

A New Hybrid of Khalouta Transform and Residual Power Series Method for Solving General Nonlinear Time-fractional Partial Differential Equations With Inhomogeneous Terms

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Abstract. The main purpose of this work is to propose a new hybrid method to obtain analytical approximate solutions for general nonlinear time-fractional partial differential equations with inhomogeneous terms. This method is called Khalouta residual power series method (KHRPSM) which is based on a combination of Khalouta transform method and residual power series method. The main advantage of KHRPSM is that it does not require making any material assumptions about the problem and requires minimum computations to solve these types of equations. The analytical approximate solutions of three types of time-fractional nonlinear partial differential equations are presented by the proposed method. The obtained results are compared with the exact solutions. Through this comparison, we conclude that the proposed method is very effective and easy to apply to different types of time-fractional nonlinear partial differential equations.

Key Words and Phrases: fractional partial differential equations, Caputo fractional derivative operator, Khalouta transform method, residual power series method, truncated series solutions.

2010 Mathematics Subject Classifications: 34A08, 26A33, 34K28, 35C10

1. Introduction

Over the past decade, nonlinear fractional partial differential equations (NF-PDEs) have been widely used to describe and help solve many phenomena in the natural sciences such as plasma physics, mechanical system, quantum mechanics, chaos, relaxation processes, mathematical biology, diffusion and damping laws [3, 5, 9, 10, 12, 13, 14]. It is very difficult to obtain exact solutions

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to real-world problems using NFPDEs and complex mathematical techniques are usually required. Therefore, approximate or numerical methods are used for such problems, such as Adomian decomposition method (ADM), homotopy perturbation method (HPM), homotopy analysis method (HAM), differential transform method (DTM), etc.

Recently, Ahmad El-Ajou et al. [2] developed a new iterative algorithm called residual power series method (RPSM) to find approximate analytical solutions to nonlinear fractional KdV–Burgers equation. RPSM is easy and efficient to find fractional series solutions for linear and nonlinear equations without perturbation, linearization and discretization through a sequence of algebraic equations. Furthermore, RPSM can be applied to calculate analytical solutions of NFPDEs. The series solution is obtained by using the concept of residual error, where this solution and their fractional derivatives are valid in the given domain for each multidimensional variable and each arbitrary point.

In 2023, the author [4] introduced the "Khalouta transform" and applied it to solve a wide range of higher-order linear ordinary differential equations, partial differential equations, system of ordinary and partial differential equations, and integral equations. The main advantage of the Khalouta transform is that it covers existing transforms such as Laplace, Aboodh, Elzaki, Sumudu, natural, Shehu, and ZZ transforms for different values of the transform variables.

The aim of this paper is to develop a new analytical approach to obtain analytical approximate solutions of general nonlinear time-fractional partial differential equations with inhomogeneous terms involving the Caputo fractional operator. We call this approach the Khalouta residual power series method (KHRPSM). This method provides analytical solutions in the form of rapid and accurate convergent series in terms of multiple fractional power series with easily computable components. The KHRPSM is based on a new version of the Taylor series that generates a convergent series as a solution.

The outline of this work is organized as follows. Firstly, we present some essential definitions, properties, and theorems related to the fractional calculus and the Khalouta transform in Section 2. Then, we establish a new formula of multiple fractional Taylor's series, which is the basis of the new method in Section 3. Next, we study the methodology of KHRPSM for solving general nonlinear time-fractional partial differential equations with inhomogeneous terms in Section 4. After that, we solve three well-known nonlinear time-fractional partial differential equations. with the help of KHRPSM in Section 5. In addition, we discuss our results which are presented in the form of figures and tables in Section 6. Finally, our results are summarized in the conclusion.

2. Basic definitions and main results

In this section, we present the essential definitions and theorems of fractional calculus. In addition, we give a brief overview of the preliminary definitions and necessary theorems concerning the Khalouta transform of fractional derivatives, which will be used in this work.

Definition 1. [9] *The Caputo time-fractional derivative of the function $\vartheta(x, t)$ of order α is defined by*

$$D_t^\alpha \vartheta(x, t) = I_t^{n-\alpha} \vartheta^{(n)}(x, t), \quad (1)$$

where $n - 1 < \alpha \leq n, n \in \mathbb{N}^*, x \in J = [a, b] \subset \mathbb{R}, t \in [0, +\infty[$ and I_t^α is the time-fractional Riemann-Liouville integral operator defined as

$$I_t^\alpha \vartheta(x, t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t - \varrho)^{\alpha-1} \vartheta(x, \varrho) d\varrho, \quad \text{if } \alpha > 0, \quad (2)$$

and

$$I_t^\alpha \vartheta(x, t) = \vartheta(x, t), \quad \text{if } \alpha = 0. \quad (3)$$

Definition 2. [9] *The Mittag-Leffler function for one parameter is described as follows*

$$E_\alpha(z) = \sum_{i=0}^{\infty} \frac{z^i}{\Gamma(i\alpha + 1)}, \quad \alpha, z \in \mathbb{C}, \operatorname{Re}(\alpha) \geq 0. \quad (4)$$

Definition 3. [4] *Let $\vartheta(x, t)$ be a piecewise continuous function defined on $J \times [0, +\infty[$ and of exponential order. Then the Khalouta transform of $\vartheta(x, t)$ is defined by*

$$\mathbb{KH}[\vartheta(x, t)] = \mathcal{K}(x, s, \gamma, \eta) = \frac{s}{\gamma\eta} \int_0^\infty \exp\left(-\frac{st}{\gamma\eta}\right) \vartheta(x, t) dt, \quad s, \gamma, \eta > 0, \quad (5)$$

and the inverse Khalouta transform is defined by

$$\vartheta(x, t) = \mathbb{KH}^{-1}[\mathcal{K}(x, s, \eta, \gamma)] = \frac{1}{2\pi i} \int_{\varphi-i\infty}^{\varphi+i\infty} \frac{1}{s} \exp\left(\frac{st}{\gamma\eta}\right) \mathcal{K}(x, s, \eta, \gamma) ds. \quad (6)$$

Theorem 1. [8] *Let $\vartheta(x, t)$ and $\rho(x, t)$ be piecewise continuous and of exponential order. Suppose that $\mathbb{KH}[\vartheta(x, t)] = \mathcal{K}(x, s, \gamma, \eta)$ and $\mathbb{KH}[\rho(x, t)] = \mathcal{H}(x, s, \gamma, \eta)$ are the Khalouta transforms of $\vartheta(x, t)$ and $\rho(x, t)$, respectively. Then we have*

$$1) \quad \mathbb{KH}[a\vartheta(x, t) + b\rho(x, t)] = a\mathcal{K}(x, s, \gamma, \eta) + b\mathcal{H}(x, s, \gamma, \eta), \quad (7)$$

where a, b are real numbers.

2)

$$\mathbb{KH} [I_t^\alpha \vartheta(x, t)] = \frac{\gamma^\alpha \eta^\alpha}{s^\alpha} \mathcal{K}(x, s, \gamma, \eta). \quad (8)$$

3)

$$\mathbb{KH} [D_t^\alpha \vartheta(x, t)] = \frac{s^\alpha}{\gamma^\alpha \eta^\alpha} \mathcal{K}(x, s, \gamma, \eta) - \sum_{i=0}^{n-1} \left(\frac{s}{\gamma \eta} \right)^{\alpha-i} \vartheta^{(i)}(x, 0), \quad n-1 < \alpha \leq n. \quad (9)$$

4)

$$\mathbb{KH} \left[\frac{t^\alpha}{\Gamma(\alpha+1)} \right] = \frac{\gamma^\alpha \eta^\alpha}{s^\alpha}, \quad \alpha > -1. \quad (10)$$

Theorem 2. Let $\vartheta(x, t)$ be a piecewise continuous function defined on $J \times [0, +\infty[$ and of exponential order. Suppose $\mathbb{KH} [\vartheta(x, t)] = \mathcal{K}(x, s, \gamma, \eta)$ is the Khalouta transform of $\vartheta(x, t)$. Then the Khalouta transform of the m^{th} time-fractional derivative in the Caputo sense of $\vartheta(x, t)$ of order $m\alpha$ with $\frac{m-1}{m} < \alpha \leq 1, m \in \mathbb{N}^*$, is given as

$$\mathbb{KH} [D_t^{m\alpha} \vartheta(x, t)] = \frac{s^{m\alpha}}{\gamma^{m\alpha} \eta^{m\alpha}} \mathcal{K}(x, s, \gamma, \eta) - \sum_{i=0}^{m-1} \left(\frac{s}{\gamma \eta} \right)^{(m-i)\alpha} D_t^{i\alpha} \vartheta(x, 0), \quad (11)$$

where $D_t^{m\alpha} = D_t^\alpha \cdot D_t^\alpha \dots D_t^\alpha$ (m -times).

Proof. To prove the result, we use the principle of mathematical induction method.

Taking $m = 1$ in equation (11), we obtain

$$\mathbb{KH} [D_t^\alpha \vartheta(x, t)] = \frac{s^\alpha}{\gamma^\alpha \eta^\alpha} \mathcal{K}(x, s, \gamma, \eta) - \left(\frac{s}{\gamma \eta} \right)^\alpha \vartheta(x, 0). \quad (12)$$

So, equation (11) is true for $m = 1$.

