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Inverse Boundary Value Problem for Two-Dimensional Pseudo Parabolic Equation of Time-Fractional Order with Additional Integral Condition

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Abstract. Inverse boundary value problem for two-dimensional pseudo parabolic equation of third order with additional integral condition is considered. We first reduce our problem to some equivalent (in some sense) one. Using the Fourier method, the equivalent problem, in turn, is reduced to the system of integral equations. Then, using contraction mapping method, we prove the existence and uniqueness for the solution of the system of integral equations, which is also a unique solution of the equivalent problem. Finally, using equivalence, we prove the existence and uniqueness for the classical solution of the original problem.

Key Words and Phrases: inverse boundary value problem, two-dimensional pseudo parabolic equation of third order, Fourier method, classical solution.

2010 Mathematics Subject Classifications: 35R30, 35K10, 35A09

1. Introduction

By the inverse problem for partial differential equations, we mean a problem that requires to find, along with a solution itself, the right-hand side and (or) some coefficient(s) of the equation. Inverse problems arise in many fields of human activities, such as seismology, mineral exploration, biology, medicine, quality control of industrial products, etc. which makes them one of the most important problems in today's mathematics. If an inverse problem requires to find not only the solution itself, but also the right-hand side of the equation, then such an inverse problem is linear. And if it requires to find both the solution and at least one of the coefficients, then such an inverse problem is nonlinear. Many mathematicians have studied various inverse problems for some types of partial

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differential equations, such as Tikhonov [1], Lavrentiev [2,3], Ivanov [4] and their students. More details about these problems can be found in the monograph by Denisov [5].

Inverse problems for pseudo-parabolic equations of third-order were studied in [6,7,8]. The existence and uniqueness of the solution of the inverse problem for the third order pseudoparabolic equation with integral over-determination condition was studied in [9]. Khompysh [10] investigated the reconstruction of unknown coefficient in pseudo-parabolic inverse problem with the integral over determination condition and studied the uniqueness and existence of solution by means of the method of successive approximations. Ramazanova et al. [11] theocratically studied the fourth-order inverse problem with nonlocal integral condition. Studies of wave propagation in cold plasma and magnetohydrodynamics also reduce to the partial differential equations of third-order, see [12]. For nonlocal boundary value problems (including integral conditions) for partial differential equations of third-order see, for example, [13,14].

It should be noted that boundary value problems with integral conditions are of particular interest. From physical considerations, the integral conditions are completely natural, and they arise in mathematical modelling in cases where it is impossible to obtain information about the process occurring at the boundary of domain.

Recently, some inverse problems concerning the determination time-dependent coefficients have been numerically solved [15–21]. In [22], free boundary coming from two new scenarios, aggregation processes and nonlocal diffusion have been considered. Snitko [23], theoretically, and Huntul [24] numerically investigated the inverse problem of determining the time-dependent reaction coefficient in a two-dimensional parabolic problem. Furthermore, Huntul et al. [25,26] numerically studied the inverse problems of reconstructing the unknown coefficients in a third-order pseudo-parabolic equation. In [29], an inverse problem of determining a time-dependent source coefficient in a one-dimensional time-fractional diffusion equation is investigated with initial-boundary and overdetermination conditions. And in [30] nonlocal problems for fractional subdiffusion equations in time are considered.

In this work, using Fourier method and contraction mapping principle, we prove the existence and uniqueness of the solution of the inverse boundary value problem for two-dimensional pseudo parabolic equation of time-fractional order with additional integral condition

2. Problem statement and its reduction to the equivalent problem

Let $D_T = Q_{xy} \times \{0 \le t \le T\}$, where $Q_{xy} = \{(x,y) : 0 \le x \le 1,\ 0 \le y \le 1\}$. Also, let $f(x,y,t)$, $\varphi(x,y)$, $\omega(x,y)$, $h(t)$ be the given functions defined for $x \in [0,1]$, $y \in [0,1]$, $t \in [0,T]$. Consider the following inverse boundary value problem: find a pair $\{u(x,t)\,,\ p(t)\}$ of functions $u(x,t)$, $p(t)$ which satisfy the equation

$$ {}_C D_{0t}^{\alpha} u(x,y,t) - {}_C D_{0t}^{\alpha} \Delta u(x,y,t) - \Delta u(x,y,t) = p(t)u(x,y,t) + f(x,y,t)\,, \quad (1) $$

initial condition

$$ u(x,y,0) = \varphi(x,y) \ \ (0 \le x \le 1,\ 0 \le y \le 1)\,, \quad (2) $$

boundary conditions

$$ u_x(0,y,t) = u(1,y,t) = 0 \ \ (0 \le y \le 1,\, 0 \le t \le T)\,, \quad (3) $$

$$ u(x,0,t) = u_y(x,1,t) = 0 \ (0 \le y \le 1,\ \ 0 \le t \le T)\,, \quad (4) $$

and the additional condition

$$ \int_0^1 \int_0^1 \omega(x,y)u(x,y,t)dxdy = h(t) \ (0 \le t \le T), \quad (5) $$

where

$$ {}_C D_{0t}^{\alpha} u(x,y,t) = \frac{1}{\Gamma(1-\alpha)} \int_0^t (t-\tau)^{-\alpha} \frac{\partial u(x,y,\tau)}{\partial \tau} d\tau, \alpha \in (0.1). $$

The notation $\Gamma\ (\alpha)$ in the last equation represents Euler's gamma function.

