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On the Variable-coefficient Harry Dym Equation With a Self-consistent Source

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Abstract. We consider the variable-coefficient Harry Dym equation with the self-consistent source. The source consists of the variable-coefficient term and the combination of the eigenfunctions of the corresponding spectral problem for the string equation which has not spectral singularities. Assuming that the spectral problem has simple eigenvalues we apply the inverse scattering transform (IST) method and the (A, B, C) triplet technique for providing the explicit form of the multisoliton solution.

Key Words and Phrases: Harry Dym equation, inverse scattering method, scattering data, variable-coefficient term, self-consistent source, triplet matrix.

2010 Mathematics Subject Classifications: 34L25, 35Q41, 37K10, 35R30, 34M46

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1. Introduction

We consider the following initial value problem for the variable-coefficient Harry Dym equation with a self-consistent source:

$$q_t(x,t) = 2\left(1\Big/\sqrt{1+q(x,t)}\right)_{xxx} - 2\sum_{n=1}^{N}(1+q(x,t))\frac{\partial}{\partial x}\left(\varphi_n^2(x,t)\right) - \\ -q_x(x,t)\sum_{n=1}^{N}\varphi_n^2(x,t) - \gamma(t)q_x(x,t), \tag{1}$$

$$\varphi_n''(x,t) - \chi_n^2\varphi_n(x,t)q(x,t) = \chi_n^2\varphi_n(x,t), \tag{2}$$

with initial data

$$q(x,0) = q_0(x), \tag{3}$$

which has following properties

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- $$\int_{-\infty}^{\infty}(1+x^2)\left(|q_0(x)|+\left|1-\frac{1}{1+q_0(x)}\right|\right)dx<\infty, \tag{4}$$
- The operator $L(0) := \frac{d^2}{dx^2} + \lambda^2 q_0(x)$ possesses exactly N simple eigenvalues $-\chi_1^2(0) > -\chi_2^2(0) > ... > -\chi_N^2(0)$ without spectral singularities.

Here, the prime means the derivative with respect to the variable x, while the dot means the derivative with respect to the variable t, $f_n(x,t)$ is an eigenfunction corresponding to the eigenvalue $-\chi_n^2$ and normalized by the following condition:

$$\int_{-\infty}^{\infty}(1+q(x,t))\varphi_k^2(x,t)dx = A_k(t), \tag{5}$$

where $A_k(t)$ and $\gamma(t)$ are the given continuous functions, and $\gamma(t)$ is bounded.

For any $t \geq 0$, $q(x,t)$ is assumed to be sufficiently smooth and sufficiently rapidly tends to zero as $|x| \to \infty$:

$$\int_{-\infty}^{\infty}(1+x^2)\left(|q(x,t)|+\left|1-\frac{1}{1+q(x,t)}\right|\right)dx<\infty. \tag{6}$$

The Harry Dym equation [1]-[3]

$$q_t(x,t) = 2\left(1\Big/\sqrt{1+q(x,t)}\right)_{xxx}$$

is an integrable nonlinear evolution equation [4]-[5] which appears in the hydrodynamics [6] and plasma physics [7]. Recall that the conversion methods from the KdV to the HD equation have been well studied in [8]. Moreover, the investigation on the variable-coefficient generalized Korteweg-de Vries model with dissipative, perturbed and external-force terms for the pulse waves in a blood vessel or dynamics in a circulatory system can be found in [9]. Further related studies can be found in [12]-[16]. The inverse scattering transform method for the string equation

$$Ly \equiv y'' + \lambda^2 q(x)y = -\lambda^2 y$$

was studied in [5], where the time dependence of the scattering data $\{R(\lambda),\ c_n,\ -\chi_n^2,\ n=\overline{1,N}\}$ was obtained:

$$R(\lambda,t) = R(\lambda,0)e^{8i\lambda^3 t},$$

$$\chi_n(t) = \chi_n(0),$$

$$c_n(t) = c_n(0)e^{8\chi_n^3 t},\ \ n=\overline{1,N}.$$

In the present work, we also apply this IST method to the integration of the variable-coefficient Harry Dym equation with the self-consistent source. First, we need to remind the following fact [10]:

Lemma 1. *If $X(x,\lambda)$ and $Y(x,\mu)$ are solutions of $LX = -\lambda^2 X$ and $LY = -\mu^2 Y$, respectively, then the following identity holds:*

$$\frac{dW(X(x,\lambda), Y(x,\mu))}{dx} = (1+q(x))(\lambda^2-\mu^2)X(x,\lambda), Y(x,\mu), \tag{7}$$

where $W(X(x,\lambda), Y(x,\mu)) = X(x,\lambda)Y'(x,\mu) - X'(x,\lambda)Y(x,\mu)$.