Taking $m = 2$ in equation (11), we obtain

$$\mathbb{KH} [D_t^{2\alpha} \vartheta(x, t)] = \frac{s^{2\alpha}}{\gamma^{2\alpha} \eta^{2\alpha}} \mathcal{K}(x, s, \gamma, \eta) - \left(\frac{s}{\gamma \eta} \right)^{2\alpha} \vartheta(x, 0) - \left(\frac{s}{\gamma \eta} \right)^\alpha D_t^\alpha \vartheta(x, 0). \quad (13)$$

Given the left-hand side (LHS) of equation (13), we get

$$LHS = \mathbb{KH} [D_t^{2\alpha} \vartheta(x, t)] = \mathbb{KH} [D_t^\alpha (D_t^\alpha \vartheta(x, t))]. \quad (14)$$

Let

$$\phi(x, t) = D_t^\alpha \vartheta(x, t). \quad (15)$$

Therefore, equation (14) becomes

$$LHS = \mathbb{KH} [D_t^{2\alpha}\vartheta(x, t)] = \mathbb{KH} [D_t^\alpha\phi(x, t)]. \quad (16)$$

Using the Caputo time-fractional derivative defined by (1), equation (16) becomes

$$LHS = \mathbb{KH} [D_t^{2\alpha}\vartheta(x, t)] = \mathbb{KH} [I_t^{1-\alpha}\phi^{(1)}(x, t)]. \quad (17)$$

Using the Khalouta transform of the Riemann-Liouville fractional integral defined by (8), we get

$$LHS = \mathbb{KH} [D_t^{2\alpha}\vartheta(x, t)] = \frac{\gamma^{1-\alpha}\eta^{1-\alpha}}{s^{1-\alpha}} \mathbb{KH} [\phi^{(1)}(x, t)]. \quad (18)$$

Using the differential property of the Khalouta transform [4], equation (18) becomes

$$\mathbb{KH} [D_t^{2\alpha}\vartheta(x, t)] = \frac{s^\alpha}{\gamma^\alpha\eta^\alpha}\phi(x, s, \gamma, \eta) - \frac{s^\alpha}{\gamma^\alpha\eta^\alpha}\phi(x, 0), \quad (19)$$

where $\mathbb{KH} [\phi(x, t)] = \phi(x, s, \gamma, \eta)$ and $\phi(x, 0) = D_t^\alpha\vartheta(x, 0)$.

According to equation (19), we have

$$\begin{aligned} \mathbb{KH} [D_t^{2\alpha}\vartheta(x, t)] &= \frac{s^\alpha}{\gamma^\alpha\eta^\alpha}\mathbb{KH} [D_t^\alpha\vartheta(x, t)] - \frac{s^\alpha}{\gamma^\alpha\eta^\alpha}D_t^\alpha\vartheta(x, 0) \\ &= \frac{s^{2\alpha}}{\gamma^{2\alpha}\eta^{2\alpha}}\mathcal{K}(x, s, \gamma, \eta) - \left(\frac{s}{\gamma\eta}\right)^{2\alpha}\vartheta(x, 0) - \left(\frac{s}{\gamma\eta}\right)^\alpha D_t^\alpha\vartheta(x, 0). \end{aligned} \quad (20)$$

Thus, from equation (20) we conclude that equation (11) is true for $m = 2$.

Now suppose that equation (11) is true for $m = l$. So we have

$$\mathbb{KH} [D_t^{l\alpha}\vartheta(x, t)] = \frac{s^{l\alpha}}{\gamma^{l\alpha}\eta^{l\alpha}}\mathcal{K}(x, s, \gamma, \eta) - \sum_{i=0}^{l-1} \left(\frac{s}{\gamma\eta}\right)^{(l-i)\alpha} D_t^{i\alpha}\vartheta(x, 0). \quad (21)$$

Now we prove that equation (11) is true for $m = l + 1$.

For this, from equation (11) we have

$$\mathbb{KH} [D_t^{(l+1)\alpha}\vartheta(x, t)] = \frac{s^{(l+1)\alpha}}{\gamma^{(l+1)\alpha}\eta^{(l+1)\alpha}}\mathcal{K}(x, s, \gamma, \eta) - \sum_{i=0}^l \left(\frac{s}{\gamma\eta}\right)^{(l+1-i)\alpha} D_t^{i\alpha}\vartheta(x, 0). \quad (22)$$

Through the LHS of equation (22), we have

$$LHS = \mathbb{KH} \left[D_t^{(l+1)\alpha} \vartheta(x, t) \right] = \mathbb{KH} \left[D_t^\alpha \left(D_t^{l\alpha} \vartheta(x, t) \right) \right]. \quad (23)$$

Let

$$\chi(x, t) = D_t^{l\alpha} \vartheta(x, t). \quad (24)$$

Therefore, equation (24) becomes

$$LHS = \mathbb{KH} \left[D_t^{(l+1)\alpha} \vartheta(x, t) \right] = \mathbb{KH} \left[D_t^\alpha \chi(x, t) \right]. \quad (25)$$

Using the Caputo time-fractional derivative defined by (1), equation (25) becomes

$$LHS = \mathbb{KH} \left[D_t^{(l+1)\alpha} \vartheta(x, t) \right] = \mathbb{KH} \left[I_t^{1-\alpha} \chi^{(1)}(x, t) \right]. \quad (26)$$

Using the Khalouta transform of the Riemann-Liouville fractional integral defined by (8), we get

$$LHS = \mathbb{KH} \left[D_t^{(l+1)\alpha} \vartheta(x, t) \right] = \frac{\gamma^{1-\alpha} \eta^{1-\alpha}}{s^{1-\alpha}} \mathbb{KH} \left[\chi^{(1)}(x, t) \right]. \quad (27)$$

Using the differential property of the Khalouta transform [4], equation (27) becomes

$$\mathbb{KH} \left[D_t^{(l+1)\alpha} \vartheta(x, t) \right] = \frac{s^\alpha}{\gamma^\alpha \eta^\alpha} \mathcal{X}(x, s, \gamma, \eta) - \frac{s^\alpha}{\gamma^\alpha \eta^\alpha} \chi(x, 0), \quad (28)$$

where $\mathbb{KH} [\chi(x, t)] = \mathcal{X}(x, s, \gamma, \eta)$ and $\chi(x, 0) = D_t^{l\alpha} \vartheta(x, 0)$.

According to equation (28), we have

$$\begin{aligned} \mathbb{KH} \left[D_t^{(l+1)\alpha} \vartheta(x, t) \right] &= \frac{s^\alpha}{\gamma^\alpha \eta^\alpha} \mathbb{KH} \left[D_t^{l\alpha} \vartheta(x, t) \right] - \frac{s^\alpha}{\gamma^\alpha \eta^\alpha} D_t^{l\alpha} \vartheta(x, 0) \\ &= \frac{s^{(l+1)\alpha}}{\gamma^{(l+1)\alpha} \eta^{(l+1)\alpha}} \mathcal{K}(x, s, \gamma, \eta) - \sum_{i=0}^{l-1} \left(\frac{s}{\gamma \eta} \right)^{(l+1-i)\alpha} D_t^{i\alpha} \vartheta(x, 0) - \frac{s^\alpha}{\gamma^\alpha \eta^\alpha} D_t^{l\alpha} \vartheta(x, 0) \\ &= \frac{s^{(l+1)\alpha}}{\gamma^{(l+1)\alpha} \eta^{(l+1)\alpha}} \mathcal{K}(x, s, \gamma, \eta) - \sum_{i=0}^l \left(\frac{s}{\gamma \eta} \right)^{(l+1-i)\alpha} D_t^{i\alpha} \vartheta(x, 0) \end{aligned} \quad (29)$$

Thus, from equation (29) we conclude that equation (11) is true for $m = l + 1$. So, equation (11) is true for all positive integers and the above proven formula

takes the following form:

$$\mathbb{KH} [D_t^{m\alpha} \vartheta(x, t)] = \frac{s^{m\alpha}}{\gamma^{m\alpha} \eta^{m\alpha}} \mathcal{K}(x, s, \gamma, \eta) - \sum_{i=0}^{m-1} \left(\frac{s}{\gamma \eta} \right)^{(m-i)\alpha} D_t^{i\alpha} \vartheta(x, 0), \quad (30)$$

where $D_t^{m\alpha} = D_t^\alpha . D_t^\alpha \dots D_t^\alpha$ (m -times).

This completes the proof. ◀

3. New formula of multiple fractional Taylor's series

In this section, we present a new formula of multiple fractional Taylor's series using the Khalouta transform.

Suppose that the multiple fractional power series representation of the function $\vartheta(x, t)$ at $t = 0$ has the form

$$\vartheta(x, t) = \sum_{m=0}^{\infty} c_m(x) t^{m\alpha}, \quad n-1 < \alpha \leq n, \quad x \in J \subset \mathbb{R}, \quad 0 \leq t \leq R, \quad (31)$$

and R is the radius of convergence of the multiple fractional power series.

Theorem 3. [1] *If $\vartheta \in C(J \times [0, R])$ and $D_t^{m\alpha} \vartheta \in C(J \times (0, R))$ for $m = 0, 1, 2, \dots$, then the coefficients $c_m(x)$ will take the form of*

$$c_m(x) = \frac{D_t^{m\alpha} \vartheta(x, 0)}{\Gamma(m\alpha + 1)}, \quad (32)$$

where $D_t^{m\alpha} = D_t^\alpha . D_t^\alpha \dots D_t^\alpha$ (m -times).

Lemma 1. *Let $\vartheta(x, t)$ be piecewise continuous and of exponential order. Then the Khalouta transform of $\vartheta(x, t)$ given by $\mathbb{KH}[\vartheta(x, t)] = \mathcal{K}(x, s, \gamma, \eta)$, has multiple fractional Taylor's series representation as*

$$\mathcal{K}(x, s, \gamma, \eta) = \sum_{m=0}^{\infty} \frac{\gamma^{m\alpha} \eta^{m\alpha}}{s^{m\alpha}} \Theta_m(x), \quad (33)$$

where $\Theta_m(x)$ represents m^{th} coefficient of the new formula of multiple fractional Taylor's series in Khalouta transform.

Proof. Consider the following fractional Taylor series:

$$\vartheta(x, t) = \Theta_0(x) + \Theta_1(x) \frac{t^\alpha}{\Gamma(\alpha + 1)} + \Theta_2(x) \frac{t^{2\alpha}}{\Gamma(2\alpha + 1)} + \Theta_3(x) \frac{t^{3\alpha}}{\Gamma(3\alpha + 1)} + \dots \quad (34)$$

Applying the Khalouta transform in both sides of equation (11) and using its linearity property, we get

$$\begin{aligned} \mathbb{K}\mathbb{H}[\vartheta(x, t)] &= \Theta_0(x) + \Theta_1(x)\mathbb{K}\mathbb{H}\left[\frac{t^\alpha}{\Gamma(\alpha+1)}\right] + \\ &\Theta_2(x)\mathbb{K}\mathbb{H}\left[\frac{t^{2\alpha}}{\Gamma(2\alpha+1)}\right] + \Theta_3(x)\mathbb{K}\mathbb{H}\left[\frac{t^{3\alpha}}{\Gamma(3\alpha+1)}\right] + \dots \end{aligned} \quad (35)$$

Using part (4) of Theorem 1, we get

$$\begin{aligned} \mathcal{K}(x, s, \gamma, \eta) &= \Theta_0(x) + \Theta_1(x)\frac{\gamma^\alpha\eta^\alpha}{s^\alpha} + \Theta_2(x)\frac{\gamma^{2\alpha}\eta^{2\alpha}}{s^{2\alpha}} + \Theta_3(x)\frac{\gamma^{3\alpha}\eta^{3\alpha}}{s^{3\alpha}} + \dots \\ &= \sum_{m=0}^{\infty} \frac{\gamma^{m\alpha}\eta^{m\alpha}}{s^{m\alpha}} \Theta_m(x), \end{aligned} \quad (36)$$

which is a new form of fractional Taylor's series in Khalouta transform form.

