,m,r)=0$ for $r\geq m+(k-n)$. ◄

In conclusion, we focus on one important consequence of Theorem 2, which plays a significant role in the construction of transformation operators (see [8]).

Let

$$\Phi\left(n,k,\lambda\right)=w^{-1}\left(\lambda\right)\left\{\Psi_n^+\left(\lambda\right)\Psi_k^-\left(\lambda\right)-\Psi_n^-\left(\lambda\right)\Psi_k^+\left(\lambda\right)\right\}. \tag{21}$$

Note that $\Phi\left(n,k,\lambda\right)$ takes real values for $\lambda\in\partial\Gamma$, since in the process of proving Theorem 3 we saw that

$$\Phi\left(n,k,\lambda\right)=\frac{\theta_k\left(\lambda\right)}{a_0}\varphi_n\left(\lambda\right)-\frac{\theta_n\left(\lambda\right)}{a_0}\varphi_k\left(\lambda\right).$$

Corollary 1. *For all n,k,m, where $k>n$, the equality*

$$\Phi\left(n,k;\lambda\right)\Psi_m^+\left(\lambda\right)=\sum_{j=m-(k-n)+1}^{m+(k-n)-1}F\left(n,k,m,j\right)\Psi_j^+\left(\lambda\right),\ \lambda\in\partial\Gamma. \tag{22}$$

holds.

Indeed, the function $\Phi\left(n,k;\lambda\right)\Psi_m^+\left(\lambda\right)$ belongs to the space $L_2\left(\partial\Gamma;\omega_-\left(\lambda\right)\right)$, since it is continuous for $\lambda\in\partial\Gamma$. Further, since $F\left(n,k,m,\cdot\right)\in\ell_2\left(-\infty,+\infty\right)$ and $\left\{\frac{\overline{\Psi_n^+(\lambda)}}{\overline{a_-(\lambda)}}\right\}_{n\in Z}$ is a Hilbert system in $L_2\left(\partial\Gamma;\omega_-\left(\lambda\right)\right)$, there is one and only one $\tilde{F}\left(\lambda\right)\in L_2\left(\partial\Gamma;\omega_-\left(\lambda\right)\right)$ for which the equalities

$$\tilde{F}\left(\lambda\right)=\sum_{j=-\infty}^{+\infty}F\left(n,k,m,j\right)\frac{\overline{\Psi_j^+\left(\lambda\right)}}{\overline{a_-\left(\lambda\right)}}=\sum_{j=m-(k-n)+1}^{m+(k-n)-1}F\left(n,k,m,j\right)\frac{\overline{\Psi_j^+\left(\lambda\right)}}{\overline{a_-\left(\lambda\right)}}, \tag{23}$$

$$F\left(n,k,m,r\right)=\frac{1}{2\pi i}\int_{\partial\Gamma}\omega_-\left(\lambda\right)\Psi_r^-\left(\lambda\right)\overline{F_\pm\left(\lambda\right)}d\lambda \tag{24}$$

are true . On the other hand, due to equalities (20), (21) we have

$$F\left(n,k,m,r\right)=\frac{1}{2\pi i}\int_{\partial\Gamma}\omega_-\left(\lambda\right)\frac{\Phi\left(n,k;\lambda\right)}{a_-\left(\lambda\right)}\Psi_m^+\left(\lambda\right)\Psi_r^-\left(\lambda\right)d\lambda. \tag{25}$$

Comparing (23),(24), (25), we find that

$$\frac{\Phi\left(n,k;\lambda\right)}{a_-\left(\lambda\right)}\Psi_m^+\left(\lambda\right)=\overline{\tilde{F}\left(\lambda\right)}.$$

Considering that $F\left(n,k,m,r\right)$ takes real values, we finally obtain

$$\frac{\Phi\left(n,k;\lambda\right)}{a_-\left(\lambda\right)}\Psi_m^+\left(\lambda\right)=\sum_{j=m-(k-n)+1}^{m+(k-n)-1}F\left(n,k,m,j\right)\frac{\Psi_j^+\left(\lambda\right)}{a_-\left(\lambda\right)}.$$

The last relation is equivalent to the equality (22).

Acknowledgment

This work was supported by the Azerbaijan Science Foundation- Grant No AEF-MGC-2025-1(54)-20/06/1-M-06

References

[1] Bari, N.K. (1951) *Biorthogonal systems and bases in Hilbert space*, Ucheniye Zapiski MGU, **4**(148), 69–107 (in Russian).

[2] Berezanskiy, Yu.M. (1986) *The integration of semi-infinite Toda chain by means of inverse spectral problem*, Reports on Mathematical Physics, **24**, 21–46.

[3] Dunford, N., Schwartz, J.T. (1963) *Linear operators, Vol. II: Spectral theory. Self-adjoint operators in Hilbert space*, Interscience Publishers (John Wiley), New York–London.

[4] Egorova, I., Michor, J., Teschl, G. (2007) *Inverse scattering transform for the Toda hierarchy with quasi-periodic background*, Proceedings of the American Mathematical Society, **135(6)**, 1817–1827.

[5] Maslov, L. (1992) *Spectral properties of the reflectionless Jacobi matrices*, Teoriya Funktsiy, Funktsionalnyy Analiz i Ikh Prilozheniya, **57**, 25–46.

[6] Maslov, L. (1993) *Description of the set of strong limits of the reflectionless Jacobi matrices*, Teoriya Funktsiy, Funktsionalnyy Analiz i Ikh Prilozheniya, **58**, 54–60.

[7] Khanmamedov, A.Kh. (2005) *Direct and inverse scattering problems for the perturbed Hill difference equation*, Sbornik: Mathematics, **196(10)**, 1529–1552.

[8] Khanmamedov, A.Kh. (2003). *Transformation operators for the perturbed Hill difference equation and one of their applications*, Siberian Mathematical Journal, **44(4)**, 729–738.

[9] Kudryavtsev, M. (2001) *On an inverse problem for finite-difference operators of second order*, arXiv:math/011027v1 [math.SP].

[10] Teschl, G. (2000). *Jacobi operators and completely integrable nonlinear lattices*, Mathematical Surveys and Monographs, **72**, American Mathematical Society.

[11] Levin, S.S. (1930) *Über einige mit der Konvergenz im Mittel verbundenen Eigenschaften von Funktionenfolgen*, Mathematische Zeitschrift, **32(4)**, 491–511.

Agil Kh. Khanmamedov
Baku State University, Baku, Azerbaijan
Institute of Mathematics, the Ministry of Science and Education of Azerbaijan, Baku, Azerbaijan
E-mail: agil_khanmamedov@yahoo.com

Ali H. Yagubov
National Aviation Academy of Azerbaijan, Baku, Azerbaijan
E-mail: aliyaqubov@naa.edu.az

Received 04 August 2025
Accepted 14 October 2025