Then, we introduce auxiliary function

$$H_0 = Y_t(x,\lambda,t) - BY(x,\lambda,t) - \lambda^2 \sum_{n=1}^{N} \varphi_n(x,t)F - \gamma(t)Y_x(x,\lambda,t)$$

for constructing the source $\frac{G}{\lambda^2}$ for the Harry Dym equation, where

$$G = -2\lambda^2 \sum_{n=1}^{N}(1+q)\frac{\partial}{\partial x}\left(\varphi_n^2(x,t)\right) - \lambda^2 q_x \sum_{n=1}^{N} \varphi_n^2(x,t) + \lambda^2\gamma(t)q_x(x,t).$$

It should be noted that the newly introduced auxiliary function also satisfy the string equation and then we get the time evolution of the scattering data for the string equation which will be shown in Section 2. Similar technique was used in [10] for the integration of Harry Dym equation with the integral type source. Finally, using the obtained results on the time evolvement, we solve the Gelfand-Levitan equation by the (A,B,C) triplet technique to derive the explicit form of the multisoliton solution for the variable-coefficient Harry Dym equation with the self-consistent source in Section 3.

2. Evolution equations

We have to mention that the equation (1) can be represented by Lax pairs [5] as

$$L_t = [B, L] + G,$$

where

$$L = \frac{d^2}{dx^2} + \lambda^2 q(x,t),$$

$$B = 2\lambda^2\left[\frac{2}{\sqrt{1+q(x,t)}}\frac{\partial}{\partial x} - \left(\frac{1}{\sqrt{1+q(x,t)}}\right)_x\right]. \tag{8}$$

Let $y(x,t)$ be a solution of the equation

$$Ly = y''(x,t) + \lambda^2 y(x,t)q(x,t) = -\lambda^2 y(x,t) \tag{9}$$

and let $F = F(x, \lambda, \lambda_n, t)$ satisfy the equation

$$F_x = (1 + q)\varphi_n(x, t)Y(x, \lambda, t). \tag{10}$$

We introduce the following function:

$$H_0 = Y_t(x, \lambda, t) - BY(x, \lambda, t) - \lambda^2 \sum_{n=1}^{N} \varphi_n(x, t)F - \gamma(t)Y_x(x, \lambda, t). \tag{11}$$

For $\lambda \in \mathbb{R}$, the function (11) will be a solution of the equation

$$LH_0 = -\lambda^2 H_0 - \lambda^2 \sum_{n=1}^{N}(1 + q(x, t))\varphi_n(x, t)\tilde{H},$$

where

$$\tilde{H} = (\lambda^2 - \lambda_n^2)F(x, \lambda, \lambda_n) - W\{\varphi_n(x, t), Y(x, \lambda, t)\}.$$

The functions

$$\begin{aligned} F^- &= -\int_{-\infty}^{x}(1 + q(\tau, t))\varphi_n(\tau, t)g(\tau, \lambda)d\tau, \\ F^+ &= \int_{x}^{\infty}(1 + q(\tau, t))\varphi_n(\tau, t)f(\tau, \lambda)d\tau, \end{aligned} \tag{12}$$

defined by the Jost solutions, satisfy the equation (10). We remind that the Jost solutions have the following asymptotics:

$$g(x, \lambda) \to e^{-i\lambda x}, \ \ x \to -\infty, \tag{13}$$

$$f(x, \lambda) \to e^{i\lambda x}, \ \ x \to \infty. \tag{14}$$

Then it is easy to show that the function (11) defined by the Jost solutions

$$H_0^-(x, \lambda, t) = g_t(x, \lambda, t) - Bg(x, \lambda, t) - \lambda^2 \sum_{n=1}^{N} \varphi_n(x, t)F^- - \gamma(t)g_x(x, \lambda, t),$$
$$H_0^+(x, \lambda, t) = f_t(x, \lambda, t) - Bf(x, \lambda, t) - \lambda^2 \sum_{n=1}^{N} \varphi_n(x, t)F^+ - \gamma(t)f_x(x, \lambda, t),$$

satisfies the equation (9). In fact, taking into account the identity (7), we have $\dfrac{\partial \tilde{H}}{\partial x} = 0$. With the help of the asymptotes of the Jost solutions and taking account of (12), for $Im\lambda \geq 0$, we have $\tilde{H}^{\pm} \to 0$ as $x \to \pm\infty$. Then, for $Im\lambda \geq 0$ and $x \in (-\infty, \infty)$ it follows that $\tilde{H}^{\pm} \equiv 0$. Hence, $LH_0^+ = -\lambda^2 H_0^+$ and $LH_0^- = -\lambda^2 H_0^-$.

