Azerbaijan Journal of Mathematics
V. 16, No 1, 2026, January
ISSN 2218-6816
https://doi.org/10.59849/2218-6816.2026.1.67

Maximal-simultaneous Approximation by Faber Series in Bergman Spaces Defined on Unbounded Continuums

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Abstract. In this study, we examine the problem of maximal simultaneous approximation in Bergman spaces defined over unbounded continuum of the complex plane, using classical Faber series. We derive upper bounds for the approximation error that explicitly depend on the best polynomial approximation numbers, as well as on structural parameters of the canonical domain under consideration. These results provide quantitative insights into how the geometry of the domain influences the convergence behavior of the Faber series.

Key Words and Phrases: quasidisc, Faber series, maximal convergence, simultaneous approximation, Bergman spaces.

2010 Mathematics Subject Classifications: 30E10, 41A10, 41A28

1. Introduction

Let $\overline{\mathbb{C}}$ be the extended complex plane and let $\mathfrak{M}$ be an unbounded continuum which contains ∞. We assume that the complementary set $\mathcal{B}:=\mathfrak{M}^c := \overline{\mathbb{C}}\diagdown\mathfrak{M}$ is an simply connected bounded domain in $\mathbb{C}$ and $0 \in \mathcal{B}$. Let $D := \{z \in \mathbb{C}\colon\ |z| < 1\}$. By the Riemann conformal mapping theorem, there exists the conformal mapping $w = \varphi(z)$ which maps the domain $\mathcal{B}$ onto $C\overline{D} := \overline{\mathbb{C}}\diagdown\overline{D}$ and is normalized by the conditions

$$\varphi\,(0) = \infty \text{ and } \lim_{z\to 0} z\varphi\,(z) > 0.$$

By ψ we denote the inverse mapping of φ.

The function φ in some neighborhood of zero has the series representation

$$\varphi(z) = \frac{\alpha}{z} + \alpha_0 + \alpha_1 z + ... + \alpha_k z^k + ...,$$

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and hence

$$[\varphi(z)]^n = F_n(1/z) - Q_n(z),\ z \in B, \tag{1}$$

where

$$F_n(1/z) = \frac{\beta_1}{z} + \frac{\beta_2}{z^2} + ... + \frac{\beta_n}{z^n}, \qquad n = 1, 2, ... \ .$$

is the Faber polynomial of degree n with respect to the negative powers of z and the function $Q_n(z)$, which contains nonnegative powers of z, is an analytic function in the domain $\mathcal{B}$.

For a given $R > 1$, we set

$$L_R := \{z \in \mathcal{B} : |\varphi(z)| = R\},\quad \mathfrak{M}_R := extL_R := \{z : z \in \mathfrak{M}^c \text{ and } |\varphi(z)| < R\} \cup \mathfrak{M}.$$

In the mathematical literature $\mathfrak{M}_R$ is called an R^{th} canonical domain of the continuum $\mathfrak{M}$.

Note that the polynomials $F_k(1/z)$, $k = 1, 2, ...$, with respect to $1/z$ can be defined in the continuum $\mathfrak{M}$ also as the coefficients of the series representation

$$\frac{\psi'(w)}{\psi(w) - z} = -\sum_{k=1}^{\infty} \frac{F_k(1/z)}{w^{k+1}},\ z \in \mathfrak{M},\ |w| > 1.$$

Differentiating this equality $m + 1 \in \mathbb{N} := \{1, 2, ...\}$ times with respect to the variable z we have

$$\frac{(m+1)!\psi'(w)}{[\psi(w) - z]^{m+2}} = -\sum_{k=1}^{\infty} \frac{F_k^{(m+1)}(1/z)}{w^{k+1}},\ z \in \mathfrak{M},\ |w| > 1. \tag{2}$$

Now let G be a simply connected bounded domain in the complex plane $\mathbb{C}$ and let $A^p(G)$, $1 \le p < \infty$, be the Bergman space of analytic functions f in G, equipped with the norm

$$\|f\|_{A^p(G)} := \left(\iint_G |f(z)|^p \, d\sigma_z \right)^{1/p} < \infty,$$

where $d\sigma_z = dxdy$ is the 2-dimensional Lebesgue measure on G. We also denote $A(G) := A^1(G)$. Note that $A^p(G)$, $1 \le p < \infty$, is a Banach space.

