Potential Operator on Carleson Curves in Local Generalized Weighted Morrey Spaces

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Abstract. In this paper we give a strong and weak type Guliyev-Spanne type boundedness criterion for the potential operator $\mathcal{I}^\alpha$ in the local generalized weighted Morrey space $LM_{p,\varphi}^{\{t_0\}}(\Gamma, w)$ and the generalized weighted Morrey space $M_{p,\varphi}(\Gamma, w)$ defined on Carleson curves Γ. For the operator $\mathcal{I}^\alpha$ we establish necessary and sufficient conditions for the strong and $LM_{p,\varphi}^{\{t_0\}}(\Gamma, w)$ and the strong and weak Guliyev-Spanne type boundedness and $M_{p,\varphi}(\Gamma, w)$.

Key Words and Phrases: Carleson curve, local generalized weighted Morrey space, potential operator, Guliyev-Spanne type boundedness.

2010 Mathematics Subject Classifications: 26A33, 42B25, 42B35, 47B38

1. Introduction

The classical Morrey spaces were originally introduced by Morrey in [29] to study the local behavior of solutions to second order elliptic partial differential equations. For the properties and applications of Morrey spaces, we refer the readers to [3, 9, 19, 20, 33]. Guliyev, Mizuhara and Nakai [12, 28, 32] introduced generalized Morrey spaces $M^{p,\varphi}(\mathbb{R}^n)$ (see also [13, 35]). Recently, Komori and Shirai [27] considered the weighted Morrey spaces $L^{p,\kappa}(w)$ and studied the boundedness of some classical operators such as the Hardy-Littlewood maximal operator, the Calderón-Zygmund operator on these spaces. Guliyev [15] gave a concept of generalized weighted Morrey space $M_{p,\varphi}(w)$ which could be viewed as extension of both generalized Morrey space $M_{p,\varphi}$ and weighted Morrey space $L^{p,\kappa}(w)$. In [15], Guliyev also studied the boundedness of classical operators and their commutators in the spaces $M_{p,\varphi}(w)$.

Let $\Gamma = \{t \in \mathbb{C} : t = t(s),\ 0 \leq s \leq l \leq \infty\}$ be a rectifiable Jordan curve in the complex plane $\mathbb{C}$ with arc-length measure $\nu(t) = s$, with $l = \nu\Gamma = $ length of Γ. We denote

$$\Gamma(t,r) := \Gamma \cap B(t,r),\ t \in \Gamma,\ r > 0,$$

where $B(t,r) = \{z \in \mathbb{C} : |z - t| < r\}$. We also denote for brevity $\nu\Gamma(t,r) = |\Gamma(t,r)|$.

A rectifiable Jordan curve Γ is called a Carleson curves if the condition

$$\nu\Gamma(t,r) \leq c_0 r$$

holds for all $t \in \Gamma$ and $r > 0$, where the constant $c_0 > 0$ does not depend on t and r.

Let $f \in L_1^{loc}(\Gamma, w)$. The maximal operator $\mathcal{M}$ and the potential operator $\mathcal{I}^\alpha$ on Γ are defined by

$$\mathcal{M}f(t) = \sup_{t>0}(\nu\Gamma(t,r))^{-1} \int_{\Gamma(t,r)} |f(\tau)| d\nu(\tau),$$

and

$$\mathcal{I}^\alpha f(t) = \int_\Gamma \frac{f(\tau)}{|t - \tau|^{1-\alpha}}\, d\nu(\tau), \quad 0 < \alpha < 1,$$

respectively.

Maximal operator and potential operator in various spaces, in particular, defined on Carleson curves, have been widely studied by many authors (see, for example, [4, 5, 6, 8, 10, 11, 22, 23, 24, 26]).

According to [2, 3, 18, 19, 20, 21, 33], the maximal estimates are vital for examining the regularity of solutions of elliptic, parabolic, and ultraparabolic partial differential equations of second order, and their boundedness can be used to describe specific function spaces.

The main purpose of this paper is to establish the boundedness of potential operator $\mathcal{I}^\alpha$, $0 < \alpha < 1$ in local generalized weighted Morrey spaces $LM_{p,\varphi}^{\{x_0\}}(\Gamma, w)$ defined on Carleson curves Γ. We shall give characterizations of the strong and weak Guliyev-Spanne type boundedness of the operator $\mathcal{I}^\alpha$ from $LM_{p,\varphi_1}^{\{x_0\}}(\Gamma, w)$ to $LM_{q,\varphi_2}^{\{x_0\}}(\Gamma, w)$, $1 < p < q < \infty$, $1/p - 1/q = \alpha$ and from the space $LM_{1,\varphi_1}^{\{x_0\}}(\Gamma, w)$ to the weak space $WLM_{q,\varphi_2}^{\{x_0\}}(\Gamma, w)$, $1 < q < \infty$, $1 - 1/q = \alpha$.

By $A \lesssim B$ we mean that $A \leq CB$ with some positive constant C independent of appropriate quantities. If $A \lesssim B$ and $B \lesssim A$, we write $A \approx B$ and say that A and B are equivalent.