Denote

$$ \tilde{C}^{2,2,1}(D_T) = \{u(x,y,t) : u(x,y,t) \in C^{2,2,0}(D_T), $$

$$ {}_C D_{0t}^{\alpha} u(x,t), {}_C D_{0t}^{\alpha} u_{xx}(x,t), {}_C D_{0t}^{\alpha} u_{yy}(x,t) \in C(D_T)\}\,. $$

Definition 1. *By the classical solution of the inverse boundary value problem* (1)-(4)*, we mean a pair* $\{u(x,y,t), p(t)\,\}$ *of functions* $u(x,t)$*,* $p(t)$ *such that* $u(x,y,t) \in \tilde{C}^{2,2,1}(D_T)$*,* $p(t) \in C[0,T]$ *and the relations* (1)-(5) *are satisfied in the usual sense.*

The following theorem is true.

Theorem 1. *Let* $\varphi(x,y) \in C(Q_{xy})$, $\omega(x,y) \in C(Q_{xy})$, $f(x,y,t) \in C(D_T)$, $h(t) \in C[0,T]$, ${}_CD_{0t}^{\alpha}h(t) \in C[0,T]$, $h(t) \neq 0 \ (0 \leq t \leq T)$, $\alpha > 0, \beta > 0$, *and the coherence condition*

$$\int_0^1 \int_0^1 \omega(x,y)\varphi(x,y)dxdy = h(0)$$

be satisfied. Then the problem of finding the classical solution of the problem (1)-(5) *is equivalent to the one of determining the functions* $u(x,y,t) \in \tilde{C}^{2,2,1}(D_T)$, $p(t) \in C[0,T]$ *from the relations* (1)-(4) *such that*

$${}_CD_{0t}^{\alpha}h(t) - {}_CD_{0t}^{\alpha}\int_0^1 \int_0^1 \omega(x,y)\Delta u(x,y,t)dxdy - \int_0^1 \int_0^1 \omega(x,y)\Delta u(x,y,t)dxdy =$$

$$= p(t)h(t) + \int_0^1 \int_0^1 \omega(x,y)f(x,y,t)dxdy \qquad (0 \leq t \leq T). \tag{6}$$

Proof. Let $\{u(x,y,t), p(t))\}$ be a classical solution of the problem (1)- (5). Multiplying the both sides of equation (1) by a function $\omega(x,y)$ and integrating from 0 to 1 with respect to x and y gives

$${}_CD_{0t}^{\alpha}\int_0^1 \int_0^1 \omega(x,y)u(x,y,t)dxdy -$$

$$-{}_CD_{0t}^{\alpha}\int_0^1 \int_0^1 \omega(x,y)\Delta u(x,y,t)dxdy - \int_0^1 \int_0^1 \omega(x,y)\Delta u(x,y,t)dxdy =$$

$$= p(t)\int_0^1 \int_0^1 \omega(x,y)u(x,y,t)dxdy + \int_0^1 \int_0^1 \omega(x,y)f(x,y,t)dxdy \, (\, 0 \leq t \leq T). \tag{7}$$

Now, taking ${}_CD_{0t}^{\alpha}h(t) \in C[0,T]$ and differentiating (5), we have

$${}_CD_{0t}^{\alpha}\int_0^1 \int_0^1 \omega(x,y)u(x,y,t)dxdy = {}_CD_{0t}^{\alpha}h(t) \ (0 \leq t \leq T). \tag{8}$$

By (5) and (8), it follows from (7) that the relation (6) is valid.

Now let's assume that $\{u(x,t), p(t)\}$ is a soluton of the problem (1)-(4), (6). Then from (6) and (7) we obtain

$${}_CD_{0t}^{\alpha}\left(\int_0^1 \int_0^1 \omega(x,y)u(x,y,t)dxdy - h(t)\right) =$$

$$= p(t)\left(\int_0^1 \int_0^1 \omega(x,y)u(x,y,t)dxdy - h(t)\right)$$

$$(\, 0 \leq t \leq T \,)\,. \tag{9}$$

By (2) and the coherence condition $\int_0^1 \int_0^1 \omega(x,y)\varphi(x,y)dxdy = h(0)$, we have

$$\int_0^1 \int_0^1 \omega(x,y)u(x,y,0)dxdy - h(0) = \int_0^1 \int_0^1 \omega(x,y)\varphi(x,y)dxdy - h(0) \, = 0. \tag{10}$$

From (9), taking into account (10) [27], it is clear that condition (5) is also satisfied.