As is well known, Faber polynomials and their various generalizations have played a significant role in approximation theory, particularly in the construction of approximation schemes (see, e.g., [32, 29, 30, 8, 3, 9, 10, 11, 12, 13, 14, 2, 1, 16, 17, 27, 21, 24, 18, 25, 26, 20, 19]). In addition, these polynomials have been effectively employed in solving various boundary value problems and basicity problems in domains of the complex plane (see, for example, [6, 7, 28]).

In mathematical literature, problems of maximal and simultaneous approximation problems, generally, have been studied separately. Mostly, maximal convergence problems have been studied on different spaces defined on various sets of complex plane (see, for example, [14, 21, 31]). The classical results in this direction can be found in the monographs [32, 29, 30, 8]. In this work, we will study maximal and simultaneous approximation property of Faber series in the norm of the Bergman spaces defined on the unbounded continuums of the complex plane. Here we investigate maximal and simultaneous approximation problems jointly, providing a unified approach to their analysis. We provide upper estimates for the approximation rates of functions from the Bergman space on a canonical domain $\mathfrak{M}_R$, and for those of their derivatives, by partial sums of the Faber series and its derivatives (simultaneous approximation) in the Bergman space norm on a set of $\mathfrak{M}$ (maximal convergence). These estimates depend on the best approximation numbers, the canonical domain parameters, the order of a derivative, and the degree of the partial sums. The considered problem and obtained results can be found in detail in the third section of this paper. In the case of bounded continuum this problem was investigated in [22] (nonweighted case) and [23] (weighted case). In the unbounded case, only maximal approximation property without considering simultaneous case was studied in [15].

2. Auxiliary Results

Let G be a bounded simply connected domain with quasiconformal boundary L. We recall that a Jordan curve L is called quasiconformal if it is the image of some quasiconformal homeomorphism of the complex plane onto itself that maps a circle onto L.

Without loss of generality, we assume that $0 \in G$.

If f is analytic and bounded in G, then the following integral representation holds:

$$f(z) = -\frac{1}{\pi} \iint\limits_{\overline{G}^c} \frac{(f \circ y)(\varsigma)\, y_{\overline{\varsigma}}(\varsigma)}{(\varsigma - z)^2} d\sigma_\varsigma, \quad z \in G. \tag{3}$$

This result was established by V.I. Belyi in [5] (see also [3, pp. 103-113]). Here, $y = y(\varsigma)$ denotes a quasiconformal reflection across the boundary L. This integral representation plays a crucial role in establishing direct approximation theorems in the uniform norm for domains with quasiconformal boundaries. It is worth noting that there also exists a canonical quasiconformal reflection $y = y(\varsigma)$ (see, for example, [3, pp. 107-109]), which is differentiable almost everywhere in $\mathbb{C}$, except possibly on L, and satisfies the following relations for any small fixed $\delta > 0$:

$$\begin{array}{lll} |y_\varsigma| + |y_{\overline{\varsigma}}| & \leq & c_1, \ \delta < |\varsigma| < 1/\delta; \quad if \quad \varsigma \notin L, \\ |y_\varsigma| + |y_{\overline{\varsigma}}| & < & c_2 \, |\varsigma|^{-2}; \ if \quad |\varsigma| \geq 1/\delta \ \text{ or } \ |\varsigma| \leq \delta, \end{array} \tag{4}$$

where $c_i = c_i(\delta)$, $i = 1, 2$, are some positive constants.

By considering only canonical quasiconformal reflections, Batchaev [4] showed that the above integral representation also holds in the space $A\,(G)$.

The following analogue of formula (3) for unbounded domains was established in [15]:

Theorem A *Let $G \ni \infty$ be an unbounded simply connect domain, with a quasiconformal boundary L, and let $f \in A(G)$. Then for every $z \in G$ the following integral representation holds:*

$$f(z) = -\frac{1}{\pi} \iint\limits_{\overline{G}^c} \frac{(f \circ y)(\varsigma)\, y_{\overline{\varsigma}}(\varsigma) \ z^2}{(\varsigma - z)^2 \, [y(\varsigma)]^2} d\sigma_\varsigma, \quad z \in G. \tag{5}$$