2. Preliminaries

Morrey spaces were introduced by C.B. Morrey [29] in 1938 in connection with some problems in elliptic partial differential equations and calculus of variations. Later, Morrey spaces found important applications to Navier-Stokes and Schrödinger equations, elliptic problems with discontinuous coefficients, and potential theory.

By a weight function, briefly weight, we mean a locally integrable function on Γ which takes values in $(0, \infty)$ almost everywhere. For a weight w and a measurable set E, we define $w(E) = \int_E w(t)d\nu(t)$, and denote the Lebesgue measure of E by $|E|$ and the characteristic function of E by χ_E.

Let w is a weight function, and $L_{p,w^p}(\Gamma)$, $1 \le p < \infty$ be the space of measurable functions on Γ with finite norm

$$\|f\|_{L_{p,w^p}(\Gamma,w)} = \left(\int_\Gamma |f(t)|^p w(t)d\nu(t) \right)^{1/p}.$$

We recall a weight function w is in the Muckenhoupt's class $A_p(\Gamma)$, $1 < p < \infty$ [30], if

$$[w]_{A_p(\Gamma)} := \sup_D [w]_{A_p(D)}$$

$$= \sup_D \left(\frac{1}{|D|} \int_D w(t)d\nu(t) \right) \left(\frac{1}{|D|} \int_D w(t)^{1-p'} d\nu(t) \right)^{p-1} < \infty, \quad (1)$$

where the supremum is taken with respect to all the balls D and $\frac{1}{p} + \frac{1}{p'} = 1$. Note that, for all balls D, Hölder's inequality is

$$[w]_{A_p(D)}^{\frac{1}{p}} = |D|^{-1} \|w\|_{L_1(D)}^{\frac{1}{p}} \|w^{-\frac{1}{p}}\|_{L_{p'}(D)} \ge 1. \tag{2}$$

For $p = 1$, $w \in A_1(\Gamma)$ is defined by the condition $Mw(t) \le Cw(t)$ with $[w]_{A_1(\Gamma)} = \sup_{t \in \Gamma} \frac{Mw(t)}{w(t)}$, and for $p = \infty$ $A_\infty(\Gamma) = \cup_{1 \le p < \infty} A_p(\Gamma)$ and $[w]_\infty = \inf_{1 \le p < \infty} [w]_{A_p}$.

A weight function w is in the Muckenhoupt-Wheeden class $A_{p,q}(\Gamma)$, $1 < p < \infty$ [31], if

$$[w]_{A_{p,q}(\Gamma)} := \sup_D [w]_{A_{p,q}(D)}$$

$$= \sup_D \left(\frac{1}{|D|} \int_B w(t)^q d\nu(t) \right)^{1/q} \left(\frac{1}{|D|} \int_D w(t)^{-p'} d\nu(t) \right)^{1/p'} < \infty,$$

where the supremum is taken with respect to all the balls D and $\frac{1}{p} + \frac{1}{p'} = 1$. Note that, for all balls D, Hölder's inequality is

$$[w]_{A_{p,q}(D)} = |D|^{\frac{1}{p} - \frac{1}{q} - 1} \|w\|_{L_q(D)} \|w^{-1}\|_{L_{p'}(D)} \ge 1. \tag{3}$$

While $p = 1$, $w \in A_{1,q}(\Gamma)$ with $1 < q < \infty$ if

$$[w]_{A_{1,q}(\Gamma)} := \sup_D [w]_{A_{1,q}(D)}$$

$$= \sup_D \left(\frac{1}{|D|} \int_D w(t)^q d\nu(t) \right)^{\frac{1}{q}} \left(\operatorname*{ess\,sup}_{t \in D} \frac{1}{w(t)} \right) < \infty. \qquad (4)$$

Definition 1. *Let $1 \leq p < \infty$, $0 \leq \lambda \leq 1$, $[r]_1 = \min\{1, r\}$. We denote by $L_{p,\lambda}(\Gamma)$ the Morrey space, and by $\widetilde{L}_{p,\lambda}(\Gamma)$ the modified Morrey space, the set of locally integrable functions f on Γ with the finite norms*

$$\|f\|_{L_{p,\lambda}(\Gamma)} = \sup_{t \in \Gamma, r > 0} r^{-\frac{\lambda}{p}} \|f\|_{L_p(\Gamma(t,r))}, \qquad \|f\|_{\widetilde{L}_{p,\lambda}(\Gamma)} = \sup_{t \in \Gamma, r > 0} [r]_1^{-\frac{\lambda}{p}} \|f\|_{L_p(\Gamma(t,r))},$$

respectively.

Note that (see [11, 14]) $L_{p,0}(\Gamma) = \widetilde{L}_{p,0}(\Gamma) = L_p(\Gamma)$,

$$\widetilde{L}_{p,\lambda}(\Gamma) = L_{p,\lambda}(\Gamma) \cap L_p(\Gamma) \quad \text{and} \quad \|f\|_{\widetilde{L}_{p,\lambda}(\Gamma)} = \max\{\|f\|_{L_{p,\lambda}(\Gamma)}, \|f\|_{L_p(\Gamma)}\}$$

and if $\lambda < 0$ or $\lambda > 1$, then $L_{p,\lambda}(\Gamma) = \widetilde{L}_{p,\lambda}(\Gamma) = \Theta$, where Θ is the set of all functions equivalent to 0 on Γ.