This completes the proof. ◀

Lemma 2. *Suppose the function $\mathbb{K}\mathbb{H}[\vartheta(x, t)] = \mathcal{K}(x, s, \gamma, \eta)$ has a multiple fractional power series representation in the new form of the Taylor's series (33). Then we have*

$$\lim_{s \rightarrow \infty} \mathcal{K}(x, s, \gamma, \eta) = \Theta_0(x) = \vartheta(x, 0). \quad (37)$$

Proof. Taking $\lim_{s \rightarrow \infty}$ of equation (36) and performing a simple calculation, we get

$$\lim_{s \rightarrow \infty} \mathcal{K}(x, s, \gamma, \eta) = \Theta_0(x) = \vartheta(x, 0). \quad (38)$$

This completes the proof. ◀

Theorem 4. *Suppose that the function $\mathbb{K}\mathbb{H}[\vartheta(x, t)] = \mathcal{K}(x, s, \gamma, \eta)$ has the following multiple fractional power series representation:*

$$\mathcal{K}(x, s, \gamma, \eta) = \sum_{m=0}^{\infty} \frac{\gamma^{m\alpha}\eta^{m\alpha}}{s^{m\alpha}} \Theta_m(x). \quad (39)$$

Then we have

$$\Theta_m(x) = D_t^{m\alpha} \vartheta(x, 0), \quad (40)$$

where $D_t^{m\alpha} = D_t^\alpha . D_t^\alpha \dots D_t^\alpha$ (m -times).

Proof. Suppose that $\mathcal{K}(x, s, \gamma, \eta)$ has multiple fractional power series representation as in equation (39). Then equation (39) becomes

$$\mathcal{K}(x, s, \gamma, \eta) = \Theta_0(x) + \frac{\gamma^\alpha \eta^\alpha}{s^\alpha} \Theta_1(x) + \frac{\gamma^{2\alpha} \eta^{2\alpha}}{s^{2\alpha}} \Theta_2(x) + \frac{\gamma^{3\alpha} \eta^{3\alpha}}{s^{3\alpha}} \Theta_3(x) + \dots \quad (41)$$

Multiplying equation (41) by $\frac{s^\alpha}{\gamma^\alpha \eta^\alpha}$, we get

$$\Theta_1(x) = \frac{s^\alpha}{\gamma^\alpha \eta^\alpha} \mathcal{K}(x, s, \gamma, \eta) - \frac{s^\alpha}{\gamma^\alpha \eta^\alpha} \Theta_0(x) - \frac{\gamma^\alpha \eta^\alpha}{s^\alpha} \Theta_2(x) - \frac{\gamma^{2\alpha} \eta^{2\alpha}}{s^{2\alpha}} \Theta_3(x) - \dots \quad (42)$$

Taking $\lim_{s \rightarrow \infty}$ of equation (42) and using Theorem 2, we get

$$\begin{aligned} \Theta_1(x) &= \lim_{s \rightarrow \infty} \left(\frac{s^\alpha}{\gamma^\alpha \eta^\alpha} \mathcal{K}(x, s, \gamma, \eta) - \frac{s^\alpha}{\gamma^\alpha \eta^\alpha} \Theta_0(x) \right) \\ &= \lim_{s \rightarrow \infty} (\mathbb{K}\mathbb{H} [D_t^\alpha \vartheta(x, t)](s, \gamma, \eta)). \end{aligned} \quad (43)$$

By Lemma 2, equation (43) becomes

$$\Theta_1(x) = D_t^\alpha \vartheta(x, 0). \quad (44)$$

Similarly, we multiply equation (41) by $\frac{s^{2\alpha}}{\gamma^{2\alpha} \eta^{2\alpha}}$ to get

$$\Theta_2(x) = \frac{s^{2\alpha}}{\gamma^{2\alpha} \eta^{2\alpha}} \mathcal{K}(x, s, \gamma, \eta) - \frac{s^{2\alpha}}{\gamma^{2\alpha} \eta^{2\alpha}} \Theta_0(x) - \frac{s^\alpha}{\gamma^\alpha \eta^\alpha} \Theta_1(x) - \frac{\gamma^\alpha \eta^\alpha}{s^\alpha} \Theta_3(x) + \dots \quad (45)$$

Taking $\lim_{s \rightarrow \infty}$ of equation (45) and using Theorem 2, we get

$$\begin{aligned} \Theta_2(x) &= \lim_{s \rightarrow \infty} \left(\frac{s^{2\alpha}}{\gamma^{2\alpha} \eta^{2\alpha}} \mathcal{K}(x, s, \gamma, \eta) - \frac{s^{2\alpha}}{\gamma^{2\alpha} \eta^{2\alpha}} \Theta_0(x) - \frac{s^\alpha}{\gamma^\alpha \eta^\alpha} \Theta_1(x) \right) \\ &= \lim_{s \rightarrow \infty} (\mathbb{K}\mathbb{H} [D_t^{2\alpha} \vartheta(x, t)](s, \gamma, \eta)). \end{aligned} \quad (46)$$

By Lemma 2, equation (46) becomes

$$\Theta_2(x) = D_t^{2\alpha} \vartheta(x, 0). \quad (47)$$

To complete the proof, we use the principle of mathematical induction method.

Suppose $\Theta_{m-1}(x) = D_t^{(m-1)\alpha} \vartheta(x, 0)$. Multiplying equation (41) by $\frac{s^{m\alpha}}{\gamma^{m\alpha} \eta^{m\alpha}}$ and using Theorem 2 and Lemma 2, we get

$$\Theta_m(x) =$$

$$\begin{aligned}
& \lim_{s \rightarrow \infty} \left(\frac{s^{m\alpha}}{\gamma^{m\alpha} \eta^{m\alpha}} \mathcal{K}(x, s, \gamma, \eta) - \frac{s^{m\alpha}}{\gamma^{m\alpha} \eta^{m\alpha}} \Theta_0(x) - \frac{s^{(m-1)\alpha}}{\gamma^{(m-1)\alpha} \eta^{(m-1)\alpha}} D_t^\alpha \vartheta(x, 0) \right. \\
& \quad \left. - \frac{s^{(m-2)\alpha}}{\gamma^{(m-2)\alpha} \eta^{(m-2)\alpha}} D_t^{2\alpha} \vartheta(x, 0) - \dots - \frac{s^\alpha}{\gamma^\alpha \eta^\alpha} D_t^{(m-1)\alpha} \vartheta(x, 0) \right) \\
& = \lim_{s \rightarrow \infty} (\mathbb{K}\mathbb{H} [D_t^{m\alpha} \vartheta(x, t)](s, \gamma, \eta)) = D_t^{m\alpha} \vartheta(x, 0). \tag{48}
\end{aligned}$$

This completes the proof. ◀

Remark 1. *The inverse Khalouta transform of the series extension in Theorem 4 has the form of the following multiple fractional power series:*

$$\vartheta(x, t) = \sum_{m=0}^{\infty} \frac{D_t^{m\alpha} \vartheta(x, 0)}{\Gamma(m\alpha + 1)} t^{m\alpha}, \quad 0 < \alpha \leq 1, t \geq 0. \tag{49}$$

In the following theorem, we explain and determine the convergence conditions of the new form of multiple fractional Taylor's formula.

Theorem 5. *Let $\vartheta(x, t)$ be a piecewise continuous function defined on $J \times [0, +\infty[$ and of exponential order, and let $\mathbb{K}\mathbb{H} [\vartheta(x, t)] = \mathcal{K}(x, s, \gamma, \eta)$ be represented as the new form of multiple fractional Taylor's formula explained in Theorem 4. If $|\mathbb{K}\mathbb{H} [D_t^{(k+1)\alpha} \vartheta(x, t)]| \leq T(x)$ on $J \times (0, d]$ with $0 < \alpha \leq 1$, then the remainder $R_k(x, s, \gamma, \eta)$ of the new form of multiple fractional Taylor's formula satisfies the following inequality:*

$$|R_k(x, s, \gamma, \eta)| \leq \frac{\gamma^{(k+1)\alpha} \eta^{(k+1)\alpha}}{s^{(k+1)\alpha}} T(x). \tag{50}$$

Proof. Consider the following:

$$\begin{aligned}
\mathcal{K}_k(x, s, \gamma, \eta) &= \Theta_0(x) + \frac{\gamma^\alpha \eta^\alpha}{s^\alpha} \Theta_1(x) \\
&+ \frac{\gamma^{2\alpha} \eta^{2\alpha}}{s^{2\alpha}} \Theta_2(x) + \frac{\gamma^{3\alpha} \eta^{3\alpha}}{s^{3\alpha}} \Theta_3(x) + \dots + \frac{\gamma^{k\alpha} \eta^{k\alpha}}{s^{k\alpha}} \Theta_k(x). \tag{51}
\end{aligned}$$

From equations (39) and (51), we get

$$R_k(x, s, \gamma, \eta) = \mathcal{K}(x, s, \gamma, \eta) - \sum_{m=0}^k \frac{\gamma^{m\alpha} \eta^{m\alpha}}{s^{m\alpha}} \Theta_m(x). \tag{52}$$

Using Theorem 4, we get

$$R_k(x, s, \gamma, \eta) = \mathcal{K}(x, s, \gamma, \eta) - \sum_{m=0}^k \frac{\gamma^{m\alpha} \eta^{m\alpha}}{s^{m\alpha}} D_t^{m\alpha} \vartheta(x, 0). \quad (53)$$

Multiplying equation (52) by $\frac{s^{(k+1)\alpha}}{\gamma^{(k+1)\alpha} \eta^{(k+1)\alpha}}$, we get

$$\begin{aligned} & \frac{s^{(k+1)\alpha}}{\gamma^{(k+1)\alpha} \eta^{(k+1)\alpha}} R_k(x, s, \gamma, \eta) = \\ & \frac{s^{(k+1)\alpha}}{\gamma^{(k+1)\alpha} \eta^{(k+1)\alpha}} \mathcal{K}(x, s, \gamma, \eta) - \sum_{m=0}^k \frac{s^{(k-m+1)\alpha}}{\gamma^{(k-m+1)\alpha} \eta^{(k-m+1)\alpha}} D_t^{m\alpha} \vartheta(x, 0). \end{aligned} \quad (54)$$