Since every analytic curve is also a quasiconformal curve, it follows from Theorem A that for any $f \in A^2(\mathfrak{M}_R)$ the following integral representation holds:

$$f(z) = -\frac{1}{\pi} \iint\limits_{\mathfrak{M}_R^c} \frac{(f \circ y_R)(\varsigma)\,(y_R)_{\overline{\varsigma}}(\varsigma) \ z^2}{(\varsigma - z)^2 \, [y_R(\varsigma)]^2} d\sigma_\varsigma, \quad z \in \mathfrak{M}_R. \tag{6}$$

Here, y_R denotes a quasiconformal reflection with respect to the level line L_R. The integral representation given in (6) can be equivalently written as

$$\frac{f(z)}{z^2} = -\frac{1}{\pi} \iint\limits_{\mathfrak{M}_R^c} \frac{(f \circ y_R)(\varsigma)}{[y_R(\varsigma)]^2} \frac{(y_R)_{\overline{\varsigma}}(\varsigma)}{(\varsigma - z)^2} d\sigma_\varsigma, \quad z \in \mathfrak{M}_R \tag{7}$$

Now, we denote by $A_0(\mathfrak{M}_R)$ the class of functions f_0 that can be represented in the form $f_0(z) = f(z)/z^2$, for some $f \in A(\mathfrak{M}_R)$. In this class the following result holds:

Theorem B *If $f_0 \in A_0(\mathfrak{M}_R)$, $R > 1$, then*

$$f_0(z) = -\frac{1}{\pi} \iint\limits_{\mathfrak{M}_R^c} (f_0 \circ y_R)(\varsigma) \, \frac{(y_R)_{\overline{\varsigma}}(\varsigma)}{(\varsigma - z)^2} d\sigma_\varsigma, \quad z \in \mathfrak{M}_R. \tag{8}$$

Proof. Since $f_0 \in A_0(\mathfrak{M}_R)$, there exists a function $f \in A(\mathfrak{M}_R)$ such that $f_0(z) = f(z)/z^2$, $z \in \mathfrak{M}_R$, for some $f \in A(\mathfrak{M}_R)$. Consequently,

$$(f_0 \circ y_R)(\varsigma) = \frac{(f \circ y_R)(\varsigma)}{[y_R(\varsigma)]^2}.$$

Substituting this identity into formula (7) gives the representation stated in (8). ◀

Differentiating formula (8) $m \in \mathbb{N}$ times, we obtain:

$$f_0^{(m)}(z) = -\frac{(m+1)!}{\pi} \iint\limits_{\mathfrak{M}_R^c} \frac{(f_0 \circ y_R)(\varsigma)\,(y_R)_{\bar{\varsigma}}(\varsigma)}{(\varsigma - z)^{m+2}} d\sigma(\varsigma), \; z \in \mathfrak{M}_R$$

Now, by substituting $\varsigma = \psi(w)$ we derive that for all $z \in \mathfrak{M}_R$

$$\begin{aligned} & f_0^{(m)}(z) \\ = \; & -\frac{(m+1)!}{\pi} \iint\limits_{|w>R|} \frac{(f_0 \circ y_R \circ \psi)(w)\,((y_R)_{\bar{\varsigma}} \circ \psi)(w)\;\; \psi'(w)\,\overline{\psi'(w)} d\sigma(\varsigma)}{[\psi(w) - z]^{m+2}}. \end{aligned} \quad (9)$$

Hence, considering (2) in (9), we have the series representation

$$f_0^{(m)}(z) = \sum_{k=1}^{\infty} a_k(f_0) F_k^{(m+1)}(1/z), \; z \in \mathfrak{M}_R, \; m \in \mathbb{N}, \quad (10)$$

where

$$a_k(f_0) := -\frac{1}{\pi} \iint\limits_{|w|>R} \frac{(f_0 \circ y_R)(\psi(w))\,((y_R)_{\bar{\varsigma}} \circ \psi))(w)\,\overline{\psi'(w)}}{w^{k+1}} d\sigma(w), \; k = 1, \; 2, \ldots .$$

Note that if $f_0 \in A_0^2(\mathfrak{M}_R)$, then the series (10) converges uniformly on the compact subsets of the canonical domain $\mathfrak{M}_R$.

Now let

$$S_n^{(m)}(f_0, 1/z) := \sum_{k=1}^{n} a_k(f_0) F_k^{(m+1)}(1/z), \; z \in \mathfrak{M}_R, \quad m \in \mathbb{N}, \; n = 1, 2, \ldots$$

be the n^{th} partial sum of the series (10).