We denote by $WL_{p,\lambda}(\Gamma)$ the weak Morrey space, and by $W\widetilde{L}_{p,\lambda}(\Gamma)$ the modified Morrey space, as the set of locally integrable functions f on Γ with finite norms

$$\|f\|_{WL_{p,\lambda}(\Gamma)} = \sup_{\beta > 0} \beta \sup_{t \in \Gamma, r > 0} \left(r^{-\lambda} \int_{\{\tau \in \Gamma(t,r): |f(\tau)| > \beta\}} w(\tau) d\nu(\tau) \right)^{1/p},$$

$$\|f\|_{W\widetilde{L}_{p,\lambda}(\Gamma)} = \sup_{\beta > 0} \beta \sup_{t \in \Gamma, r > 0} \left([r]_1^{-\lambda} \int_{\{\tau \in \Gamma(t,r): |f(\tau)| > \beta\}} w(\tau) d\nu(\tau) \right)^{1/p}.$$

N. Samko [34] studied the boundedness of the maximal operator $\mathcal{M}$ defined on quasimetric measure spaces, in particular on Carleson curves in Morrey spaces $L_{p,\lambda}(\Gamma)$:

Theorem A. *Let Γ be a Carleson curve, $1 < p < \infty$, $0 < \alpha < 1$ and $0 \leq \lambda < 1$. Then $\mathcal{M}$ is bounded from $L_{p,\lambda}(\Gamma)$ to $L_{p,\lambda}(\Gamma)$.*

V. Kokilashvili and A. Meskhi [25] studied the boundedness of the operator $\mathcal{I}^\alpha$ defined on quasimetric measure spaces, in particular on Carleson curves in Morrey spaces and proved the following:

Theorem B. *Let Γ be a Carleson curve, $1 < p < q < \infty$, $0 < \alpha < 1$, $0 < \lambda_1 < \frac{p}{q}$, $\frac{\lambda_1}{p} = \frac{\lambda_2}{q}$ and $\frac{1}{p} - \frac{1}{q} = \alpha$. Then the operator $\mathcal{I}^\alpha$ is bounded from the spaces $L_{p,\lambda_1}(\Gamma)$ to $L_{q,\lambda_2}(\Gamma)$.*

The following Adams boundedness (see [1]) of the operator $\mathcal{I}^\alpha$ in Morrey space defined on Carleson curves was proved in [10].

Theorem C. *Let Γ be a Carleson curve, $0 < \alpha < 1$, $0 \leq \lambda < 1 - \alpha$ and $1 \leq p < \frac{1-\lambda}{\alpha}$.*

1) If $1 < p < \frac{1-\lambda}{\alpha}$, then the condition $\frac{1}{p} - \frac{1}{q} = \frac{\alpha}{1-\lambda}$ is sufficient and in the case of infinite curve also necessary for the boundedness of the operator $\mathcal{I}^\alpha$ from $L_{p,\lambda}(\Gamma)$ to $L_{q,\lambda}(\Gamma)$.

2) If $p = 1$, then the condition $1 - \frac{1}{q} = \frac{\alpha}{1-\lambda}$ is sufficient and in the case of infinite curve also necessary for the boundedness of the operator $\mathcal{I}^\alpha$ from $L_{1,\lambda}(\Gamma)$ to $WL_{q,\lambda}(\Gamma)$.

The following Adams boundedness of the operator $\mathcal{I}^\alpha$ in modified Morrey space $\widetilde{L}_{p,\lambda}(\Gamma)$ defined on Carleson curves was proved in [11], see also [14].

Theorem D. *Let Γ be a Carleson curve, $0 < \alpha < 1$, $0 \leq \lambda < 1 - \alpha$ and $1 \leq p < \frac{1-\lambda}{\alpha}$.*

1) If $1 < p < \frac{1-\lambda}{\alpha}$, then the condition $\alpha \leq \frac{1}{p} - \frac{1}{q} \leq \frac{\alpha}{1-\lambda}$ is sufficient and in the case of infinite curve also necessary for the boundedness of the operator $\mathcal{I}^\alpha$ from $\widetilde{L}_{p,\lambda}(\Gamma)$ to $\widetilde{L}_{q,\lambda}(\Gamma)$.

2) If $p = 1$, then the condition $\alpha \leq 1 - \frac{1}{q} \leq \frac{\alpha}{1-\lambda}$ is sufficient and in the case of infinite curve also necessary for the boundedness of $\mathcal{I}^\alpha$ from $\widetilde{L}_{1,\lambda}(\Gamma)$ to $W\widetilde{L}_{q,\lambda}(\Gamma)$.

We will use the following statement on the boundedness of the weighted Hardy operator

$$H_w g(t) := \int_t^\infty g(s)w(s)ds, \qquad 0 < t < \infty,$$

where w is a weight.

The following theorem was proved in [17], see also [16].