By Theorem 2, we have

$$\frac{s^{(k+1)\alpha}}{\gamma^{(k+1)\alpha} \eta^{(k+1)\alpha}} R_k(x, s, \gamma, \eta) = \mathbb{KH} \left[D_t^{(k+1)\alpha} \vartheta(x, t) \right]. \quad (55)$$

Using the absolute sign in equation (55), we get

$$\left| \frac{s^{(k+1)\alpha}}{\gamma^{(k+1)\alpha} \eta^{(k+1)\alpha}} R_k(x, s, \gamma, \eta) \right| = \left| \mathbb{KH} \left[D_t^{(k+1)\alpha} \vartheta(x, t) \right] \right|. \quad (56)$$

Under the given assumption, equation (56) becomes

$$-T(x) \leq \frac{s^{(k+1)\alpha}}{\gamma^{(k+1)\alpha} \eta^{(k+1)\alpha}} R_k(x, s, \gamma, \eta) \leq T(x). \quad (57)$$

From equation (57), we have

$$|R_k(x, s, \gamma, \eta)| \leq \frac{\gamma^{(k+1)\alpha} \eta^{(k+1)\alpha}}{s^{(k+1)\alpha}} T(x). \quad (58)$$

This completes the proof. ◀

4. The methodology of KHRPSM

In this section, we present the fundamental idea of KHRPSM for solving general nonlinear fractional partial differential equations with inhomogeneous terms.

Theorem 6. *Consider the following general time-fractional nonlinear partial dif-*

ferential equation:

$$D_t^{m\alpha}\vartheta(x, t) = L(\vartheta(x, t)) + N(\vartheta(x, t)) + f(x, t), \quad (59)$$

with the initial conditions

$$D_t^{i\alpha}\vartheta(x, 0) = \vartheta_i(x), i = 0, 1, 2, \dots, m - 1, \quad (60)$$

where the symbol $D_t^{m\alpha}$ represents the m^{th} time-fractional derivative operator in the Caputo sense of order $m\alpha$ with $\frac{m-1}{m} < \alpha \leq 1, m \in \mathbb{N}^*$, L is a linear operator, N is a nonlinear operator, and $f(x, t)$ is an inhomogeneous term.

According to the KHRPSM, the solution of (59), (60) is described as an infinite series expansion which rapidly converges to the exact solution as follows:

$$\vartheta(x, t) = \lim_{k \rightarrow \infty} \vartheta_k(x, t), \quad (61)$$

where $\vartheta_k(x, t)$ is the k^{th} -approximate solution given by

$$\vartheta_k(x, t) = \sum_{i=0}^k \vartheta_i(x) \frac{t^{i\alpha}}{\Gamma(i\alpha + 1)}. \quad (62)$$

Proof.

To prove this theorem, consider the general nonlinear time-fractional partial differential equation with inhomogeneous term (59) under the initial conditions (60).

Operating the Khalouta transform on both sides of equation (59) and using part (1) of Theorem 1, we get

$$\mathbb{KH} [D_t^{m\alpha}\vartheta(x, t)] = \mathbb{KH} [L\vartheta(x, t)] + \mathbb{KH} [N\vartheta(x, t)] + \mathbb{KH} [f(x, t)]. \quad (63)$$

According to Theorem 2 and the initial conditions (60), equation (63) can be written as

$$\begin{aligned} \mathcal{K}(x, s, \gamma, \eta) &= \sum_{i=0}^{m-1} \left(\frac{\gamma\eta}{s}\right)^{i\alpha} D_t^{i\alpha}\vartheta(x, 0) + \frac{\gamma^{m\alpha}\eta^{m\alpha}}{s^{m\alpha}} L(\mathcal{K}(x, s, \gamma, \eta)) + \\ &+ \frac{\gamma^{m\alpha}\eta^{m\alpha}}{s^{m\alpha}} \mathbb{KH} [N(\mathbb{KH}^{-1}[\mathcal{K}(x, s, \gamma, \eta)])] + \frac{\gamma^{m\alpha}\eta^{m\alpha}}{s^{m\alpha}} \mathcal{F}(x, s, \gamma, \eta), \end{aligned} \quad (64)$$

where $\mathbb{KH}^{-1}\mathcal{K}(x, s, \gamma, \eta) = \vartheta(x, t)$ and $\mathbb{KH} [f(x, t)] = \mathcal{F}(x, s, \gamma, \eta)$.

Based on Theorem 4, we assume that the solution of equation (64) has the following form:

$$\mathcal{K}(x, s, \gamma, \eta) = \sum_{m=0}^{\infty} \frac{\gamma^{m\alpha} \eta^{m\alpha}}{s^{m\alpha}} \Theta_m(x). \quad (65)$$

Taking into account that $\lim_{s \rightarrow \infty} \mathcal{K}(x, s, \gamma, \eta) = \Theta_0(x) = \vartheta(x, 0)$ and using Theorem 4, we have

$$\begin{aligned} \Theta_1(x) &= D_t^\alpha \vartheta(x, 0), \\ \Theta_2(x) &= D_t^{2\alpha} \vartheta(x, 0), \\ &\vdots \\ \Theta_i(x) &= D_t^{i\alpha} \vartheta(x, 0), \end{aligned} \quad (66)$$

and the k^{th} -truncated series solutions of equation (64) will be written as follows

$$\begin{aligned} \mathcal{K}_k(x, s, \gamma, \eta) &= \sum_{m=0}^k \frac{\gamma^{m\alpha} \eta^{m\alpha}}{s^{m\alpha}} \Theta_m(x) = \Theta_0(x) + \frac{\gamma^\alpha \eta^\alpha}{s^\alpha} \Theta_1(x) + \frac{\gamma^{2\alpha} \eta^{2\alpha}}{s^{2\alpha}} \Theta_2(x) + \dots \\ &+ \frac{\gamma^{i\alpha} \eta^{i\alpha}}{s^{i\alpha}} \Theta_i(x) + \sum_{m=i+1}^k \frac{\gamma^{m\alpha} \eta^{m\alpha}}{s^{m\alpha}} \Theta_m(x). \end{aligned} \quad (67)$$

Now, we consider separately the Khalouta residual function of equation (64) and the k^{th} -truncated Khalouta residual function, so that

$$\begin{aligned} \mathbb{K}\mathbb{H}\mathcal{R}es(x, s, \gamma, \eta) &= \mathcal{K}(x, s, \gamma, \eta) - \sum_{i=0}^{m-1} \left(\frac{\gamma \eta}{s} \right)^{i\alpha} D_t^{i\alpha} \vartheta(x, 0) - \frac{\gamma^{m\alpha} \eta^{m\alpha}}{s^{m\alpha}} L(\mathcal{K}(x, s, \gamma, \eta)) \\ &- \frac{\gamma^{m\alpha} \eta^{m\alpha}}{s^{m\alpha}} \mathbb{K}\mathbb{H} [N(\mathbb{K}\mathbb{H}^{-1}[\mathcal{K}(x, s, \gamma, \eta)])] - \frac{\gamma^{m\alpha} \eta^{m\alpha}}{s^{m\alpha}} \mathcal{F}(x, s, \gamma, \eta), \end{aligned} \quad (68)$$

and

$$\begin{aligned} \mathbb{K}\mathbb{H}\mathcal{R}es_k(x, s, \gamma, \eta) &= \mathcal{K}_k(x, s, \gamma, \eta) - \\ &\sum_{i=0}^{m-1} \left(\frac{\gamma \eta}{s} \right)^{i\alpha} D_t^{i\alpha} \vartheta(x, 0) - \frac{\gamma^{m\alpha} \eta^{m\alpha}}{s^{m\alpha}} L(\mathcal{K}_k(x, s, \gamma, \eta)) \\ &- \frac{\gamma^{m\alpha} \eta^{m\alpha}}{s^{m\alpha}} \mathbb{K}\mathbb{H} [N(\mathbb{K}\mathbb{H}^{-1}[\mathcal{K}_k(x, s, \gamma, \eta)])] - \frac{\gamma^{m\alpha} \eta^{m\alpha}}{s^{m\alpha}} \mathcal{F}(x, s, \gamma, \eta). \end{aligned} \quad (69)$$

Substituting the series form of $\mathcal{K}_k(x, s, \gamma, \eta)$ in equation (69) and multiplying both sides of equation (69) by $\frac{s^{k\alpha}}{\gamma^{k\alpha}\eta^{k\alpha}}$, we get

$$\begin{aligned} & \frac{s^{k\alpha}}{\gamma^{k\alpha}\eta^{k\alpha}} \mathbb{K}\mathbb{H}\mathcal{R}es_k(x, s, \gamma, \eta) \\ = & \frac{s^{k\alpha}}{\gamma^{k\alpha}\eta^{k\alpha}} \left(\mathcal{K}_k(x, s, \gamma, \eta) - \sum_{i=0}^{m-1} \left(\frac{\gamma\eta}{s}\right)^{i\alpha} D_t^{i\alpha} \vartheta(x, 0) - \frac{\gamma^{m\alpha}\eta^{m\alpha}}{s^{m\alpha}} L(\mathcal{K}_k(x, s, \gamma, \eta)) \right. \\ & \left. - \frac{\gamma^{m\alpha}\eta^{m\alpha}}{s^{m\alpha}} \mathbb{K}\mathbb{H} [N(\mathbb{K}\mathbb{H}^{-1}[\mathcal{K}_k(x, s, \gamma, \eta)])] - \frac{\gamma^{m\alpha}\eta^{m\alpha}}{s^{m\alpha}} \mathcal{F}(x, s, \gamma, \eta) \right). \end{aligned} \quad (70)$$

Taking $\lim_{s \rightarrow \infty}$ on both sides of equation (70), we obtain

$$\begin{aligned} & \lim_{s \rightarrow \infty} \frac{s^{k\alpha}}{\gamma^{k\alpha}\eta^{k\alpha}} \mathbb{K}\mathbb{H}\mathcal{R}es_k(x, s, \gamma, \eta) \\ = & \lim_{s \rightarrow \infty} \frac{s^{k\alpha}}{\gamma^{k\alpha}\eta^{k\alpha}} \left(\mathcal{K}_k(x, s, \gamma, \eta) - \sum_{i=0}^{m-1} \left(\frac{\gamma\eta}{s}\right)^{i\alpha} D_t^{i\alpha} \vartheta(x, 0) - \frac{\gamma^{m\alpha}\eta^{m\alpha}}{s^{m\alpha}} L(\mathcal{K}_k(x, s, \gamma, \eta)) \right. \\ & \left. - \frac{\gamma^{m\alpha}\eta^{m\alpha}}{s^{m\alpha}} \mathbb{K}\mathbb{H} [N(\mathbb{K}\mathbb{H}^{-1}[\mathcal{K}_k(x, s, \gamma, \eta)])] - \frac{\gamma^{m\alpha}\eta^{m\alpha}}{s^{m\alpha}} \mathcal{F}(x, s, \gamma, \eta) \right). \end{aligned} \quad (71)$$

Solving the following equation for $\Theta_k(x)$

$$\lim_{s \rightarrow \infty} \frac{s^{k\alpha}}{\gamma^{k\alpha}\eta^{k\alpha}} \mathbb{K}\mathbb{H}\mathcal{R}es_k(x, s, \gamma, \eta) = 0, \quad k = i + 1, i + 2, i + 3, \dots \quad (72)$$

and substituting the obtained values of $\Theta_k(x)$ into k^{th} -truncated series of $\mathcal{K}(x, s, \gamma, \eta)$, we get the k^{th} -approximate solution of equation (64).