Lemma 1. *Let* $f_0 \in A_0^2(\mathfrak{M}_R)$, $R > 1$, *and* y_R *be a canonical quasiconformal reflection across the level line* L_R *of the unbounded continuum* $\mathfrak{M}$. *Then*

$$\iint\limits_{\mathfrak{M}_R^c} \left|(f_0 \circ y_R)(\varsigma)\,(y_R)_{\bar{\zeta}}(\varsigma)\right|^2 d\sigma(\varsigma) \le \frac{\|f_0\|^2_{A^2(\mathfrak{M}_R)}}{1-k_R^2},$$

where $k_R = (K_R - 1)\,(K_R + 1)$ *and* K_R *is a quasiconformality coefficient of the level line* L_R.

Proof. It is seen that the function $\overline{y}_R(\varsigma)$ is a canonical K quasiconformal mapping of the extended complex plane onto itself. Then we have $\left|(\overline{y}_R)_{\bar{\varsigma}}\right| \diagup \left|(\overline{y}_R)_{\varsigma}\right| \le k$ and $\left|(\overline{y}_R)_{\varsigma}\right|^2 - \left|(\overline{y}_R)_{\bar{\varsigma}}\right|^2 > 0$. Considering the relations $\left|(\overline{y}_R)_{\bar{\varsigma}}\right| = \left|(y_R)_{\varsigma}\right|$ and $\left|(\overline{y}_R)_{\varsigma}\right| = \left|(y_R)_{\bar{\varsigma}}\right|$, we see that $\left|(y_R)_{\varsigma}\right| \diagup \left|(y_R)_{\bar{\varsigma}}\right| \le k$ and $\left|(y_R)_{\bar{\varsigma}}\right|^2 - \left|(y_R)_{\varsigma}\right|^2 > 0$. Then

$$\begin{aligned}
&\iint\limits_{\mathfrak{M}_R^c} \left|(f_0 \circ y_R)(\varsigma)\,(y_R)_{\bar{\zeta}}(\zeta)\right|^2 d\sigma(\varsigma) \\
&= \iint\limits_{\mathfrak{M}_R^c} |(f_0 \circ y_R)(\varsigma)|^2 \left(1 - \left|(y_R)_{\varsigma}\right|^2 \diagup \left|(y_R)_{\bar{\varsigma}}\right|^2\right)\left(\left|(y_R)_{\bar{\varsigma}}\right|^2 - \left|(y_R)_{\varsigma}\right|^2\right) d\sigma(\varsigma) \\
&\le \frac{1}{1-k_R^2} \iint\limits_{\mathfrak{M}_R^c} |(f_0 \circ y_R)(\varsigma)|^2 \left(\left|(y_R)_{\bar{\varsigma}}\right|^2 - \left|(y_R)_{\varsigma}\right|^2\right) d\sigma(\varsigma).
\end{aligned}$$

Since $\left|(y_R)_{\varsigma}\right|^2 - \left|(y_R)_{\bar{\varsigma}}\right|^2$ is the Jacobian of y_R, substituting ς by y_R on the right-hand side of the last inequality, we obtain

$$\iint\limits_{\mathfrak{M}_R^c} \left|(f_0 \circ y_R)(\varsigma)\,(y_R)_{\bar{\zeta}}(\zeta)\right|^2 d\sigma(\varsigma) \le \frac{1}{1-k_R^2}\,\frac{\|f_0\|^2_{A^2(\mathfrak{M}_R)}}{1-k_R^2}.$$

◄

Lemma 2. *If* $F_n(1/z)$, $n \in N$, *is the Faber polynomial of degree* n *with respect to* $1/z$, *then* $a_k\left(F_n'\right) = 0$ *for every* $k \ge n+1$.