Theorem 1. *Let v_1, v_2 and w be weights on $(0,\infty)$ and $v_1(t)$ be bounded outside a neighborhood of the origin. The inequality*

$$\operatorname*{ess\,sup}_{t>0} v_2(t)H_w g(t) \leq C \operatorname*{ess\,sup}_{t>0} v_1(t)g(t) \tag{5}$$

holds for some $C > 0$ for all non-negative and non-decreasing g on $(0,\infty)$ if and only if

$$B := \sup_{t>0} v_2(t) \int_t^\infty \frac{w(s)ds}{\operatorname*{ess\,sup}_{s<\tau<\infty} v_1(\tau)} < \infty.$$

Moreover, the value $C = B$ is the best constant for (5).

3. Local generalized weighted Morrey spaces

We find it convenient to define the local generalized weighted Morrey spaces in the form as follows, see [17, 18].

Definition 2. *Let* $1 \leq p < \infty$, $\varphi(t,r)$ *be a positive measurable function on* $\Gamma \times (0, \infty)$ *and* w *be a non-negative measurable function on* Γ. *Fixed* $t_0 \in \Gamma$, *we denote by* $LM_{p,\varphi}^{\{t_0\}}(\Gamma, w)$ $(WLM_{p,\varphi}^{\{t_0\}}(\Gamma, w))$ *the local generalized weighted Morrey space (the weak local generalized weighted Morrey space), the space of all functions* $f \in L_p^{\mathrm{loc}}(\Gamma, w)$ *with finite quasinorm*

$$\|f\|_{LM_{p,\varphi}^{\{t_0\}}(\Gamma, w)} = \sup_{r>0} \frac{1}{\varphi(t_0, r)} \frac{1}{w(\Gamma(t_0, r))^{\frac{1}{p}}} \|f\|_{L_p(\Gamma(t_0, r), w)},$$

$$\left(\|f\|_{WLM_{p,\varphi}^{\{t_0\}}(\Gamma, w)} = \sup_{r>0} \frac{1}{\varphi(t_0, r)} \frac{1}{w(\Gamma(t_0, r))^{\frac{1}{p}}} \|f\|_{WL_p(\Gamma(t_0, r), w)} \right).$$

Definition 3. *Let* $1 \leq p < \infty$, $\varphi(t,r)$ *be a positive measurable function on* $\Gamma \times (0, \infty)$ *and* w *be a non-negative measurable function on* Γ. *The generalized weighted Morrey space* $M_{p,\varphi}(\Gamma, w)$ *is defined as the set of all functions* $f \in L_p^{loc}(\Gamma, w)$ *with the finite norm*

$$\|f\|_{M_{p,\varphi}(w)} = \sup_{t \in \Gamma, r>0} \frac{1}{\varphi(t, r)} \frac{1}{w(\Gamma(t, r))^{\frac{1}{p}}} \|f\|_{L_p(\Gamma(t, r), w)}.$$

Also the weak generalized Morrey space $WM_{p,\varphi}(\Gamma, w)$ *is defined as the set of all functions* $f \in L_p^{loc}(\Gamma, w)$ *with the finite norm*

$$\|f\|_{WM_{p,\varphi}(w)} = \sup_{t \in \Gamma, r>0} \frac{1}{\varphi(t, r)} \frac{1}{w(\Gamma(t, r))^{\frac{1}{p}}} \|f\|_{WL_p(\Gamma(t, r), w)}.$$

It is natural, first of all, to find conditions ensuring that the spaces $LM_{p,\varphi}^{\{t_0\}}(\Gamma, w)$ and $M_{p,\varphi}(\Gamma, w)$ are nontrivial, that is consist not only of functions equivalent to 0 on Γ.

Lemma 1. *[4] Let* $t_0 \in \Gamma$, $\varphi(t,r)$ *be a positive measurable function on* $\Gamma \times (0, \infty)$ *and* w *be a non-negative measurable function on* Γ. *If*

$$\sup_{r < \tau < \infty} \frac{1}{\varphi(t_0, r)} \frac{1}{w(\Gamma(t_0, r))^{\frac{1}{p}}} = \infty \quad \text{for some } r > 0, \tag{6}$$

then $LM_{p,\varphi}^{\{t_0\}}(\Gamma, w) = \Theta$.

Remark 1. *Let $t_0 \in \Gamma$ and w be a non-negative measurable function on Γ. We denote by $\Omega_{p,w}^{\mathrm{loc}}$ the set of all positive measurable functions φ on $\Gamma \times (0, \infty)$ such that for all $r > 0$,*

$$\left\| \frac{1}{\varphi(t_0, \tau)} \frac{1}{w(\Gamma(t_0, \tau))^{\frac{1}{p}}} \right\|_{L_\infty(r, \infty)} < \infty.$$

In what follows, keeping in mind Lemma 1, for the non-triviality of the space $LM_{p,\varphi}^{\{t_0\}}(\Gamma, w)$ we always assume that $\varphi \in \Omega_{p,w}^{\mathrm{loc}}$.