The theorem is proved. ◄

3. The proof of the existence and uniqueness of the classical solution of the inverse boundary value problem

We will search for the first component $u(x,y,t)$ of the solution $\{u(x,y,t), p(t)\,\}$ of the problem (1)-(4), (6) in the following form:

$$u(x,y,t) = \sum_{n=1}^{\infty}\sum_{k=1}^{\infty} u_{k,n}(t) \cos\lambda_k x \sin\gamma_n y \ \ , \tag{11}$$

where

$$\lambda_k = \frac{\pi}{2}(2k-1) \ (k = 1,2,...), \quad \gamma_n = \frac{\pi}{2}(2n-1) \ (n = 1,2,...)\,,$$

$$u_{k,n}(t) = 4\int_0^1 \int_0^1 u(x,y,t) \cos\lambda_k x \sin\gamma_n y dxdy \ \ (k = 1,2,...; \ n = 1,2,...). \tag{12}$$

Using the method of separation of variables to define the sought coefficients $u_{k,n}(t) \ \ (k = 1,2,...; n = 1,2,...)$ of the function $u(x,t)$, from (1), (2) we obtain

$${}_C D_{0t}^{\alpha} u_{k,n}(t) + \frac{\mu_{k,n}^2}{1+\mu_{k,n}^2}\, u_{k,n}(t) =$$

$$= \frac{1}{1+\mu_{k,n}^2} F_{k,n}(t;u,p) \ (k = 1,2,...,n = 1,2,...; \ 0 \leq t \ \leq T)\,, \tag{13}$$

$$u_{k,n}(0) = \varphi_{k,n}(k = 1,2,...; n = 1,2,...)\,, \tag{14}$$

where

$$\mu_{k,n}^2 = \lambda_k^2 + \gamma_n^2 \ (k = 1,2,...; n = 1,2,...),$$

$$F_{k,n}(t;u,p) = f_{k,n}(t) + p(t)u_{k,n}(t)\,(k = 1,2,...; n = 1,2,...)\,,$$

$$f_{k,n}(t) = 4\int_0^1\int_0^1 f(x,y,t)\cos\lambda_k x\sin\gamma_n ydxdy \;\; (k=1,2,...;\; n=1,2,...),$$

$$\varphi_{k,n} = 4\int_0^1\int_0^1 \varphi(x,y)\cos\lambda_k x\sin\gamma_n ydxdy \;\; (k=1,2,...;\; n=1,2,...).$$

Solving the problem (13), (14), we find [27]

$$u_{k,n}(t) = \varphi_{k,n}E_\alpha(-\beta_{k,n}^2\, t^\alpha)+$$

$$+\frac{1}{1+\mu_{k,n}^2}\int_0^t (t-\tau)^{\alpha-1}E_{\alpha,\alpha}(-\beta_{k,n}^2(t-\tau)^\alpha)F_{k,n}(\tau;\, u,\, p)d\tau$$

$$(k=1,2,...;n=1,2,...), \tag{15}$$

where $E_\alpha(-\beta_{k,n}^2 t^\alpha)$ and $E_{\alpha,\alpha}(-\beta_{k,n}^2 t-\tau)^\alpha)$ are the Mittag- Leffler functions, $\beta_{k,n}^2 = \frac{\mu_{k,n}^2}{1+\mu_{k,n}^2}$ $(k=1,2,...;n=1,2,...)$.

Definition 2 ([27])**.** *The generalized Mittag-Leffler function is defined by*

$$E_{\alpha,\beta}(z) = \sum_{k=0}^{\infty}\frac{z^k}{\Gamma(\alpha k+\beta)}\;, z\in C,$$

where α*and* $\beta \in R$.

Remark 1 ([28])**.** *Let* $0<\alpha<2$, *and* $\beta\in R$ *be arbitrary numbers. If* $\frac{\pi\alpha}{2}<\mu<\min\{\pi,\pi\alpha\}$ *, then there exists a constant* $C_{\alpha,\beta}$ *such that*

$$|E_{\alpha,\beta}(z)| \le \frac{C_{\alpha,\beta}}{1+|z|}, \mu\le|\arg(z)|\le\pi.$$

Substituting the expressions $u_{k,n}(t)$ $(k=1,2,\ldots;n=1,2,...)$ in (11), we have

$$u(x,y,t) = \sum_{n=1}^{\infty}\sum_{k=1}^{\infty}\{\varphi_{k,n}E_\alpha(-\beta_{k,n}^2\, t^\alpha)+$$

$$\left.+\frac{1}{1+\mu_{k,n}^2}\int_0^t (t-\tau)^{\alpha-1}E_{\alpha,\alpha}(-\beta_{k,n}^2(t-\tau)^\alpha)F_{k,n}(\tau;\, u,\, p)d\tau\right\}\cos\lambda_k x\sin\gamma_n y. \tag{16}$$