Using the inverse Khalouta transform on $\mathcal{K}_k(x, s, \gamma, \eta)$, we obtain the k^{th} -approximate solution of equation (59) as follows:

$$\vartheta_k(x, t) = \sum_{i=0}^k \vartheta_i(x) \frac{t^{i\alpha}}{\Gamma(i\alpha + 1)}, \quad (73)$$

where $\vartheta_i(x) = D_t^{i\alpha} \vartheta(x, 0)$.

Finally, the solution of (59), (60) is given by

$$\begin{aligned}
\vartheta(x, t) &= \lim_{k \rightarrow \infty} \vartheta_k(x, t) \\
&= \lim_{k \rightarrow \infty} \sum_{i=0}^k \vartheta_i(x) \frac{t^{i\alpha}}{\Gamma(i\alpha + 1)} \\
&= \sum_{i=0}^{\infty} \vartheta_i(x) \frac{t^{i\alpha}}{\Gamma(i\alpha + 1)}. \tag{74}
\end{aligned}$$

This completes the proof. ◀

Now, we analyse the convergence of the KHRPSM when it is applied to solve general nonlinear time-fractional partial differential equations with inhomogeneous terms.

Theorem 7. (*Convergence theorem*) For $k \in \mathbb{N}$, let $\vartheta_k(x, t)$ and $\vartheta(x, t)$ be respectively the approximate and exact solutions of equations (59) and (60). If there exists a positive constant $\tau \in [0, 1]$ such that $\|\vartheta_{k+1}(x, t)\| \leq \tau \|\vartheta_k(x, t)\|$ for all $(x, t) \in J \times [0, +\infty[$ as well as $\|\vartheta_0(x)\| < \infty$ for all $x \in J$, then $\vartheta_k(x, t)$ converges to $\vartheta(x, t)$ as soon as $k \rightarrow \infty$.

Proof. For all $(x, t) \in J \times [0, +\infty[$, we have $\|\vartheta_{k+1}(x, t)\| \leq \tau \|\vartheta_k(x, t)\|$, which gives $\|\vartheta_1(x, t)\| \leq \tau \|\vartheta_0(x, t)\| = \tau \|\vartheta_0(x)\|$. So, $\|\vartheta_2(x, t)\| \leq \tau^2 \|\vartheta_1(x, t)\|$ and therefore $\|\vartheta_k(x, t)\| \leq \tau^k \|\vartheta_0(x)\|$. Consequently, we get $\sum_{m=k+1}^{\infty} \|\vartheta_m(x, t)\| \leq \sum_{m=k+1}^{\infty} \tau^m \|\vartheta_0(x)\| = \|\vartheta_0(x)\| \sum_{m=k+1}^{\infty} \tau^m$.

Therefore, we conclude that

$$\begin{aligned}
\|\vartheta(x, t) - \vartheta_k(x, t)\| &= \left\| \sum_{m=k+1}^{\infty} \|\vartheta_m(x, t)\| \right\| \\
&\leq \sum_{m=k+1}^{\infty} \|\vartheta_m(x, t)\| \\
&\leq \|\vartheta_0(x)\| \sum_{m=k+1}^{\infty} \tau^m \\
&\leq \frac{\tau^{k+1}}{1 - \tau} \|\vartheta_0(x)\| \rightarrow 0,
\end{aligned}$$

as soon as $k \rightarrow \infty$.

This completes the proof. ◀

5. Numerical tests

In this section, we show the efficiency and applicability of KHRPSM which is achieved by testing three well-known nonlinear time-fractional partial differential equations.

Example 1. *Let's consider the nonlinear time-fractional reaction-diffusion-convection equation [6]*

$$D_t^\alpha \vartheta = (\vartheta \vartheta_x)_x + 3\vartheta \vartheta_x + 2(\vartheta - \vartheta^2), \quad (75)$$

with the initial condition

$$\vartheta(x, 0) = 2\sqrt{\exp(x) - \exp(-4x)}, \quad (76)$$

where D_t^α represents the Caputo time-fractional derivative operator of order α with $0 < \alpha \leq 1$ and $\vartheta = \{\vartheta(x, t), (x, t) \in \mathbb{R}^+ \times \mathbb{R}^+\}$.

By applying the Khalouta transform operator on equation (75) and using part (1) of Theorem 1, we get

$$\mathbb{KH} [D_t^\alpha \vartheta] = \mathbb{KH} [(\vartheta \vartheta_x)_x] + 3\mathbb{KH} [\vartheta \vartheta_x] + 2(\mathbb{KH} [\vartheta] - \mathbb{KH} [\vartheta^2]). \quad (77)$$

By using Theorem 2 for $m = 1$ and the initial condition (76), we rewrite the equation (77) as follows:

$$\begin{aligned} \mathcal{K}(x, s, \gamma, \eta) &= 2\sqrt{\exp(x) - \exp(-4x)} + \\ &+ \frac{\gamma^\alpha \eta^\alpha}{s^\alpha} \mathbb{KH} [D_x (\mathbb{KH}^{-1} [\mathcal{K}(x, s, \gamma, \eta)] D_x \mathbb{KH}^{-1} [\mathcal{K}(x, s, \gamma, \eta)])] \\ &+ 3 \frac{\gamma^\alpha \eta^\alpha}{s^\alpha} \mathbb{KH} [\mathbb{KH}^{-1} [\mathcal{K}(x, s, \gamma, \eta)] D_x \mathbb{KH}^{-1} [\mathcal{K}(x, s, \gamma, \eta)]] \\ &+ 2 \frac{\gamma^\alpha \eta^\alpha}{s^\alpha} \mathcal{K}(x, s, \gamma, \eta) - 2 \frac{\gamma^\alpha \eta^\alpha}{s^\alpha} \mathbb{KH} \left[[\mathbb{KH}^{-1} [\mathcal{K}(x, s, \gamma, \eta)]]^2 \right]. \end{aligned} \quad (78)$$

According to the construction of KHRPSM presented in Section 4, the series solution of equation (78) can be expressed as follows:

$$\mathcal{K}(x, s, \gamma, \eta) = \sum_{m=0}^{\infty} \frac{\gamma^{m\alpha} \eta^{m\alpha}}{s^{m\alpha}} \Theta_m(x), \quad (79)$$

and the k^{th} -truncated series of equation (78) is given by

$$\mathcal{K}_k(x, s, \gamma, \eta) = \sum_{m=0}^k \frac{\gamma^{m\alpha} \eta^{m\alpha}}{s^{m\alpha}} \Theta_m(x). \quad (80)$$

By using Lemma 2, we have

$$\lim_{s \rightarrow \infty} \mathcal{K}(x, s, \gamma, \eta) = \vartheta(x, 0) = 2\sqrt{\exp(x) - \exp(-4x)}. \quad (81)$$

Therefore, the k^{th} -truncated series (80) becomes

$$\mathcal{K}_k(x, s, \gamma, \eta) = 2\sqrt{\exp(x) - \exp(-4x)} + \sum_{m=1}^k \frac{\gamma^{m\alpha} \eta^{m\alpha}}{s^{m\alpha}} \Theta_m(x). \quad (82)$$

From Theorem 5, we establish the Khalouta residual function of equation (78) as follows:

$$\begin{aligned} \mathbb{KH}Res(x, s, \gamma, \eta) &= \mathcal{K}(x, s, \gamma, \eta) - 2\sqrt{\exp(x) - \exp(-4x)} \\ &- \frac{\gamma^\alpha \eta^\alpha}{s^\alpha} \mathbb{KH} [D_x (\mathbb{KH}^{-1} [\mathcal{K}(x, s, \gamma, \eta)] D_x \mathbb{KH}^{-1} [\mathcal{K}(x, s, \gamma, \eta)])] \\ &- 3 \frac{\gamma^\alpha \eta^\alpha}{s^\alpha} \mathbb{KH} [\mathbb{KH}^{-1} [\mathcal{K}(x, s, \gamma, \eta)] D_x \mathbb{KH}^{-1} [\mathcal{K}(x, s, \gamma, \eta)]] \\ &- 2 \frac{\gamma^\alpha \eta^\alpha}{s^\alpha} \mathcal{K}(x, s, \gamma, \eta) + 2 \frac{\gamma^\alpha \eta^\alpha}{s^\alpha} \mathbb{KH} \left[[\mathbb{KH}^{-1} [\mathcal{K}(x, s, \gamma, \eta)]]^2 \right], \end{aligned} \quad (83)$$

and the k^{th} -truncated Khalouta residual function of equation (78) becomes

$$\begin{aligned} \mathbb{KH}Res_k(x, s, \gamma, \eta) &= \mathcal{K}_k(x, s, \gamma, \eta) - 2\sqrt{\exp(x) - \exp(-4x)} \\ &- \frac{\gamma^\alpha \eta^\alpha}{s^\alpha} \mathbb{KH} [D_x (\mathbb{KH}^{-1} [\mathcal{K}_k(x, s, \gamma, \eta)] D_x \mathbb{KH}^{-1} [\mathcal{K}_k(x, s, \gamma, \eta)])] \\ &- 3 \frac{\gamma^\alpha \eta^\alpha}{s^\alpha} \mathbb{KH} [\mathbb{KH}^{-1} [\mathcal{K}_k(x, s, \gamma, \eta)] D_x \mathbb{KH}^{-1} [\mathcal{K}_k(x, s, \gamma, \eta)]] \\ &- 2 \frac{\gamma^\alpha \eta^\alpha}{s^\alpha} \mathcal{K}_k(x, s, \gamma, \eta) + 2 \frac{\gamma^\alpha \eta^\alpha}{s^\alpha} \mathbb{KH} \left[[\mathbb{KH}^{-1} [\mathcal{K}_k(x, s, \gamma, \eta)]]^2 \right]. \end{aligned} \quad (84)$$