Lemma 2. *[4] Let $\varphi(t, r)$ be a positive measurable function on $\Gamma \times (0, \infty)$ and w be a non-negative measurable function on Γ.*
  (i)   If

$$\sup_{r < \tau < \infty} \frac{1}{\varphi(t, \tau)} \frac{1}{w(\Gamma(t, r))^{\frac{1}{p}}} = \infty \quad \text{for some } r > 0 \text{ and for all } t \in \Gamma, \quad (7)$$

then $M_{p,\varphi}(\Gamma, w) = \Theta$.
  (ii)   If

$$\sup_{0 < \tau < r} \varphi(t, \tau)^{-1} = \infty \quad \text{for some } r > 0 \text{ and for all } t \in \Gamma, \quad (8)$$

then $M_{p,\varphi}(\Gamma, w) = \Theta$.

Remark 2. *We denote by $\Omega_{p,w}$ the set of all positive measurable functions φ on $\Gamma \times (0, \infty)$ such that for all $r > 0$,*

$$\sup_{t \in \Gamma} \left\| \frac{1}{\varphi(t, \tau)} \frac{1}{w(\Gamma(t, r))^{\frac{1}{p}}} \right\|_{L_\infty(r, \infty)} < \infty, \quad \text{and} \quad \sup_{t \in \Gamma} \left\| \varphi(t, \tau)^{-1} \right\|_{L_\infty(0, r)} < \infty.$$

In what follows, keeping in mind Lemma 2, we always assume that $\varphi \in \Omega_{p,w}$.

Let $\mathcal{G} = \{\Gamma(t, r) : t \in \Gamma, r > 0\}$. For a non-negative measurable function w, we denote by $\mathcal{G}_w^p$ the set of all almost decreasing functions $\varphi : \Gamma \times (0, \infty) \to (0, \infty)$ such that

$$\inf_{\Gamma(t,r) \in \mathcal{D} : r \leq r_0} \varphi(\Gamma(t, r)) \gtrsim \varphi(\Gamma(t_0, r_0)) \quad \text{for all} \quad \Gamma(t_0, r_0) \in \mathcal{D}$$

and

$$\inf_{\Gamma(t,r) \in \mathcal{D} : r \geq r_0} \varphi(\Gamma(t, r)) \, w^p(\Gamma(t, r))^{\frac{1}{p}} \gtrsim \varphi(\Gamma(t_0, r_0)) \, w^p(\Gamma(t_0, r_0))^{\frac{1}{p}},$$

where r and r_0 denote the radius of the Γ-balls $\Gamma(t, r)$ and $\Gamma(t_0, r_0)$, respectively.

For proving our main results, we need the following estimate.

Lemma 3. *Let $\Gamma_0 := \Gamma(t_0, r_0)$. If $\varphi \in \mathcal{G}_w^p$, then there exists $C > 0$ such that*

$$\frac{1}{\varphi(\Gamma(t_0, r_0))} \leq \|\chi_{\Gamma(t_0, r_0)}\|_{M_{p,\varphi}(\Gamma, w^p)} \leq \frac{C}{\varphi(\Gamma(t_0, r_0))}.$$

Proof.

$$\|\chi_{\Gamma(t_0, r_0)}\|_{M_{p,\varphi}(\Gamma, w^p)} = \sup_{\Gamma(t, r)} \varphi(\Gamma(t, r))^{-1} w^p(\Gamma(t, r))^{-\frac{1}{p}} (w^p(\Gamma(t, r) \cap \Gamma(t_0, r_0)))^{\frac{1}{p}}$$

$$\geq \varphi(\Gamma(t_0, r_0))^{-1} w^p(\Gamma(t_0, r_0))^{-\frac{1}{p}} w^p(\Gamma(t_0, r_0))^{\frac{1}{p}} = \varphi(\Gamma(t_0, r_0))^{-1}.$$

Now if $r \leq r_0$, then $\varphi(\Gamma(t_0, r_0)) = \varphi(t_0, r_0) \leq C\varphi(t, r) = C\varphi(\Gamma(t, r))$ and

$$\varphi(\Gamma(t, r))^{-1} w^p(\Gamma(t, r))^{-\frac{1}{p}} \|\chi_{\Gamma(t, r)_0}\|_{L_{p, w^p}(\Gamma(t, r))}$$

$$= \varphi(\Gamma(t, r))^{-1} w^p(\Gamma(t, r))^{-\frac{1}{p}} (w^p(\Gamma(t, r) \cap \Gamma(t_0, r_0)))^{\frac{1}{p}}$$

$$\leq \varphi(\Gamma(t, r))^{-1} \leq C\varphi(\Gamma(t_0, r_0))^{-1}.$$

Therefore, $\|\chi_{\Gamma(t_0, r_0)}\|_{M_{p,\varphi}(\Gamma, w^p)} \leq C\varphi(\Gamma(t_0, r_0))^{-1}$.
On the other hand, if $r \geq r_0$, then

$$\varphi(\Gamma(t, r)) \, w^p(\Gamma(t, r))^{\frac{1}{p}} \geq C\varphi(\Gamma(t_0, r_0)) \, w^p(\Gamma(t_0, r_0))^{\frac{1}{p}}$$

and

$$\varphi(\Gamma(t_0, r_0))^{-1} \geq C \, \varphi(\Gamma(t, r))^{-1} w^p(\Gamma(t, r))^{-\frac{1}{p}} \, w^p(\Gamma(t_0, r_0))^{\frac{1}{p}}$$

$$\geq C \, \varphi(\Gamma(t, r))^{-1} \, w^p(\Gamma(t, r))^{-\frac{1}{p}} \, w^p(\Gamma(t, r) \cap \Gamma(t_0, r_0))^{\frac{1}{p}}.$$