Proof. In fact, for every $k \ge n+1$ and $R > 1$ we have

$$a_k\left(F_n'\right) = \frac{1}{\pi} \iint\limits_{|w|>R} \frac{\left(F_n' \circ y_R\right)(\psi(w))\left(\left(y_R\right)_{\bar{\varsigma}} \circ \psi\right)(w)\,\overline{\psi'(w)}}{w^{k+1}} d\sigma(w)$$

$$= \frac{1}{\pi} \iint\limits_{|w|>R} -\frac{\partial}{\partial \overline{w}} \left[\frac{(F_n \circ y_R)(\psi(w))}{w^{k+1}} \right] d\sigma(w)$$

$$= \frac{1}{2\pi i} \int\limits_{|w|=R} \frac{F_n(1/\psi(w))}{w^{k+1}} dw.$$

Since by (1)

$$F_n(1/z) = [\varphi(z)]^n + Q_n(z), \ z \in B,$$

where $Q_n(z)$ is an analytic function in $B = \mathfrak{M}^c$ and $Q_n(0) = const$, we get

$$a_k\left(F_n'\right) = \frac{1}{2\pi i} \int\limits_{|w|=R} \frac{w^n + Q_n(\psi(w))}{w^{k+1}} dw$$
$$= \frac{1}{2\pi i} \int\limits_{|w|=R} \frac{w^n}{w^{k+1}} dw + \frac{1}{2\pi i} \int\limits_{|w|=R} \frac{Q_n(\psi(w))}{w^{k+1}} dw.$$

Here the first integral on the right-hand side vanishes by the Cauchy integral formula.

Since the function $Q_n(z)$ contains only non-negative powers of z and the conformal mapping $\psi(w)$ admits the Laurent expansion

$$\psi(w) = \frac{\alpha_1}{w} + \frac{\alpha_2}{w^2} + \frac{\alpha_3}{w^3} + ...$$

in a neighborhood of infinity, the second integral also vanishes by the Cauchy integral theorem for unbounded domains. ◄

Lemma 3. *Let $F_k(1/z)$, $k = 1, 2, 3, ...$ be the Faber polynomials of order k with respect to $1/z$ in an unbounded continuum $\mathfrak{M}$. Let also $R > 1$ and $1 < r < R$. Then for every $m = 1, 2, ...$the following inequality holds:*

$$\sum_{k=n+1}^{\infty} \frac{\left\| F_k^{(m+1)}(1/z) \right\|^2_{A^2(\mathfrak{M})}}{kR^{2k}} \le c(r, m, \mathfrak{M}) \left(\frac{r}{R}\right)^{2(n+1)} \frac{[(n+m)!]^2}{(n!)^2 (n+1)}.$$

Proof. Consider the Faber polynomial of degree k with respect to $1/z$:

$$F_k(1/z) = \frac{\beta_1}{z} + \frac{\beta_2}{z^2} + \frac{\beta_3}{z^3} ... + \frac{\beta_k}{z^k}, \quad k = 1, 2, ... \ .$$

Differentiating by z we get:

$$\begin{aligned} F_k^{'}(1/z) &= -\frac{\beta_1}{z^2} - \frac{2\beta_2}{z^3} - \frac{3\beta_3}{z^4} - ... - \frac{k\beta_k}{z^{k+1}} \\ &= \frac{k}{z^{k+1}}(c_1^{(1)}z^{k-1} + c_2^{(1)}z^{k-2} + c_3^{(1)}z^{k-3} ... + c_k^{(1)}), \quad k = 1, 2, ... \end{aligned}$$

Similarly, the second derivative is:

$$\begin{aligned} F_k^{''}(1/z) &= \frac{1\cdot 2\beta_1}{z^3} + \frac{2\cdot 3\beta_2}{z^4} + \frac{3\cdot 4\beta_3}{z^5} + ... + \frac{k\,(k+1)\,\beta_k}{z^{k+2}} \\ &= \frac{k\,(k+1)}{z^{k+2}}(c_1^{(2)}z^{k-1} + c_2^{(2)}z^{k-2} + c_3^{(2)}z^{k-3} ... + c_k^{(2)}). \end{aligned}$$

For higher derivatives of order $m > 2$:

$$F_k^{(m)}(1/z) = (-1)^m \left[\frac{1\cdot 2\cdot 3\cdot ...\cdot (1+m-1)\,\beta_1}{z^{1+m}} + \frac{2\cdot 3\cdot ...\cdot (2+m-1)\,\beta_2}{z^{2+m}} \right.$$

$$\left. + \frac{3\cdot 4\,(3+m-1)\,\beta_3}{z^{3+m}} + ... + \frac{k\,(k+1)\cdot ...\cdot (k+m-1)\,\beta_k}{z^{k+m}} \right]$$