To find the unknown coefficients, using equations (82) and (84) and by solving

$$\lim_{s \rightarrow \infty} \frac{s^{k\alpha}}{\gamma^{k\alpha} \eta^{k\alpha}} \mathbb{KH}Res_k(x, s, \gamma, \eta) = 0, k = 1, 2, 3, \dots \quad (85)$$

we get

$$\begin{aligned}
\Theta_1(x) &= 4\sqrt{\exp(x) - \exp(-4x)}, \\
\Theta_2(x) &= 8\sqrt{\exp(x) - \exp(-4x)}, \\
\Theta_3(x) &= 16\sqrt{\exp(x) - \exp(-4x)}, \\
\Theta_4(x) &= 32\sqrt{\exp(x) - \exp(-4x)}, \\
&\vdots \\
\Theta_k(x) &= 2^{k+1}\sqrt{\exp(x) - \exp(-4x)}.
\end{aligned} \tag{86}$$

So, the k^{th} -the approximate solution of equation (78) is formulated as follows:

$$\begin{aligned}
\mathcal{K}_k(x, s, \gamma, \eta) &= 2\sqrt{\exp(x) - \exp(-4x)} + 4\frac{\gamma^\alpha \eta^\alpha}{s^\alpha} \sqrt{\exp(x) - \exp(-4x)} \\
&+ 8\frac{\gamma^{2\alpha} \eta^{2\alpha}}{s^{2\alpha}} \sqrt{\exp(x) - \exp(-4x)} + \dots + 2^{k+1} \frac{\gamma^{k\alpha} \eta^{k\alpha}}{s^{k\alpha}} \sqrt{\exp(x) - \exp(-4x)}. \tag{87}
\end{aligned}$$

Finally, by applying the inverse Khalouta transform of the obtained equation (87), we derive that the k^{th} -approximate solution for the nonlinear time-fractional reaction-diffusion-convection equation (75), can be written as follows:

$$\begin{aligned}
\vartheta_k(x, t) &= 2\sqrt{\exp(x) - \exp(-4x)} + 2\sqrt{\exp(x) - \exp(-4x)} \frac{2t^\alpha}{\Gamma(\alpha + 1)} \\
&+ 2\sqrt{\exp(x) - \exp(-4x)} \frac{2^2 t^{2\alpha}}{\Gamma(2\alpha + 1)} + \dots + 2\sqrt{\exp(x) - \exp(-4x)} \frac{2^k t^{k\alpha}}{\Gamma(k\alpha + 1)} \\
&= 2\sqrt{\exp(x) - \exp(-4x)} \sum_{i=0}^k \frac{(2t^\alpha)^i}{\Gamma(i\alpha + 1)}. \tag{88}
\end{aligned}$$

By taking $\lim_{k \rightarrow \infty}$ on both sides of equation (88), we obtain

$$\begin{aligned}
\vartheta(x, t) &= 2\sqrt{\exp(x) - \exp(-4x)} \sum_{i=0}^{\infty} \frac{(2t^\alpha)^i}{\Gamma(i\alpha + 1)} \\
&= 2\sqrt{\exp(x) - \exp(-4x)} \sum_{i=0}^{\infty} E_\alpha(2t^\alpha), \tag{89}
\end{aligned}$$

where $E_\alpha(2t^\alpha)$ is the Mittag-Leffler function defined in equation (4).

If we put $\alpha = 1$ in equation (89), then we get

$$\vartheta(x, t) = 2e^{2t} \sqrt{\exp(x) - \exp(-4x)}, \quad (90)$$

which represents the exact solution of the classical nonlinear reaction-diffusion-convection equation, and is in complete agreement with the results in [6].

Example 2. Let's consider the nonlinear time-fractional gas dynamics equation with inhomogeneous term [11]

$$D_t^\alpha \vartheta + \vartheta \vartheta_x + (1+t)^2 \vartheta^2 = x^2, \quad (91)$$

with the initial condition

$$\vartheta(x, 0) = x, \quad (92)$$

where D_t^α represents the Caputo time-fractional derivative operator of order α with $0 < \alpha \leq 1$ and $\vartheta = \{\vartheta(x, t), (x, t) \in \mathbb{R} \times \mathbb{R}^+\}$.

By applying the Khalouta transform operator on equation (91) and using part (1) of Theorem 1, we get

$$\mathbb{KH} [D_t^\alpha \vartheta] + \mathbb{KH} [\vartheta \vartheta_x] + \mathbb{KH} [(1+t)^2 \vartheta^2] = \mathbb{KH} [x^2]. \quad (93)$$

By using Theorem 2 for $m = 1$ and the initial condition (92), we rewrite the equation (93) as follows:

$$\begin{aligned} \mathcal{K}(x, s, \gamma, \eta) &= x - \frac{\gamma^\alpha \eta^\alpha}{s^\alpha} \mathbb{KH} [\mathbb{KH}^{-1} [\mathcal{K}(x, s, \gamma, \eta)] D_x \mathbb{KH}^{-1} [\mathcal{K}(x, s, \gamma, \eta)]] \\ &- \frac{\gamma^\alpha \eta^\alpha}{s^\alpha} \mathbb{KH} [\mathbb{KH}^{-1} [\mathbb{KH} [(1+t)^2]] [\mathbb{KH}^{-1} [\mathcal{K}(x, s, \gamma, \eta)]]^2] + \frac{\gamma^\alpha \eta^\alpha}{s^\alpha} \mathbb{KH} [x^2]. \end{aligned} \quad (94)$$

According to the construction of KHRPSM presented in Section 4, the series solution of equation (94) can be expressed as follows:

$$\mathcal{K}(x, s, \gamma, \eta) = \sum_{m=0}^{\infty} \frac{\gamma^{m\alpha} \eta^{m\alpha}}{s^{m\alpha}} \Theta_m(x), \quad (95)$$

and the k^{th} -truncated series of equation (94) is given by

$$\mathcal{K}_k(x, s, \gamma, \eta) = \sum_{m=0}^k \frac{\gamma^{m\alpha} \eta^{m\alpha}}{s^{m\alpha}} \Theta_m(x). \quad (96)$$

By using Lemma 2, we have

$$\lim_{s \rightarrow \infty} \mathcal{K}(x, s, \gamma, \eta) = \vartheta(x, 0) = x. \quad (97)$$

Therefore, the k^{th} -truncated series (96) becomes

$$\mathcal{K}_k(x, s, \gamma, \eta) = x + \sum_{m=1}^k \frac{\gamma^{m\alpha} \eta^{m\alpha}}{s^{m\alpha}} \Theta_m(x). \quad (98)$$

From Theorem 5, we establish the Khalouta residual function of equation (94) as follows:

$$\begin{aligned} \mathbb{KH} [\mathcal{R}es(x, s, \gamma, \eta)] &= \mathcal{K}(x, s, \gamma, \eta) - x + \\ &+ \frac{\gamma^\alpha \eta^\alpha}{s^\alpha} \mathbb{KH} [\mathbb{KH}^{-1} [\mathcal{K}(x, s, \gamma, \eta)] D_x \mathbb{KH}^{-1} [\mathcal{K}(x, s, \gamma, \eta)]] \\ &+ \frac{\gamma^\alpha \eta^\alpha}{s^\alpha} \mathbb{KH} \left[\mathbb{KH}^{-1} \left[\left(1 + 2 \frac{\gamma \eta}{s} + 2 \frac{\gamma^2 \eta^2}{s^2} \right) \right] [\mathbb{KH}^{-1} [\mathcal{K}(x, s, \gamma, \eta)]]^2 \right] - \\ &\quad - \frac{\gamma^\alpha \eta^\alpha}{s^\alpha} x^2 \end{aligned} \quad (99)$$

and the k^{th} -truncated Khalouta residual function of equation (94) becomes

$$\begin{aligned} \mathbb{KH} [\mathcal{R}es_k(x, s, \gamma, \eta)] &= \mathcal{K}_k(x, s, \gamma, \eta) - x + \\ &+ \frac{\gamma^\alpha \eta^\alpha}{s^\alpha} \mathbb{KH} [\mathbb{KH}^{-1} [\mathcal{K}_k(x, s, \gamma, \eta)] D_x \mathbb{KH}^{-1} [\mathcal{K}_k(x, s, \gamma, \eta)]] + \\ &+ \frac{\gamma^\alpha \eta^\alpha}{s^\alpha} \mathbb{KH} \left[\mathbb{KH}^{-1} \left[\left(1 + 2 \frac{\gamma \eta}{s} + 2 \frac{\gamma^2 \eta^2}{s^2} \right) \right] [\mathbb{KH}^{-1} [\mathcal{K}_k(x, s, \gamma, \eta)]]^2 \right] - \\ &\quad - \frac{\gamma^\alpha \eta^\alpha}{s^\alpha} x^2. \end{aligned} \quad (100)$$

To find the unknown coefficients, using equations (98) and (100) and by solving

$$\lim_{s \rightarrow \infty} \frac{s^{k\alpha}}{\gamma^{k\alpha} \eta^{k\alpha}} \mathbb{KH} [\mathcal{R}es_k(x, s, \gamma, \eta)] = 0, k = 1, 2, 3, \dots \quad (101)$$

we get

$$\Theta_1(x) = -x,$$

$$\begin{aligned}
\Theta_2(x) &= 2x, \\
\Theta_3(x) &= -6x, \\
\Theta_4(x) &= 24x, \\
&\vdots \\
\Theta_k(x) &= (-1)^k k!x.
\end{aligned} \tag{102}$$