Then

$$\|\chi_{\Gamma(t_0, r_0)}\|_{M_{p,\varphi}(\Gamma, w^p)} = \sup_{\Gamma(t, r) \in \mathcal{D}} \varphi(\Gamma(t, r))^{-1} \, w^p(\Gamma(t, r))^{-\frac{1}{p}} \, w^p(\Gamma(t, r) \cap \Gamma(t_0, r_0))^{\frac{1}{p}}$$

$$\leq C\varphi(\Gamma(t_0, r_0))^{-1}.$$

Because of this, $\|\chi_{\Gamma(t_0, r_0)}\|_{M_{p,\varphi}(\Gamma, w^p)} \leq C\varphi(\Gamma(t_0, r_0))^{-1}$. ◄

By elementary calculations we have the following, which shows in particular that the spaces $LM_{p,\varphi}^{\{t_0\}}$, $WLM_{p,\varphi}^{\{t_0\}}$, $M_{p,\varphi}(\Gamma, w)$ and $WM_{p,\varphi}(\Gamma, w)$ are not trivial, see, for example, [4].

Lemma 4. *[4] Let $\varphi \in \mathcal{G}_w^p$, $1 \leq p < \infty$, $\Gamma_0 = \Gamma(t_0, r_0)$ and χ_{Γ_0} be the character-istic function of the ball Γ_0. Then $\chi_{\Gamma_0} \in LM_{p,\varphi}^{\{t_0\}}(\Gamma, w^p) \cap M_{p,\varphi}(\Gamma, w^p)$. Moreover, there exists $C > 0$ such that*

$$\frac{1}{\varphi(r_0)} \leq \|\chi_{\Gamma_0}\|_{WLM_{p,\varphi}^{\{t_0\}}(\Gamma, w^p)} \leq \|\chi_{\Gamma_0}\|_{LM_{p,\varphi}^{\{t_0\}}(\Gamma, w^p)} \leq \frac{C}{\varphi(r_0)}$$

and

$$\frac{1}{\varphi(r_0)} \leq \|\chi_{\Gamma_0}\|_{WM_{p,\varphi}(\Gamma, w^p)} \leq \|\chi_{\Gamma_0}\|_{M_{p,\varphi}(\Gamma, w^p)} \leq \frac{C}{\varphi(r_0)}.$$

The following Guliyev type local estimate for the maximal operator $\mathcal{M}$ is true, see, for example, [13].

Lemma 5. *[7, Lemma 4.2] Let Γ be a Carleson curve, $1 \leq p < \infty$, $t_0 \in \Gamma$ and $w \in A_p(\Gamma)$. Then for $p > 1$ and any $r > 0$ in Γ, the inequality*

$$\|\mathcal{M}f\|_{L_{p,w}(\Gamma(t_0,r))} \lesssim [w]_{A_p(\Gamma)} \left(w(\Gamma(t_0,r))\right)^{\frac{1}{p}} \sup_{\tau > 2r} \left(w(\Gamma(t_0,r))\right)^{-\frac{1}{p}} \|f\|_{L_{p,w}(\Gamma(t_0,\tau))} \tag{9}$$

holds for all $f \in L_{p,w}^{\mathrm{loc}}(\Gamma)$.
Moreover, for $p = 1$ the inequality

$$\|\mathcal{M}f\|_{WL_{1,w}(\Gamma(t_0,r))} \lesssim [w]_{A_1(\Gamma)} w(\Gamma(t_0,r)) \sup_{\tau > 2r} \left(w(\Gamma(t_0,r))\right)^{-1} \|f\|_{L_{1,w}(\Gamma(t_0,\tau))} \tag{10}$$

holds for all $f \in L_{1,w}^{\mathrm{loc}}(\Gamma)$.

The following theorem is valid.

Theorem 2. *[7, Theorem 4.3] Let Γ be a Carleson curve, $1 \leq p < \infty$, $t_0 \in \Gamma$, $w \in A_p(\Gamma)$ and (φ_1, φ_2) satisfy the condition*

$$\sup_{r < \tau < \infty} \left(w(\Gamma(t_0,r))\right)^{-\frac{1}{p}} \operatorname*{ess\,inf}_{\tau < s < \infty} \varphi_1(t_0, s) \, s^{\frac{1}{p}} \leq C \, \varphi_2(t_0, r), \tag{11}$$

where C does not depend on r. Then for $p > 1$, the operator $\mathcal{M}$ is bounded from $LM_{p,\varphi_1}^{\{t_0\}}(\Gamma, w)$ to $LM_{p,\varphi_2}^{\{t_0\}}(\Gamma, w)$ and for $p = 1$, the operator $\mathcal{M}$ is bounded from $LM_{1,\varphi_1}^{\{t_0\}}(\Gamma, w)$ to $WLM_{1,\varphi_2}^{\{t_0\}}(\Gamma, w)$.