$$= (-1)^m \frac{k\,(k+1)\cdot ...\cdot (k+m-1)\beta_k}{z^{k+m}} (c_1^{(m)}z^{k-1} + c_2^{(m)}z^{k-2} + ... + c_k^{(m)})$$

$$= \frac{(k+m-1)!}{(k-1)!z^{k+m}} \lambda_m(z),$$

where

$$\lambda_m(z) := (-1)^m \beta_k (c_1^{(m)}z^{k-1} + c_2^{(m)}z^{k-2} + ... + c_k^{(m)}), \quad m = 1, 2, ...$$

are analytic functions in $\mathfrak{M}^c$. Using the conformal mapping substitution

$$\begin{aligned} z = \psi(w) = & \quad \frac{\alpha_1}{w} + \frac{\alpha_2}{w^2} + \frac{\alpha_3}{w^3} + ... \\ = & \quad \frac{1}{w}(\alpha_1 + \frac{\alpha_2}{w^1} + \frac{\alpha_3}{w^2} + ...) =: \frac{1}{w}\mu\,(w)\,, \end{aligned}$$

where $\mu(w)$ is analytic and nonzero at infinity, we rewrite

$$\begin{aligned} F_k^{(m)}(1/\psi(w)) &= \frac{(k+m-1)!}{(k-1)!\,[\mu(w)/w]^{k+m}} \lambda_m(\psi(w)) \\ &=: \frac{(k+m-1)!}{(k-1)!} \sigma_{k,m}(w), \end{aligned}$$

with

$$\sigma_{k.m}(w) := \frac{w^{k+m}\lambda_m(\psi(w))}{[\mu(w)]^{k+m}}$$

analytic function in $\overline{D}^c$, having a pole of order $k+m$ at infinity.

Using the Laurent expansion of $\sigma_{k.m}(w)$ in some neighborhood of ∞, we have

$$F_k^{(m)}(1/\psi(w)) = \frac{(k+m-1)!}{(k-1)!}\left(\sum_{j=0}^{k+m}\gamma_j^{(k,m)}w^j + \sum_{j=1}^{\infty}\delta_j^{(k,m)}/w^j\right), \tag{11}$$

where the coefficients $\gamma_j^{(k,m)}$, $j = 0, 1, ..., k+m$, can be defined as

$$\gamma_j^{(k,m)} = \frac{1}{2\pi i}\int_{|w|=r}\frac{\sigma_{k,m}(w)dw}{w^{j+1}}$$

for some $r \in (1, R)$ and can be estimated as

$$\left|\gamma_j^{(k,m)}\right| \leq \frac{r^{k+m}M(r, \mathfrak{M}, m)}{r^j}. \tag{12}$$

Since the function $F_k^{(m)}(1/z)$ is analytic on $\mathfrak{M}$ with the connected complement $\mathcal{B}$, considering the representation (11) and using the area theorem due to Lebedev-Milin (given in [30, p.170]), we see that the area of the Riemann surface onto which the function $F_k^{(m)}(1/z)$ maps the continuum $\mathfrak{M}$ can be given by the formula

$$S_{k,m} = \pi\left[\frac{(k+m-1)!}{(k-1)!}\right]^2\left(\sum_{j=0}^{k+m}j\left|\gamma_j^{(k,m)}\right|^2 - \sum_{j=1}^{\infty}j\left|\delta_j^{(k,m)}\right|^2\right) \geq 0,$$

and hence using (12) we have

$$\begin{aligned}
S_{k,m} &\leq \pi\left[\frac{(k+m-1)!}{(k-1)!}\right]^2\sum_{j=0}^{k+m}j\left|\gamma_j^{(k,m)}\right|^2 \\
&\leq \pi\left[\frac{(k+m-1)!}{(k-1)!}\right]^2 M^2(r, \mathfrak{M}, m)\sum_{j=1}^{k+m}jr^{2(k+m-j)} \\
&= \pi\left[\frac{(k+m-1)!}{(k-1)!}\right]^2 M^2(r, \mathfrak{M}, m)\, r^{2(k+m)}\sum_{j=1}^{k+m}jr^{-2j)}
\end{aligned}$$

$$\leq \quad c(r,m,\mathfrak{M})M^2\left(r,\mathfrak{M},m\right)r^{2k}\left[\frac{(k+m-1)!}{(k-1)!}\right]^2. \qquad (13)$$