So, the k^{th} - approximate solution of equation (94) is formulated as follows:

$$\begin{aligned}
\mathcal{K}_k(x, s, \gamma, \eta) &= x - \frac{\gamma^\alpha \eta^\alpha}{s^\alpha} x + 2 \frac{\gamma^{2\alpha} \eta^{2\alpha}}{s^{2\alpha}} x - \\
&- 6 \frac{\gamma^{3\alpha} \eta^{3\alpha}}{s^{3\alpha}} x + 24 \frac{\gamma^{4\alpha} \eta^{4\alpha}}{s^{4\alpha}} x + \dots + (-1)^k k! \frac{\gamma^{k\alpha} \eta^{k\alpha}}{s^{k\alpha}} x.
\end{aligned} \tag{103}$$

Finally, by applying the inverse Khalouta transform of the obtained equation (103), we derive that the k^{th} -approximate solution for the nonlinear time-fractional gas dynamics equation with inhomogeneous term (91) can be written as follows:

$$\begin{aligned}
\vartheta_k(x, t) &= x - \frac{t^\alpha}{\Gamma(\alpha + 1)} x + 2 \frac{t^{2\alpha}}{\Gamma(2\alpha + 1)} x - \\
&- 6 \frac{t^{3\alpha}}{\Gamma(3\alpha + 1)} x + 24 \frac{t^{4\alpha}}{\Gamma(4\alpha + 1)} x + \dots + (-1)^k k! \frac{t^{k\alpha}}{\Gamma(k\alpha + 1)} x \\
&= x \left(1 - \frac{1}{\Gamma(\alpha + 1)} t^\alpha + \frac{2!}{\Gamma(2\alpha + 1)} t^{2\alpha} - \right. \\
&\quad \left. - \frac{3!}{\Gamma(3\alpha + 1)} t^{3\alpha} + \frac{4!}{\Gamma(4\alpha + 1)} t^{4\alpha} + \dots + \frac{(-1)^k k!}{\Gamma(k\alpha + 1)} t^{k\alpha} \right) \\
&= x \sum_{i=0}^k \frac{(-1)^i i!}{\Gamma(i\alpha + 1)} t^{i\alpha}.
\end{aligned} \tag{104}$$

By taking $\lim_{k \rightarrow \infty}$ on both sides of equation (104), we obtain

$$\vartheta(x, t) = x \sum_{i=0}^{\infty} \frac{(-1)^i i!}{\Gamma(i\alpha + 1)} t^{i\alpha}. \tag{105}$$

If we put $\alpha = 1$ in equation (105), then we get

$$\vartheta(x, t) = \frac{x}{1+t}, \quad (106)$$

which represents the exact solution of the classical nonlinear gas dynamics equation, and is in complete agreement with the results in [11].

Example 3. Let's consider the nonlinear time-fractional wave-like equation with variable coefficients [7]

$$D_t^{2\alpha}\vartheta = x^2(\vartheta_x\vartheta_{xx})_x - x^2(\vartheta_{xx})^2 - \vartheta, \quad (107)$$

with the initial conditions

$$\vartheta(x, 0) = 0, \quad D_t^\alpha\vartheta(x, 0) = x^2, \quad (108)$$

where $D_t^{2\alpha}$ represents the Caputo time-fractional derivative operator of order α with $\frac{1}{2} < \alpha \leq 1$ and $\vartheta = \{\vartheta(x, t), (x, t) \in]0, 1[\times \mathbb{R}^+\}$.

By applying the Khalouta transform operator on equation (75) and using part (1) of Theorem 1, we get

$$\mathbb{KH} [D_t^\alpha\vartheta] = x^2\mathbb{KH} [(\vartheta_x\vartheta_{xx})_x] - x^2\mathbb{KH} [(\vartheta_{xx})^2] - \mathbb{KH} [\vartheta]. \quad (109)$$

By using Theorem 2 for $m = 2$ and the initial conditions (108), we rewrite the equation (109) as follows:

$$\begin{aligned} \mathcal{K}(x, s, \gamma, \eta) &= \frac{\gamma^\alpha\eta^\alpha}{s^\alpha}x^2 + \\ &+ x^2\frac{\gamma^{2\alpha}\eta^{2\alpha}}{s^{2\alpha}}\mathbb{KH} [D_x(D_x\mathbb{KH}^{-1}[\mathcal{K}(x, s, \gamma, \eta)]D_{xx}\mathbb{KH}^{-1}[\mathcal{K}(x, s, \gamma, \eta)])] - \\ &- x^2\frac{\gamma^{2\alpha}\eta^{2\alpha}}{s^{2\alpha}}\mathbb{KH} [(D_{xx}\mathbb{KH}^{-1}[\mathcal{K}(x, s, \gamma, \eta)])^2] - \frac{\gamma^{2\alpha}\eta^{2\alpha}}{s^{2\alpha}}\mathcal{K}(x, s, \gamma, \eta). \end{aligned} \quad (110)$$

According to the construction of KHRPSM presented in Section 4, the series solution of equation (110) can be expressed as follows:

$$\mathcal{K}(x, s, \gamma, \eta) = \sum_{m=0}^{\infty} \frac{\gamma^{m\alpha}\eta^{m\alpha}}{s^{m\alpha}}\Theta_m(x), \quad (111)$$

and the k^{th} -truncated series of equation (110) is given by

$$\mathcal{K}_k(x, s, \gamma, \eta) = \sum_{m=0}^k \frac{\gamma^{m\alpha} \eta^{m\alpha}}{s^{m\alpha}} \Theta_m(x). \quad (112)$$

By using Lemma 2, we have

$$\lim_{s \rightarrow \infty} \mathcal{K}(x, s, \gamma, \eta) = \vartheta(x, 0) = 0, \Theta_1(x) = D_t^\alpha \vartheta(x, 0) = x^2. \quad (113)$$

Therefore, the k^{th} -truncated series (112) becomes

$$\mathcal{K}_k(x, s, \gamma, \eta) = \frac{\gamma^\alpha \eta^\alpha}{s^\alpha} x^2 + \sum_{m=2}^k \frac{\gamma^{m\alpha} \eta^{m\alpha}}{s^{m\alpha}} \Theta_m(x). \quad (114)$$

From Theorem 5, we establish the Khalouta residual function of equation (110) as follows:

$$\begin{aligned} \mathbb{K}\mathbb{H}\mathbb{R}es(x, s, \gamma, \eta) &= \mathcal{K}(x, s, \gamma, \eta) - \frac{\gamma^\alpha \eta^\alpha}{s^\alpha} x^2 \\ &- x^2 \frac{\gamma^{2\alpha} \eta^{2\alpha}}{s^{2\alpha}} \mathbb{K}\mathbb{H} [D_x(D_x \mathbb{K}\mathbb{H}^{-1} [\mathcal{K}(x, s, \gamma, \eta)] D_{xx} \mathbb{K}\mathbb{H}^{-1} [\mathcal{K}(x, s, \gamma, \eta)])] \\ &+ x^2 \frac{\gamma^{2\alpha} \eta^{2\alpha}}{s^{2\alpha}} \mathbb{K}\mathbb{H} [(D_{xx} \mathbb{K}\mathbb{H}^{-1} [\mathcal{K}(x, s, \gamma, \eta)])^2] + \frac{\gamma^{2\alpha} \eta^{2\alpha}}{s^{2\alpha}} \mathcal{K}(x, s, \gamma, \eta), \end{aligned} \quad (115)$$

and the k^{th} -truncated Khalouta residual function of equation (110) becomes

$$\begin{aligned} \mathbb{K}\mathbb{H}\mathbb{R}es_k(x, s, \gamma, \eta) &= \mathcal{K}_k(x, s, \gamma, \eta) - \frac{\gamma^\alpha \eta^\alpha}{s^\alpha} x^2 \\ &- x^2 \frac{\gamma^{2\alpha} \eta^{2\alpha}}{s^{2\alpha}} \mathbb{K}\mathbb{H} [D_x(D_x \mathbb{K}\mathbb{H}^{-1} [\mathcal{K}_k(x, s, \gamma, \eta)] D_{xx} \mathbb{K}\mathbb{H}^{-1} [\mathcal{K}_k(x, s, \gamma, \eta)])] \\ &+ x^2 \frac{\gamma^{2\alpha} \eta^{2\alpha}}{s^{2\alpha}} \mathbb{K}\mathbb{H} [(D_{xx} \mathbb{K}\mathbb{H}^{-1} [\mathcal{K}_k(x, s, \gamma, \eta)])^2] + \frac{\gamma^{2\alpha} \eta^{2\alpha}}{s^{2\alpha}} \mathcal{K}_k(x, s, \gamma, \eta). \end{aligned} \quad (116)$$

To find the unknown coefficients, using equations (114) and (116) and by solving

$$\lim_{s \rightarrow \infty} \frac{s^{k\alpha}}{\gamma^{k\alpha} \eta^{k\alpha}} \mathbb{K}\mathbb{H}\mathbb{R}es_k(x, s, \gamma, \eta) = 0, k = 2, 3, 4, \dots \quad (117)$$

we get

$$\begin{aligned}
\Theta_2(x) &= 0, \\
\Theta_3(x) &= -x^2, \\
\Theta_4(x) &= 0, \\
\Theta_5(x) &= x^2 \\
&\vdots \\
\Theta_{2k}(x) &= 0, \\
\Theta_{2k+1}(x) &= (-1)^k x^2.
\end{aligned} \tag{118}$$

So, the k^{th} - approximate solution of equation (110) is formulated as follows

$$\mathcal{K}_k(x, s, \gamma, \eta) = \frac{\gamma^\alpha \eta^\alpha}{s^\alpha} x^2 - \frac{\gamma^{3\alpha} \eta^{3\alpha}}{s^{3\alpha}} x^2 + \frac{\gamma^{5\alpha} \eta^{5\alpha}}{s^{5\alpha}} x^2 + \dots + (-1)^k \frac{\gamma^{(2k+1)\alpha} \eta^{(2k+1)\alpha}}{s^{(2k+1)\alpha}} x^2. \tag{119}$$

Finally, by applying the inverse Khalouta transform of the obtained equation (119), we derive that the k^{th} -approximate solution for the nonlinear time-fractional wave-like equation with variable coefficients (107) can be written as follows:

$$\begin{aligned}
\vartheta_k(x, t) &= \frac{t^\alpha}{\Gamma(\alpha+1)} x^2 - \frac{t^{3\alpha}}{\Gamma(3\alpha+1)} x^2 \\
&+ \frac{t^{5\alpha}}{\Gamma(5\alpha+1)} x^2 + \dots + (-1)^k \frac{t^{(2k+1)\alpha}}{\Gamma((2k+1)\alpha+1)} x^2 \\
&= x^2 \left(\frac{t^\alpha}{\Gamma(\alpha+1)} - \frac{t^{3\alpha}}{\Gamma(3\alpha+1)} + \frac{t^{5\alpha}}{\Gamma(5\alpha+1)} + \dots + (-1)^k \frac{t^{(2k+1)\alpha}}{\Gamma((2k+1)\alpha+1)} \right) \\
&= x^2 \sum_{i=0}^k (-1)^i \frac{t^{(2i+1)\alpha}}{\Gamma((2i+1)\alpha+1)}. \tag{120}
\end{aligned}$$

By taking $\lim_{k \rightarrow \infty}$ on both sides of equation (120), we obtain

$$\begin{aligned}
\vartheta(x, t) &= x^2 \sum_{i=0}^{\infty} (-1)^i \frac{t^{(2i+1)\alpha}}{\Gamma((2i+1)\alpha+1)} \\
&= x^2 \sin(t^\alpha, \alpha). \tag{121}
\end{aligned}$$

If we put $\alpha = 1$ in equation (121), then we get

$$\vartheta(x, t) = x^2 \sin(t), \quad (122)$$

which represents the exact solution of the classical nonlinear wave-like equation with variable coefficients, and is in complete agreement with the results in [7].