From Theorem 2 we get the following

Corollary 1. *[7, Corollary 4.4] Let* Γ *be a Carleson curve,* $1 \le p < \infty$, $w \in A_p(\Gamma)$ *and* φ_1, φ_2 *satisfy the condition*

$$\sup_{\tau > r}(w(\Gamma(t,r)))^{-\frac{1}{p}} \operatorname*{ess\,inf}_{s > \tau} \varphi_1(t,s)\, s^{\frac{1}{p}} \le C\, \varphi_2(t,r), \tag{12}$$

where C *does not depend on* t *and* r. *Then for* $p > 1$, *the operator* $\mathcal{M}$ *is bounded from* $M_{p,\varphi_1}(\Gamma, w)$ *to* $M_{p,\varphi_2}(\Gamma, w)$ *and for* $p = 1$, *the operator* $\mathcal{M}$ *is bounded from* $M_{1,\varphi_1}(\Gamma, w)$ *to* $WM_{1,\varphi_2}(\Gamma, w)$.

4. Fractional integral operator in the spaces $LM_{p,\varphi}^{\{t_0\}}(\Gamma, w)$ and $M_{p,\varphi}(\Gamma, w)$

The following Guliyev type local estimate for the potential operator $\mathcal{I}^\alpha$ is true, see, for example, [13, 15].

Theorem 3. *Let* Γ *be a Carleson curve,* $1 \le p < q < \infty$, $t_0 \in \Gamma$, $0 < \alpha < \frac{1}{p}$, $\frac{1}{q} = \frac{1}{p} - \alpha$, $f \in L_{p,w^p}^{loc}(\Gamma, w)$ *and* $\omega \in A_{p,q}(\Gamma)$. *Then for* $p > 1$ *the inequality*

$$\|\mathcal{I}^\alpha f\|_{L_{q,w^q}(\Gamma(t_0,r))}$$

$$\lesssim (w^q(\Gamma(t_0,r)))^{\frac{1}{q}} \int_{2r}^{\infty} \|f\|_{L_{p,w^p}(\Gamma(t_0,\tau))} (w^q(\Gamma(t_0,\tau)))^{-\frac{1}{q}} \frac{d\tau}{\tau} \tag{13}$$

holds for any ball $\Gamma(t_0,r)$ *and for all* $f \in L_{p,w}^{loc}(\Gamma)$, *and for* $p = 1$ *the inequality*

$$\|\mathcal{I}^\alpha f\|_{WL_{q,w^q}(\Gamma(t_0,r))}$$

$$\lesssim (w^q(\Gamma(t_0,r)))^{\frac{1}{q}} \int_{2r}^{\infty} \|f\|_{L_1(\Gamma(t_0,\tau))} (w^q(\Gamma(t_0,\tau)))^{-\frac{1}{q}} \frac{d\tau}{\tau} \tag{14}$$

holds for any ball $\Gamma(t_0,r)$ *and for all* $f \in L_{1,w}^{loc}(\Gamma)$.

Proof. For a given ball $\Gamma(t_0, r)$, we split the function f as $f = f_1 + f_2$, where $f_1 = f\chi_{\Gamma(t_0,2r)}$, $f_2 = f\chi_{\complement(\Gamma(t_0,2r))}$. Then

$$\mathcal{I}^\alpha f(t) = \mathcal{I}^\alpha f_1(t) + \mathcal{I}^\alpha f_2(t).$$

Let $1 < p < \infty$, $0 < \alpha < \frac{1}{p}$, $\frac{1}{q} = \frac{1}{p} - \alpha$. Since $f_1 \in L_{p,w^p}(\Gamma)$, by the boundedness of the operator $\mathcal{I}^\alpha$ from $L_{p,w^p}(\Gamma)$ to $L_{q,w^q}(\Gamma)$ (see Theorem B) it follows that

$$\|\mathcal{I}^\alpha f_1\|_{L_{q,w^q}(\Gamma, w)} \le C\|f_1\|_{L_{p,w^p}(\Gamma, w)} = C\|f\|_{L_{p,w^p}(\Gamma(t_0,2r))}$$

$$\leq C(w^q(\Gamma(t_0,r)))^{\frac{1}{q}} \int_{2r}^{\infty} \|f\|_{L_{p,w^p}(\Gamma(t_0,\tau))} (w^q(\Gamma(t_0,\tau)))^{-\frac{1}{q}} \frac{d\tau}{\tau}, \qquad (15)$$

where the constant C is independent of f.

Note that the conditions $z \in \Gamma(t_0,r)$, $y \in {}^{\complement}(\Gamma(t_0,2r))$ imply

$$\frac{1}{2}|z-y| \leq |t-y| \leq \frac{3}{2}|t-z|.$$

Then for all $z \in \Gamma(t_0,r)$ we get

$$\left| \mathcal{I}^{\alpha} f_2(z) \right| \leq \left(\frac{3}{2}\right)^{1-\alpha} \int_{\complement(\Gamma(t_0,2r))} |t-y|^{\alpha-1}|f(y)|d\nu(y).$$