Now, from (6), by (11), we obtain

$${}_C D_{0t}^\alpha h(t) + \sum_{k=1}^{\infty}\sum_{n=1}^{\infty}\mu_{k,n}^2\;\left({}_C D_{0t}^\alpha u_{k,n}(t)+u_{k,n}(t)\right)q_{k\,,n} =$$

$$= p(t)h(t) + \int_0^1 \int_0^1 \omega(x,y) f(x,y,t)dxdy \qquad (0 \le t \le T), \tag{17}$$

where

$$q_{k,n} = \int_0^1 \int_0^1 \omega(x,y) \cos \lambda_k x \sin \gamma_n y dxdy.$$

Further, from (13) we have

$$\mu_{k,n}^2 \left({}_C D_{0t}^{\alpha} u_{k,n}(t) + u_{k,n}(t) \right) = F_{k,n}(t;u,p) \quad - {}_C D_{0t}^{\alpha} u_{k,n}(t) = \frac{\mu_{k,n}^2}{1+\mu_{k,n}^2} u_{k,n}(t) +$$

$$+ \frac{\mu_{k,n}^2}{1+\mu_{k,n}^2} F_{k,n}(t;u,p)(k = 1,2,...,n = 1,2,...;\, 0 \le t \le T). \tag{18}$$

From (17), taking into account (18), we obtain

$$p(t) = [h(t)]^{-1} \left\{ {}_C D_{0t}^{\alpha} h(t) - \int_0^1 \int_0^1 \omega(x,y) f(x,y,t) dxdy \right. +$$

$$\left. + \sum_{k=1}^{\infty} \sum_{n=1}^{\infty} \left(\frac{\mu_{k,n}^2}{1+\mu_{k,n}^2} u_{k,n}(t) + \frac{\mu_{k,n}^2}{1+\mu_{k,n}^2} F_{k,n}(t;u,p) \right) q_{k,n} \right\}. \tag{19}$$

To obtain the equation for the second component $p(t)$ of the solution $\{u(x,t), p(t)\}$ of the problem (1)-(4), (6), we substitute the expression (15) in (19) to get

$$p(t) = [h(t)]^{-1} \left\{ {}_C D_{0t}^{\alpha} h(t) - \int_0^1 \int_0^1 \omega(x,y) f(x,y,t) dxdy \right. +$$

$$+ \sum_{k=1}^{\infty} \sum_{n=1}^{\infty} \left[\frac{\mu_{k,n}^2}{1+\mu_{k,n}^2} \left(\varphi_{k,n} E_{\alpha}(-\beta_{k,n}^2\, t^{\alpha}) + \frac{1}{1+\mu_{k,n}^2} \times \right. \right.$$

$$\left. \left. \times \int_0^t (t-\tau)^{\alpha-1} E_{\alpha,\alpha}(-\beta_{k,n}^2 (t-\tau)^{\alpha}) F_{k,n}(\tau;\, u,\, p) d\tau \right) + \frac{\mu_{k,n}^2}{1+\mu_{k,n}^2} F_{k,n}(t;u,p) \right] q_{k,n} \right\}. \tag{20}$$

Thus, the solution of the problem (1)-(4), (6) is reduced to the solution of the system (16), (20) with respect to the unknown functions $u(x,y,t)$ and $p(t)$.

To treat the uniqueness of the solution of (1)-(4), (6), we will significantly use the following lemma.

Lemma 1. *If* $\{u(x,y,t), p(t)\}$ *is any solution of the problem* (1)-(4), (6), *then the functions*

$$u_{k,n}(t) = 4\int_0^1\int_0^1 u(x,y,t)\cos\lambda_k x\sin\gamma_n y dxdy \ \ (k = 1,2,...;\ n = 1,2,...)$$

satisfy the system (15) *on* $[0, T]$.

Proof. Let $\{u(x,y,t), p(t)\}$ be any solution of the problem (1)-(4), (6). Then, multiplying both sides of the equation (1) by the function $4\cos\lambda_k x\sin\gamma_n y$ $(k = 1,2,...;\ n = 1,2,...)$, integrating the obtained equality with respect to x and y from 0 to 1 and using the relations

$$4\int_0^1\int_0^1 {}_C D_{0t}^{\alpha} u(x,y,t)\cos\lambda_k x\sin\gamma_n y dxdy =$$

$$= {}_C D_{0t}^{\alpha}\left(4\int_0^1\int_0^1 u(x,y,t)\cos\lambda_k x\sin\gamma_n y dxdy\right) = {}_C D_{0t}^{\alpha} u_{k,n}(t)$$

$$(k = 1,2,...;\ n = 1,2,...),$$

$$4\int_0^1\int_0^1 u_{xx}(x,y,t)\cos\lambda_k x\sin\gamma_n y dxdy =$$

$$= -\lambda_k^2\left(4\int_0^1\int_0^1 u(x,y,t)\cos\lambda_k x\sin\gamma_n y dxdy\right) = -\lambda_k^2 u_{k,n}(t)$$

$$(k = 1,2,...;\ n = 1,2,...),$$

$$4\int_0^1\int_0^1 u_{yy}(x,y,t)\cos\lambda_k x\sin\gamma_n y dxdy =$$

$$-\gamma_n^2\left(4\int_0^1\int_0^1 u(x,y,t)\cos\lambda_k x\sin\gamma_n y dxdy\right) = -\gamma_k^2 u_{k,n}(t)$$