On the other hand, the area of the Riemann surface onto which the function $w=F_k^{(m)}(1/z)$ maps the continuum $\mathfrak{M}$ is defined by the formula

$$S_{k,m} =_{\mathfrak{M}} \left|F_k^{(m+1)}(1/z)\right|^2 dxdy = \left\|F_k^{(m+1)}(1/z)\right\|^2_{A^2(\mathfrak{M})}.$$

Hence, considering (13), we have the desired inequality

$$\sum_{k=n+1}^{\infty}\frac{\left\|F_k^{(m+1)}\left(1/z\right)\right\|^2_{A^2(\mathfrak{M})}}{kR^{2k}} \leq c_1(r,m,\mathfrak{M})\sum_{k=n+1}^{\infty}\left(\frac{r}{R}\right)^{2k}\frac{\left[\frac{(k+m-1)!}{(k-1)!}\right]^2}{k}$$

$$= c_2(r,m,M)\left(\frac{r}{R}\right)^{2(n+1)}\frac{\left[\frac{(n+m)!}{n!}\right]^2}{n+1} = c(r,m,M)\left(\frac{r}{R}\right)^{2(n+1)}\frac{\left[(n+m)!\right]^2}{\left(n!\right)^2(n+1)}.$$

◄

3. Main Result

In this section, we will estimate the approximation error $\left|f_0^{(m)}(z) - S_n^{(m)}(f_0,1/z)\right|$, $m\in\mathbb{N}$, in the Bergman space $A^2(\mathfrak{M})$, in terms of the best approximation number

$$E_n(f_0,\mathfrak{M}_R) := \inf_{p\in\Pi_n}\|f_0-p\|_{A^2(\mathfrak{M}_R)},$$

where Π_n is the class of algebraic polynomials in the variable $1/z$ of degree at most n, and the parameters involve n,m, $\mathfrak{M}$ and R.

The main result of this paper is the following theorem, which characterizes the maximal-simultaneous approximation property of the Faber partial sums:

Theorem 1. *If $f_0\in A_0^2(\mathfrak{M}_R)$ for some $R>1$ and*

$$S_n^{(m)}(f_0,1/z) = \sum_{k=1}^{n}a_k(f_0)F_k^{(m+1)}\left(1/z\right),\ z\in\mathfrak{M},\ n,m=1,2,...$$

is the n^{th} partial sum of the generalized Faber series

$$f_0^{(m)}(z) = \sum_{k=1}^{\infty}a_k(f_0)F_k^{(m+1)}\left(1/z\right),\ z\in\mathfrak{M}$$

of $f_0^{(m)}$, *then*

$$\left\| f_0^{(m)}(z) - S_n^{(m)}(f_0, 1/z) \right\|_{A^2(\mathfrak{M})} \leq c(r, m, \mathfrak{M}) E_n(f_0, \mathfrak{M}_R) \left(\frac{r}{R}\right)^{n+1} \frac{(n+m)!}{(n!)\,(n+1)}.$$

Proof. Let P_n^* be the best approximation polynomial with respect to $1/z$ for $f_0 \in A^2(\mathfrak{M}_R)$ of order $\leq n$, in the norm $\|\cdot\|_{A^2(\mathfrak{M}_R)}$, i.e.,

$$\|f_0 - P_n^*\|_{A^2(\mathfrak{M}_R)} = E_n(f_0, \mathfrak{M}_R) := \inf_{p \in \Pi_n} \|f_0 - p\|_{A^2(\mathfrak{M}_R)},$$

where Π_n is the class of algebraic polynomials of degree at most n with respect to $1/z$.

Then for every $z \in \mathfrak{M}$ and $n, m = 1, 2, ...$ using Lemma 2 we have

$$\left| R_n(z, f_0^{(m)}) \right| =$$

$$:= \left| f_0^{(m)}(z) - S_n^{(m)}(f_0, 1/z) \right| = \left| \sum_{k=n+1}^{\infty} a_k(f_0)\, F_k^{(m+1)}(1/z) \right|$$

$$= \frac{1}{\pi} \left| \sum_{k=n+1}^{\infty} \iint_{|w|>R} (f_0 \circ y_R)(\psi(w))\, ((y_R)_{\overline{\varsigma}} \circ \psi))(w)\, \overline{\psi'(w)} \frac{F_k^{(m+1)}(1/z)}{w^{k+1}} d\sigma(w) \right|$$