Theorem 1. *Let $q(x,t), \varphi_n(x,t)$ be a solution of the problem* (1)-(6). *Then the scattering data related to the equation* (9) *fulfill the following relations:*

$$\dot{R}(\lambda,t) = \left(8i\lambda^3 + 2i\lambda\gamma(t)\right) R(\lambda,t),$$

$$\frac{d\chi_j(t)}{dt} = 0, \quad j = \overline{1,N}. \tag{15}$$

$$\dot{c}_k(t) = \left(8i\lambda_k^3 + 2i\lambda_k\gamma(t) + \lambda_k^2 A_k(t)\right), \quad k = \overline{1,N}. \tag{16}$$

Proof. Using the result for $H_0^{\pm}(x,\lambda,t)$, we introduce the following auxiliary function for $\lambda \neq 0$:

$$S = H_0^-(x,\lambda,t) - a(\lambda,t)H_0^+(x,-\lambda,t) - b(\lambda)\,H_0^+(x,\lambda,t). \tag{17}$$

With the help of the relation

$$g(x,\lambda) = a(\lambda)f(x,-\lambda) + b(\lambda)f(x,\lambda), \tag{18}$$

and substituting the representations (12) for $F^-(x,\lambda,t)$, $F^+(x,\lambda,t)$ into (17), we obtain

$$S = \dot{a}(\lambda,t)f(x,-\lambda,t) + \dot{b}(\lambda,t)\,f(x,\lambda,t). \tag{19}$$

On the other hand, taking account the uniqueness of the Jost solutions, the representation (8) and asymptotics (13)-(14), we obtain

$$H_0^-(x,\lambda,t) = 4i\lambda^3 g(x,\lambda,t) + i\lambda\gamma(t)g(x,\lambda,t), \tag{20}$$

$$H_0^+(x,\lambda,t) = -4i\lambda^3 f(x,\lambda,t) - i\lambda\gamma(t)f(x,\lambda,t). \tag{21}$$

$$H_0^+(x,-\lambda,t) = 4i\lambda^3 f(x,-\lambda,t) + i\lambda\gamma(t)f(x,-\lambda,t). \tag{22}$$

Substituting the expressions (20)-(22) into (17) and using the relation (18), we have

$$S = 8i\lambda^3 b(\lambda,t)f(x,\lambda,t) + 2i\lambda\gamma(t)b(\lambda,t)f(x,\lambda,t), \tag{23}$$

and, comparing (19) and (23), we get

$$\begin{aligned} &\dot{a}(\lambda,t) = 0, \\ &\dot{b}(\lambda,t) = 8i\lambda^3 b(\lambda,t) + 2i\lambda\gamma(t)b(\lambda,t). \end{aligned} \tag{24}$$

Using the equalities (24) and differentiating the relation $R(\lambda,t) = b(\lambda,t)/a(\lambda,t)$ with respect to t, we have the equality

$$\dot{R}(\lambda,t) = \left(8i\lambda^3 + 2i\lambda\gamma(t)\right) R(\lambda,t).$$

Assuming that $a(\lambda,t)$ has a finite number of simple zeros $\lambda_n(t) = i\chi_n(t)$, in the upper half-plane $Im\lambda > 0$, which correspond to the eigenvalues $-\chi_n^2(t)$, $(\chi_n(t) > 0)$ $n = \overline{1,N}$, from the equality $\dot{a}(\lambda,t) = 0$ it follows that $\chi_n(t)$ do not depend on t. So (15) holds.

Now, using the analytic properties of $a(\lambda)$, the Jost solutions [5] and the relation

$$g(x,\lambda_k) = c_k f(x,\lambda_k), \quad k = 1,\ 2,\ ...,\ N$$

we can define the function

$$h(x,\lambda_k,t) = H_0^-(x,\lambda_k,t) - c_k(t)H_0^+(x,\lambda_k,t). \tag{25}$$

Here, by the definitions of $H_0^-(x,\lambda_k,t)$ and $H_0^+(x,\lambda_k,t)$ and using the asymptotics for the Jost solutions, we obtain

$$h(x,\lambda_k,t) = 8i\lambda_k^3 c_k(t)f(x,\lambda_k,t) + 2i\lambda_k\gamma(t)c_k(t)f(x,\lambda_k,t). \tag{26}$$

On the other hand, substituting the representations for $H_0^-(x,\lambda_k,t)$ and $H_0^+(x,\lambda_k,t)$ into the expression (25), we get

$$h(x,\lambda_k,t) = \dot{c}_k(t)f(x,\lambda_k,t)+$$

$$+\lambda_k^2\sum_{n=1}^{N}\varphi_n(x,t)\int_{-\infty}^{\infty}(1+q(x,t))\varphi_n(x,t)c_k(t)f(x,\lambda_k,t)dx.$$