$$(k = 1,2,...;\ n = 1,2,...),$$

$$4\int_0^1\int_0^1 {}_C D_{0t}^{\alpha} u_{xx}(x,y,t)\cos\lambda_k x\sin\gamma_n y dxdy = -\lambda_k^2 {}_C D_{0t}^{\alpha} u_{k,n}(t)$$

$$(k = 1,2,...;\ n = 1,2,...),$$

$$4\int_0^1\int_0^1 {}_C D_{0t}^{\alpha} u_{yy}(x,y,t)\cos\lambda_k x\sin\gamma_n y dxdy = -\gamma_n^2 {}_C D_{0t}^{\alpha} u_{k,n}(t)$$

$$(k = 1,2,...;\ n = 1,2,...),$$

we get the validity of the equation (13).

Similarly, from (2) it follows that the condition (14) holds.

Thus, $u_{k,n}(t)\,(k=1,2,...;\ n=1,2,...)$ are the solutions of the problem (13), (14). Hence it directly follows that the functions $u_{k,n}(t)\,(k=1,2,...;\ n=1,2,...)$ satisfy the system (15) on $[0,T]$.

The lemma is proved. ◄

It is clear that if

$$u_{k,n}(t)=4\int_0^1\int_0^1 u(x,y,t)\cos\lambda_k x\sin\gamma_n y dxdy\ \ (k=1,2,...;\ n=1,2,...)$$

are the solutions of the system (15), then the pair $\{u(x,y,t),p(t)\}$ of the functions $u(x,y,t)=\sum_{n=1}^{\infty}\sum_{k=1}^{\infty}u_{k,n}(t)\cos\lambda_k x\sin\gamma_n y$ and $p(t)$ is a solution of the system (16), (20).

Lemma 1 has the following corollary.

Corollary 1. *Let the system* (16), (20) *have a unique solution. Then the problem* (1)-(4)*,* (6) *cannot have more than one solution, i.e. if the problem* (1)-(4),(6) *has a solution, then it is unique.*

1. Denote by $B_{2,T}^3$ [31] the totality of all functions $u(x,y,t)$ of the form

$$u(x,y,t)=\sum_{n=1}^{\infty}\sum_{k=1}^{\infty}u_{k,n}(t)\cos\lambda_k x\sin\gamma_n y$$

in D_T, where each of the functions $u_{k,n}(t)\,(k=1,2,...;\ n=1,.2,..)$ is continuously differentiable on $[0,T]$ and

$$\left\{\sum_{n=1}^{\infty}\sum_{k=1}^{\infty}\left(\mu_{k,n}^3\left\|u_{k,n}(t)\right\|_{C[0,T]}\right)^2\right\}^{\frac{1}{2}}<+\infty.$$

Define the norm on this set as follows:

$$\left\|u(x,y,t)\right\|_{B_{2,T}^3}=\left\{\sum_{n=1}^{\infty}\sum_{k=1}^{\infty}\left(\mu_{k,n}^3\left\|u_{k,n}(t)\right\|_{C[0,T]}\right)^2\right\}^{\frac{1}{2}}.$$

2. Denote by E_T^3 the space consisting of topological product

$$B_{2,T}^3\times C[0,T].$$

The norm of the element $z=\{u,p\}$ is defined by the formula

$$\left\|z\right\|_{E_T^3}=\left\|u(x,y,t)\right\|_{B_{2,T}^3}+\left\|p(t)\right\|_{C[0,T]}.$$

It is known that $B_{2,T}^3$ and $E_T^{3;}$ are Banach spaces.
Now let's consider in the space E_T^3 the operator

$$\Phi(u,p) = \{\Phi_1(u,p), \Phi_2(u,p)\},$$

where

$$\Phi_1(u,p) = \tilde{u}(x,y,t) \equiv \sum_{n=1}^{\infty}\sum_{k=1}^{\infty} \tilde{u}_{k,n}(t)\cos\lambda_k x \sin\gamma_n y \ ,$$

$$\Phi_2(u,p) = \tilde{p}(t), ,$$

and $\tilde{u}_{k,n}(t)$ $(k = 1,2,...;\ n = 1,2,...)$ and $\tilde{p}(t)$ are equal to the right-hand sides of (15) and (20), respectively.