6. Numerical results and discussion

In this section, we evaluate the numerical results obtained using KHRPSM for different nonlinear time-fractional partial differential equations discussed in Examples 1, 2, and 3. Figures 1, 3 and 5 show the 3D graph comparison between the approximate solutions obtained by KHRPSM for different values of time-fractional order $\alpha = 0.8, 0.9, 1$ and exact solutions for equations (75), (91) and (107), respectively. While, figures 2, 4 and 6 show the 2D graph comparison between the approximate solutions obtained by KHRPSM for different values of time-fractional order $\alpha = 0.7, 0.8, 0.9, 1$ and exact solutions for the same equations. Similar to these graphs, approximate solutions converge to exact solutions. Also, in this case the graphs of the exact solutions and the graphs of the approximate solutions are identical at $\alpha = 1$. This implies that the results obtained by KHRPSM are in good agreement with the exact solution at $\alpha = 1$. The analysis of numerical values of the approximate solutions obtained by KHRPSM and the exact solutions are summarized in Tables 1, 2 and 3 at some selected grid points of t . These tables show that the solutions of the KHRPSM are in good agreement with the exact solutions. Thus, we can conclude from the above numerical results, that KHRPSM is too accurate to deal with nonlinear time-fractional partial differential equations.

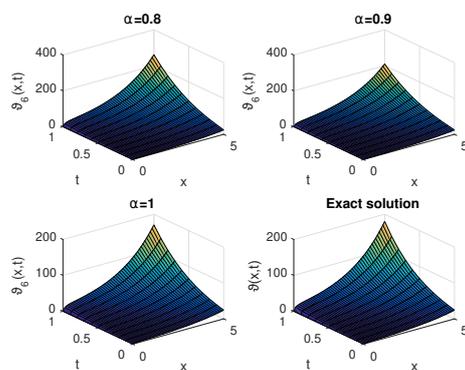


Figure 1: 3D plots of $\vartheta_6(x, t)$ obtained by KHRPSM and $\vartheta(x, t)$ for equation (75)

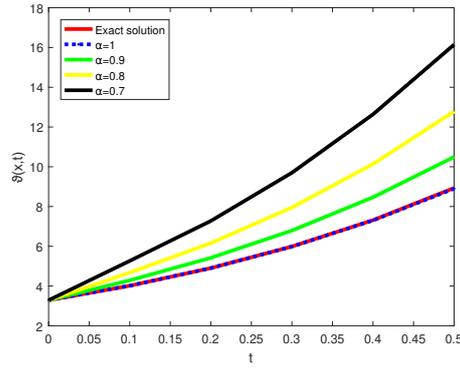


Figure 2: 2D plots of $\vartheta_6(x, t)$ obtained by KHRPSM and $\vartheta(x, t)$ at $x = 1$ for equation (75)

t	$\alpha = 0.7$	$\alpha = 0.8$	$\alpha = 0.9$	$\alpha = 1$	exact solution	$ \vartheta_{exact} - \vartheta_{KHRPSM} $
0.01	3.8409	3.7928	3.7635	3.7456	3.7456	2.2690×10^{-12}
0.03	3.9922	3.9020	3.8417	3.8011	3.8011	5.5322×10^{-10}
0.05	4.1212	4.0005	3.9165	3.8577	3.8577	7.1383×10^{-9}
0.07	4.2412	4.0947	3.9904	3.9154	3.9154	3.8520×10^{-8}
0.09	4.3565	4.1868	4.0642	3.9743	3.9743	1.3579×10^{-7}

Table 1: Numerical values of $\vartheta_6(x, t)$ obtained by KHRPSM and $\vartheta(x, t)$ at $x = 1$ for equation (75)

t	$\alpha = 0.7$	$\alpha = 0.8$	$\alpha = 0.9$	$\alpha = 1$	exact solution	$ \vartheta_{exact} - \vartheta_{KHRPSM} $
0.01	0.95858	0.97388	0.98382	0.99010	0.99010	9.9010×10^{-13}
0.03	0.91584	0.91252	0.95777	0.97087	0.97087	7.0777×10^{-10}
0.05	0.88493	0.91252	0.93488	0.95238	0.95238	1.4881×10^{-8}
0.07	0.85963	0.88904	0.91401	0.93458	0.93458	1.0995×10^{-7}
0.09	0.83773	0.86812	0.89469	0.91743	0.91743	4.8756×10^{-7}

Table 2: Numerical values of $\vartheta_6(x, t)$ obtained by KHRPSM and $\vartheta(x, t)$ at $x = 1$ for equation (91)

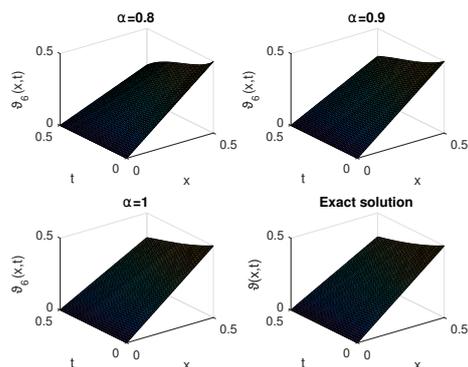


Figure 3: 3D plots of $\vartheta_6(x,t)$ obtained by KHRPSM and $\vartheta(x,t)$ at $x = 1$ for equation (91)

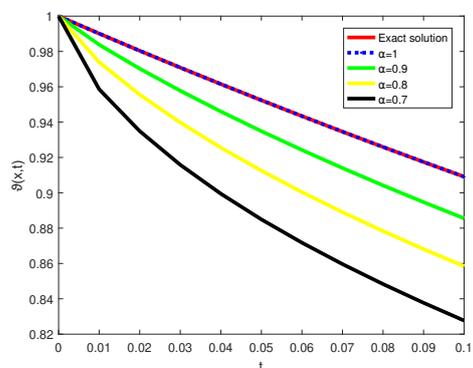


Figure 4: 2D plots of $\vartheta_6(x,t)$ obtained by KHRPSM and $\vartheta(x,t)$ at $x = 1$ for equation (91)

t	$\alpha = 0.7$	$\alpha = 0.8$	$\alpha = 0.9$	$\alpha = 1$	exact solution	$ u_{exact} - u_{KHRPSM} $
0.1	0.05400	0.042209	0.032605	0.024958	0.024958	4.9596×10^{-12}
0.3	0.10969	0.097871	0.085658	0.07388	0.07388	1.0835×10^{-8}
0.5	0.14473	0.13893	0.13028	0.11986	0.11986	3.8618×10^{-7}
0.7	0.16673	0.16866	0.16664	0.16106	0.16105	4.0574×10^{-6}
0.9	0.17926	0.18843	0.19429	0.19586	0.19583	2.346×10^{-5}

Table 3: Numerical values of $\vartheta_6(x,t)$ obtained by KHRPSM and $\vartheta(x,t)$ at $x = 0.5$ for equation (107)

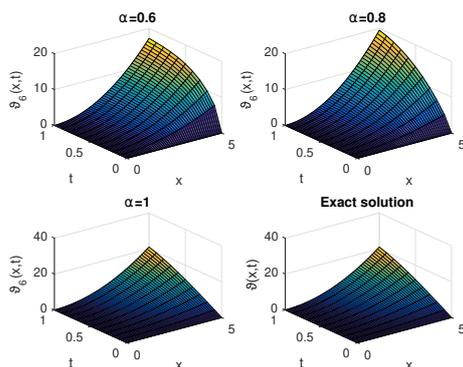


Figure 5: 3D plots of $\vartheta_6(x,t)$ obtained by KHRPSM and $\vartheta(x,t)$ at $x = 1$ for equation (107)

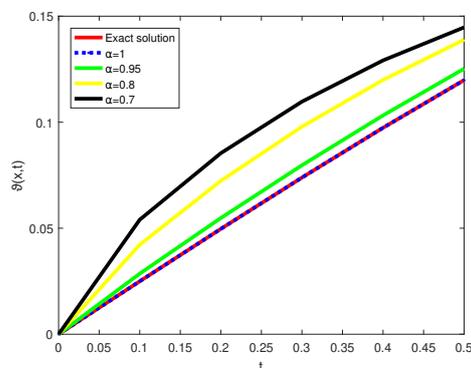


Figure 6: 2D plots of $\vartheta_6(x,t)$ obtained by KHRPSM and $\vartheta(x,t)$ at $x = 0.5$ for equation (107)

7. Conclusions

In this work, we introduced, discussed, and investigated analytical approximate solutions for general nonlinear time-fractional partial differential equations with inhomogeneous terms using a new hybrid method called Khalouta residual power series method (KHRPSM). Using the proposed method, we were able to solve three different time-fractional nonlinear partial differential equations. The solutions obtained using KHRPSM agree well with the exact solutions in the classical case of alpha value and in continuous consistency with each other for different orders of fractional cases. Moreover, the obtained results led us to conclude that the current method is easy to implement, accurate, flexible and effective. In future, the stability, existence, and uniqueness of the results obtained for nonlinear

fractional partial differential equations will be investigated.

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Received 02 June 2025

Accepted 15 October 2025