Using the orthogonality of the eigenfunctions corresponding to the different eigenvalues for $n \neq k$, it is easy to show that the integral on the right-hand side of the above equality equals to zero. Therefore,

$$h(x,\lambda_k,t) = \dot{c}_k(t)f(x,\lambda_k,t) + \lambda_k^2\varphi_k(x,t)\int_{-\infty}^{\infty}(1+q(x,t))\varphi_k(x,t)c_k(t)f(x,\lambda_k,t)dx.$$

As $c_k(t)f(x,\lambda_k,t)$ is the eigenfunction of the equation (9), we can write

$$\alpha_k\varphi_k(x,t) = c_k(t)f(x,\lambda_k,t).$$

As a result, the following expression for the function $h(x,\lambda_k,t)$

$$h(x,\lambda_k,t) = \dot{c}_k(t)f(x,\lambda_k,t) + \lambda_k^2 c_k(t)f(x,\lambda_k,t)\int_{-\infty}^{\infty}(1+q(x,t))\varphi_k^2(x,t)dx \tag{27}$$

is valid. Comparing (26) and (27), we get (16). ◄

Remark 1. *The obtained results completely specify the time evolution of the scattering data, which allows to find the solution of the considered problem* (1)-(6) *by the inverse scattering method.*

3. Multi-soliton solution

In order to derive the explicit form of the multi-soliton solution to the variable-coefficient Harry Dym equation with the self consistent source, we apply the (A, B, C) triplet matrix technique [11] for the reflectionless case, i.e. the case where $R(\lambda, t) = 0$. Then, since the operator $L(t)$ has simple eigenvalues, (A, B, C) triplet matrix takes the following form:

$$A = \begin{pmatrix} \chi_1 & 0 & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & \chi_N \end{pmatrix}, B = \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix}, C = \begin{pmatrix} c_1(t) & \dots & c_N(t) \end{pmatrix},$$

Here $c_j(t)$, $j = \overline{1, N}$, satisfy the equations (16). Hence, the kernel of Gelfand-Levitan-Marchenko equation can be written as

$$\Omega(x + y, t) = -Ce^{-A(x+y+2\varepsilon_+)}e^{8A^3t-2Ap(t)-A^2\tilde{A}_k(t)}A^{-1}B.$$

Inserting this into the Gelfand-Levitan equation

$$K(x, y, t) - \Omega(x + y, t) - \int_x^\infty K(x, s, t)\Omega'(s + y, t)ds = 0$$

and solving it, we obtain

$$K(x, x, t) = -Ce^{-A(x+2\varepsilon_+)+8A^3t-2Ap(t)-A^2\tilde{A}_k(t)}A^{-1}\Gamma^{-1}(x, t)e^{-Ax}B,$$

and by the formula $1 + q(x) = [1 - K(x, x)]^{-4}$ [5] we get

$$q(x, t) = [1 + Ce^{-A(x+2\varepsilon_+)+8A^3t-2Ap(t)-A^2\tilde{A}_k(t)}A^{-1}\Gamma^{-1}(x, t)e^{-Ax}B]^{-4} - 1,$$

where

$$\Gamma(x, t) = I - e^{-2A\varepsilon_+ +8A^3t-2Ap(t)-A^2\tilde{A}_k(t)}Q(x),$$

$$p(t) = \int_0^t \gamma(t)dt,$$

$$\tilde{A}_k(t) = \int_0^t A_k(t)dt,$$

$$Q(x) = \int_x^\infty e^{-As}BCe^{-As}ds.$$

Using the integral representation for the Jost solution $f(x, \lambda)$

$$f(x, \lambda) = e^{i\lambda(x+\varepsilon_+)} + \int_x^\infty K(x, s)e^{i\lambda s}dse^{i\lambda\varepsilon_+},$$

we define $f(x,\lambda_j)$ as a column vector $f = (f(x,\lambda_1), f(x,\lambda_2), ..., f(x,\lambda_N))^T$,

$$f = e^{-A(x+\varepsilon_+)}B - Ce^{-A(x+2\varepsilon_+)+8A^3t-2Ap(t)-A^2\tilde{A}_k(t)}\times$$

$$\times A^{-1}\Gamma^{-1}(x,t)\int_x^{\infty} e^{-As}Be^{-As}dse^{-A\varepsilon_+}B.$$

Then we find eigenfunctions $\varphi_j(x,t)$ as

$$(\varphi_1(x,t), \varphi_2(x,\lambda_2), ..., \varphi_N(x,t))^T = CD(f/\alpha).$$

Here, $D(f/\alpha)$ is a diagonal matrix which is defined as

$$D(f/\alpha) = diag(f(x,\lambda_1)/\alpha_1, f(x,\lambda_2)/\alpha_2, ..., f(x,\lambda_N)/\alpha_N),$$

where α_j can be found by the normalizing condition (5).

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