It is not difficult to see that

$$\left|E_\alpha(-\beta_{k,n}^2 t^\alpha)\right| \le \frac{M_1}{1+\beta_{k,n}^2 t^\alpha}, \left|E_{\alpha,\alpha}(-\beta_{k,n}^2 (t-\tau)^\alpha)\right| \le \frac{M_2}{1+\beta_{k,n}^2 (t-\tau)^\alpha}(0 \le \tau \le t),$$

$$|q_{k,n}| \le \|\omega(x,y)\|_{C(Q_{xy})} \equiv q\,, \ \ \frac{\mu_{k,n}^2}{1+\mu_{k,n}^2} < 1\,,$$

$$\mu_{k,n}^3 \le \ (\lambda_k^2 + \gamma_k^2)(\lambda_k + \gamma_n) = \lambda_k^3 + \lambda_k^2\gamma_n + \gamma_k^2\lambda_k + \gamma_k^3.$$

From these relations we obtain

$$\left\{\sum_{n=1}^{\infty}\sum_{k=1}^{\infty}\left(\mu_{k,n}^3 \|\tilde{u}_{k,n}(t)\|_{C[0,T]}\right)^2\ \right\}^{\frac{1}{2}} \le \sqrt{7}M_1\left(\sum_{n=1}^{\infty}\sum_{k=1}^{\infty}\left(\lambda_k^3\ \ |\varphi_{k,n}|\right)^2\right)^{\frac{1}{2}} +$$

$$+\sqrt{7}M_1\left(\sum_{n=1}^{\infty}\sum_{k=1}^{\infty}\left(\lambda_k^2\gamma_n\ \ |\varphi_{k,n}|\right)^2\right)^{\frac{1}{2}} + \sqrt{7}M_1\left(\sum_{n=1}^{\infty}\sum_{k=1}^{\infty}\left(\lambda_k\gamma_n^2\ |\varphi_{k,n}|\right)^2\right)^{\frac{1}{2}} +$$

$$+\sqrt{7}M_1\left(\sum_{n=1}^{\infty}\sum_{k=1}^{\infty}\left(\gamma_n^3\ |\varphi_{k,n}|\right)^2\right)^{\frac{1}{2}} + \frac{\sqrt{7}T^\alpha M_2}{\alpha}\times$$

$$\times\left[\left(\sum_{n=1}^{\infty}\sum_{k=1}^{\infty}\left(\lambda_k \|f_{k,n}(t)\|_{C[0,T]}\right)^2\right)^{\frac{1}{2}} + \left(\sum_{n=1}^{\infty}\sum_{k=1}^{\infty}\left(\gamma_n \|f_{k,n}(t)\|_{C[0,T]}\right)^2\right)^{\frac{1}{2}} +\right.$$

$$\left.+\|p(t)\|_{C[0,T]}\left(\sum_{k=1}^{\infty}\sum_{k=1}^{\infty}\left(\mu_{k,n}^3\ \ \|u_{k,n}(t)\|_{C[0,T]}\right)^2\right)^{\frac{1}{2}}\right], \qquad (21)$$

$$\|\tilde{p}(t)\|_{C[0,T]} \leq \left\|[h(t)]^{-1}\right\|_{C[0,T]} \left\{ \left\| {}_C D_{0t}^{\alpha} h(t) - \int_0^1 \int_0^1 \omega(x,y) f(x,y,t) dx dy \right\|_{C[0,T]} + \right.$$

$$+q \left(\sum_{k=1}^{\infty} \sum_{k=1}^{\infty} \mu_k^{-2} \right)^{\frac{1}{2}} \left[M_1 \left(\sum_{n=1}^{\infty} \sum_{k=1}^{\infty} \left(\lambda_k^3 \ \ |\varphi_{k,n}| \right)^2 \right)^{\frac{1}{2}} + M_1 \left(\sum_{n=1}^{\infty} \sum_{k=1}^{\infty} \left(\lambda_k^2 \gamma_n \ \ |\varphi_{k,n}| \right)^2 \right)^{\frac{1}{2}} + \right.$$

$$+M_1 \left(\sum_{n=1}^{\infty} \sum_{k=1}^{\infty} \left(\lambda_k \gamma_n^2 \ |\varphi_{k,n}| \right)^2 \right)^{\frac{1}{2}} + M_1 \left(\sum_{n=1}^{\infty} \sum_{k=1}^{\infty} \left(\gamma_n^3 \ |\varphi_{k,n}| \right)^2 \right)^{\frac{1}{2}} +$$

$$+\frac{T^{\alpha} M_2}{\alpha} \left[\left(\sum_{n=1}^{\infty} \sum_{k=1}^{\infty} \left(\lambda_k \left| f_{k,n}(t) \right| \right)^2 \right)^{\frac{1}{2}} + \left(\sum_{n=1}^{\infty} \sum_{k=1}^{\infty} \left(\gamma_n \left| f_{k,n}(t) \right| \right)^2 \right)^{\frac{1}{2}} + \right.$$

$$\left. \left. + \|p(t)\|_{C[0,T]} \left(\sum_{k=1}^{\infty} \sum_{k=1}^{\infty} \left(\mu_{k,n}^3 \ \ \|u_{k,n}(t)\|_{C[0,T]} \right)^2 \right)^{\frac{1}{2}} \right] + \left(\sum_{n=1}^{\infty} \sum_{k=1}^{\infty} \left(\lambda_k \left| f_{k,n}(t) \right| \right)^2 d\tau \right)^{\frac{1}{2}} + \right.$$

$$\left. + \left(\sum_{n=1}^{\infty} \sum_{k=1}^{\infty} \left(\gamma_n \left| f_{k,n}(t) \right| \right)^2 d\tau \right)^{\frac{1}{2}} + \|p(t)\|_{C[0,T]} \left(\sum_{n=1}^{\infty} \sum_{k=1}^{\infty} (\mu_k^3 \ \|u_k(t)\|_{C[0,T]})^2 \right)^{\frac{1}{2}} \right\}. \tag{22}$$