$$= \frac{1}{\pi} \left| \sum_{k=n+1}^{\infty} \iint_{|w|>R} [(f_0 - P_n^*) \circ y_R](\psi(w)) \left[(y_R)_{\overline{\varsigma}} \circ \psi) \right](w)\, \overline{\psi'(w)} \frac{F_k^{(m+1)}(1/z)}{w^{k+1}} d\sigma(w) \right|,$$

and then by the Hölder inequality

$$\left| R_n(z, f^{(m)}) \right|^2 \leq$$

$$\leq \frac{1}{\pi^2} \iint_{|w|>R} \left| (f_0 - P_n^*) \circ y_R [\psi(w)]\ \overline{\psi'(w)}\ (y_R)_{\overline{\varsigma}} [\psi(w)] \right|^2 d\sigma(w)$$
$$\cdot \iint_{|w|>R} \left| \sum_{k=n+1}^{\infty} \frac{F_k^{(m+1)}(z)}{w^{k+1}} \right|^2 d\sigma(w). \qquad (14)$$

By Lemma 1,

$$\iint\limits_{|w|>R} \left|(f_0 - P_n^*) \circ y_R\left[\psi(w)\right] \overline{\psi'(w)}\ \left(y_R\right)_{\overline{\varsigma}}\left[\psi(w)\right]\right|^2 d\sigma(w)$$

$$= \iint\limits_{\mathfrak{M}_R^c} \left|\left[(f_0 - P_n^*) \circ y_R\right](\varsigma)\left(y_R\right)_{\overline{\varsigma}}(\varsigma)\right|^2 d\sigma(\varsigma) \le \frac{\|f_0 - P_n^*\|^2_{A^2(\mathfrak{M}_R)}}{1-k_R^2}. \quad (15)$$

On the other hand, since

$$\iint\limits_{|w|>R} \frac{d\sigma(w)}{w^{k+1}\overline{w}^{l+1}} = \begin{cases} \frac{\pi}{kR^{2k}}, & k=l \\ 0, & k \ne l \end{cases},$$

we have

$$\iint\limits_{|w|>R} \left|\sum_{k=n+1}^{\infty} \frac{F_k^{(m+1)}(z)}{w^{k+1}}\right|^2 d\sigma(w) = \pi \sum_{k=n+1}^{\infty} \frac{\left|F_k^{(m+1)}(z)\right|^2}{kR^{2k}}, \quad (16)$$

and then, using (15) and (16) in (14), we get

$$\left|R_n(z, f_0^{(m)})\right|^2 \le \frac{1}{\pi}\frac{\|f_0 - P_n^*\|^2_{A^2(\mathfrak{M}_R)}}{1-k_R^2} \sum_{k=n+1}^{\infty} \frac{\left|F_k^{(m+1)}(z)\right|^2}{kR^{2k}}.$$

Integrating both sides of this inequality over $\mathfrak{M}$ and applying Lemma 3, after some calculation we get

$$\begin{aligned} \left\|R_n(z, f_0^{(m)})\right\|^2_{A^2(\mathfrak{M})} &\le \frac{1}{\pi}\frac{\left[E_n(f_0, \mathfrak{M}_R)\right]^2}{1-k_R^2} \sum_{k=n+1}^{\infty} \frac{\left\|F_k^{(m+1)}(1/z)\right\|^2_{A^2(\mathfrak{M})}}{kR^{2k}} \\ &\le \frac{1}{\pi}\frac{c(r,m,\mathfrak{M})\left[E_n(f_0, \mathfrak{M}_R)\right]^2}{1-k_R^2}\left(\frac{r}{R}\right)^{2(n+1)} \frac{\left[(n+m)!\right]^2}{(n!)^2(n+1)}, \end{aligned}$$

and hence

$$\left\|R_n(z, f_0^{(m)})\right\|_{A^2(\mathfrak{M})} \le c(r,m,\mathfrak{M}) E_n(f_0, \mathfrak{M}_R)\left(\frac{r}{R}\right)^{n+1} \frac{(n+m)!}{n!\sqrt{n+1}}.$$

◀

Remark 1. *Let us emphasize that if the boundary of the continuum* $\mathfrak{M}$ *is sufficiently smooth, then one can take* $r = 1$ *in the statement of the above theorem.*

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Received 30 July 2025
Accepted 19 October 2025