Assume that the data of the problem (1)-(4), (6) satisfy the following conditions:

1.

$$\varphi(x,y), \varphi_x(x,y), \varphi_{xx}(x,y), \varphi_y(x,y), \varphi_{xy}(x,y), \varphi_{yy}(x,y) \in C(\bar{Q}_{xy}),$$
$$\varphi_{xxy}(x,y), \varphi_{xyy}(x,y), \varphi_{xxx}(x,y), \varphi_{yyy}(x,y) \in L_2(Q_{xy}),$$
$$\varphi_x(0,y) = \varphi(1,y) = \varphi_{xx}(1,y) = 0 \, (0 \leq y \leq 1),$$
$$\varphi(x,0) = \varphi_y(x,1) = \varphi_{yy}(x,0) = 0 \, (0 \leq x \ \leq 1).$$

2.

$$f(x,y,t) \in C(D_T), \ f_x(x,y,t), f_y(x,y,t) \in C(0,T; L_2(Q_{xy})),$$
$$f \ (1,y,t) = f \ (x,0,t) = 0 \, (0 \leq x, y \leq 1, \ \ 0 \leq t \leq T) \, .$$

3.

$$h(t) \in C[0,T] \, , {}_C D_{0t}^{\alpha} h(t) \in C[0,T] \ , h(t) \neq 0 \ (0 \leq t \leq T),$$

Then from (21)- (22) we obtain

$$\|\tilde{u}(x,y,t)\|_{B_{2,T}^3} = \left\{ \sum_{n=1}^{\infty} \sum_{k=1}^{\infty} \left(\mu_k^3 \, \|\tilde{u}_{k,n}(t)\|_{C[0,T]} \right)^2 \ \right\}^{\frac{1}{2}} \ \leq$$

$$\leq A_1(T) + B_1(T)\,\|p(t)\|_{C[0,T]}\,\|u(x,y,t)\|_{B^3_{2,T}}\,, \tag{23}$$

$$\|\tilde{p}(t)\|_{C[0,T]} \leq A_2(T) + B_2(T)\,\|p(t)\|_{C[0,T]}\,\|u(x,y,t)\|_{B^3_{2,T}}\,, \tag{24}$$

where

$$A_1(T) = \sqrt{7}M_1\,\|\varphi_{xxx}(x,y)\|_{L_2(Q_{xy})}$$

$$+\sqrt{7}M_1\,\|\varphi_{xyy}(x,y)\|_{L_2(Q_{xy})}+\sqrt{7}M_1\,\|\varphi_{xxy}(x,y)\|_{L_2(Q_{xy})}+\sqrt{7}M_1\,\|\varphi_{xxx}(x,y)\|_{L_2(Q_{xy})}$$

$$+\frac{\sqrt{7}T^{\alpha}M_2}{\alpha}\left(\left\|\|f_x(x,y,t)\|_{L_2(Q_{xy})}\right\|_{C[0,T]}+\left\|\|f_y(x,y,t)\|_{L_2(Q_{xy})}\right\|_{C[0,T]}\right),$$

$$B_1\,(T) = \frac{\sqrt{7}T^{\alpha}M_2}{\alpha},$$

$$A_2(T) = \left\|[h(t)]^{-1}\right\|_{C[0,T]}\left\{\left\|{}_C D^{\alpha}_{0t}h(t) - \int_0^1\int_0^1 \omega(x,y)f(x,y,t)dxdy\right\|_{C[0,T]}\right.$$

$$+q\left(\sum_{k=1}^{\infty}\sum_{k=1}^{\infty}\mu_k^{-2}\right)^{\frac{1}{2}}\left[M_1\,\|\varphi_{xxx}(x,y)\|_{L_2(Q_{xy})}+M_1\,\|\varphi_{xyy}(x,y)\|_{L_2(Q_{xy})}+\right.$$

$$+M_1\,\|\varphi_{xxy}(x,y)\|_{L_2(Q_{xy})}+M_1\,\|\varphi_{xxx}(x,y)\|_{L_2(Q_{xy})}+$$

$$\left.\left.+\left(\frac{T^{\alpha}M_2}{\alpha}+1\right)\left(\left\|\|f_x(x,y,t)\|_{L_2(Q_{xy})}\right\|_{C[0,T]}+\left\|\|f_y(x,y,t)\|_{L_2(Q_{xy})}\right\|_{C[0,T]}\right)\right]\right\},$$

$$B_2\,(T) = \left\|[h(t)]^{-1}\right\|_{C[0,T]} q\left(\sum_{k=1}^{\infty}\sum_{k=1}^{\infty}\mu_k^{-2}\right)^{\frac{1}{2}}\left(\frac{T^{\alpha}M_2}{\alpha}+1\right).$$

From the inequalities (23)-(24) it follows

$$\|\tilde{u}(x,y,t)\|_{B^3_{2,T}} + \|\tilde{p}(t)\|_{C[0,T]} \leq$$

$$\leq A(T) + B(T)\,\|p(t)\|_{C[0,T]}\,\|u(x,t)\|_{B^3_{2,T}}\;, \tag{25}$$

where

$$A(T) = \sum_{i=1}^{2} A_i(T), \;\; B(T) = \sum_{i=1}^{2} B_i(T), \,.$$

So we can prove the following theorem.

Theorem 2. *Let the conditions 1-3 be satisfied and*

$$(A(T)+2)^2 B(T) < 1. \tag{26}$$

Then the problem (1)-(4), (6) has a unique solution in the ball $K = K_R(\|z\|_{E^3_T}$ $\leq R = A(T)+2)$ *of the space* E^3_T.

Proof. Consider in the space E_T^3 the equation

$$z = \$z, \tag{27}$$

where $z = \{u, p\}$, and the components $\Phi_i(u,p)(i = 1, 2)$ of the operator $\$(u, p)$ are defined by the right-hand sides of the equations (16), (20), respectively. Consider the operator $\$(u, p)$ in the ball $K = K_R(\|z\|_{E_T^3} \leq R = A(T) + 2)$ of E_T^3.

Similar to (25), we obtain the following estimates for every $z, z_1, z_2 \in K_R$:

$$\|\Phi z\|_{E_T^3} \leq A(T) + B(T) \|p(t)\|_{C[0,T]} \|u(x, y, t)\|_{B_{2,T}^3}, \tag{28}$$

$$\|\Phi z_1 - \Phi z_2\|_{E_T^2} \leq B(T)R \left(\|p_1(t) - p_2(t)\|_{C[0,T]} + \|u_1(x, y, t) - u_2(x, y, t)\|_{B_{2,T}^3} \right). \tag{29}$$

Then from the estimates (28) and (29), by (26), it follows that the operator \$ acts in the ball $K = K_R$ and is a contraction operator. Therefore, the operator \$ has a unique fixed point $\{u, p\}$ in the ball $K = K_R$, which is a unique solution of the equation (27), i.e. a unique solution of the system (16), (20) in the ball $K = K_R$.

As an element of the space $B_{2,T}^3$, the function $u(x, y, t)$ is continuous and has continuous derivatives $u_x(x, y, t)$, $u_{xx}(x, y, t)$, $u_y(x, y, t)$, $u_{xy}(x, y, t)$, $u_{yy}(x, y, t)$ in D_T.

Now it is not difficult to see from (13) that

$$\left\{ \sum_{n=1}^{\infty} \sum_{k=1}^{\infty} \left(\mu_{k,n}^3 \|{}_C D_{0t}^{\alpha} u_{k,n}(t)\|_{C[0,T]} \right)^2 \right\}^{\frac{1}{2}} \leq \sqrt{2} \left[\|u(x, y, t)\|_{B_{2,T}^3} + \right.$$

$$\left. + \left\| \|f_x(x, y, t) + f_y(x, y, t) + p(t) \left(u_x(x, y, t) + u_y(x, y, t) \right)\|_{L_2(Q_{xy})} \right\|_{C[0,T]} \right].$$

Hence, it is clear that $u_t(x, y, t)$, $u_{txx}(x, y, t)$, $u_{tyy}(x, y, t)$ are continuous in D_T.

It is not difficult to verify that the equation (1) and the conditions (2)-(4), (6) are satisfied in the usual sense. Thus, the solution of the problem (1)–(4), (6) is a pair of functions $\{u(x, t), p(t)\}$. By the corollary of Lemma 1, this solution is unique in the ball $K = K_R$.

The theorem is proved. ◄

Using Theorems 1 and 2, we obtain the unique solvability of the problem (1)–(5).

Theorem 3. *Let all the conditions of Theorem 2 be satisfied and the coherence conditions*

$$\int_0^1 \int_0^1 \omega(x,y)\varphi(x,y)dxdy = h(0)$$

hold. Then the problem (1)-(5) *has a unique classical solution in the ball* $K = K_R(\|z\|_{E_T^3} \leq R = A(T) + 2)$ *of the space* E_T^3.

4. Conclusions

In this paper, a nonlinear inverse boundary value problem for a time fractional two-dimensional pseudo parabolic equation with the integral type overdetermination conditions is studied and the classical solvability of the considered inverse boundary problem is investigated. To examine the solvability of the problem, the considered problem is first reduced to an auxiliary inverse boundary value problem and its equivalence to the original problem is shown. Then, the existence and uniqueness theorems for the auxiliary problem are proved using the Fourier method and the contraction mappings principle. Further, based on the equivalence of these problems, the existence and uniqueness theorem for the classical solution of the original inverse problem is established

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