By Fubini's theorem we have

$$\int_{\complement(\Gamma(t_0,2r))} |t-y|^{\alpha-1}|f(y)|d\nu(y) \approx \int_{\complement(\Gamma(t_0,2r))} |f(y)|d\nu(y) \int_{|t-y|}^{\infty} \tau^{\alpha-2}d\tau$$

$$\approx \int_{2r}^{\infty} \int_{2r \leq |t-y| < \tau} |f(y)|d\nu(y)\tau^{\alpha-2}d\tau \lesssim \int_{2r}^{\infty} \int_{\Gamma(t_0,\tau)} |f(y)|d\nu(y)\tau^{\alpha-2}d\tau.$$

Applying Hölder's inequality we get

$$\int_{\complement(\Gamma(t_0,2r))} |t-y|^{\alpha-1}|f(y)|d\nu(y) \lesssim \int_{2r}^{\infty} \|f\|_{L_{p,w^p}(\Gamma(t_0,\tau))} \|w^{-1}\|_{L_{p'}(\Gamma(t_0,\tau))} \tau^{\alpha-2}d\tau$$

$$\lesssim \int_{2r}^{\infty} \|f\|_{L_{p,w^p}(\Gamma(t_0,\tau))}(w^q(\Gamma(t_0,\tau)))^{-\frac{1}{q}} \frac{d\tau}{\tau}$$

and for all $z \in \Gamma(t_0,r)$

$$\left| \mathcal{I}^{\alpha} f_2(z) \right| \lesssim \int_{2r}^{\infty} \|f\|_{L_{p,w^p}(\Gamma(t_0,\tau))} (w^q(\Gamma(t_0,\tau)))^{-\frac{1}{q}} \frac{d\tau}{\tau}. \qquad (16)$$

Moreover, for all $p \in [1,\infty)$ the inequality

$$\|\mathcal{I}^{\alpha} f_2\|_{L_{q,w^q}(\Gamma(t_0,r))} \lesssim (w^q(\Gamma(t_0,r)))^{\frac{1}{q}} \int_{2r}^{\infty} \|f\|_{L_{p,w^p}(\Gamma(t_0,\tau))} (w^q(\Gamma(t_0,\tau)))^{-\frac{1}{q}} \frac{d\tau}{\tau}$$
$$(17)$$

is valid. Thus from (15) and (17) we get the inequality (13).

Finally, in the case $p = 1$ by the weak $(1,q)$-boundedness of the operator $\mathcal{I}^{\alpha}$ (see Theorem B) it follows that

$$\|\mathcal{I}^{\alpha} f_1\|_{WL_{q,w^q}(\Gamma(t_0,r))} \leq C\|f_1\|_{L_{1,w}(\Gamma)}$$

$$\leq C(w^q(\Gamma(t_0,r)))^{\frac{1}{q}} \int_{2r}^{\infty} \|f\|_{L_{1,w}(\Gamma(t_0,\tau))} (w^q(\Gamma(t_0,\tau)))^{-\frac{1}{q}} \frac{d\tau}{\tau}, \qquad (18)$$

where C does not depend on t_0 and r. Then from (17) and (18) we get the inequality (14). ◄

The following Spanne-Guliyev type result on the space $LM_{p,\varphi}^{\{t_0\}}(\Gamma,w)$ is valid.

Theorem 4. *Let Γ be a Carleson curve, $1 \leq p < q < \infty$, $t_0 \in \Gamma$, $0 < \alpha < \frac{1}{p}$, $\frac{1}{q} = \frac{1}{p} - \alpha$, $w \in A_{p,q}(\Gamma)$, and the pair (φ_1,φ_2) satisfy the condition*

$$\int_r^{\infty} \frac{\operatorname*{ess\,inf}_{\tau<s<\infty} \varphi_1(t_0,s)\,(w^p(\Gamma(t_0,s)))^{1/p}}{(w^q(\Gamma(t_0,\tau)))^{1/q}} \frac{d\tau}{\tau} \leq C\,\varphi_2(t_0,r), \qquad (19)$$

where C does not depend on r.

Then for $p > 1$ the operator $\mathcal{I}^\alpha$ is bounded from $LM_{p,\varphi_1}^{\{t_0\}}(\Gamma,w)$ to $LM_{q,\varphi_2}^{\{t_0\}}(\Gamma,w)$ and for $p = 1$ the operator $\mathcal{I}^\alpha$ is bounded from $LM_{1,\varphi_1}^{\{t_0\}}(\Gamma,w)$ to $WLM_{q,\varphi_2}^{\{t_0\}}(\Gamma,w)$.

Proof. By Theorems 1 and 3 with $v_2(r) = \varphi_2(t_0,r)^{-1}$, $v_1(r) = \varphi_1(t_0,r)^{-1}\,r^{-\frac{1}{p}}$ and $w(r) = r^{-\frac{1}{q}}$, we have for $p > 1$

$$\|\mathcal{I}^\alpha f\|_{LM_{q,\varphi_2}^{\{t_0\}}(\Gamma,w)} \lesssim \sup_{r>0} \varphi_2(t_0,r)^{-1} \int_r^{\infty} \|f\|_{L_{p,w^p}(\Gamma(t_0,\tau))} (w^q(\Gamma(t_0,\tau)))^{-\frac{1}{q}} \frac{d\tau}{\tau}$$

$$\lesssim \sup_{r>0} \varphi_1(t_0,r)^{-1} (w^p(\Gamma(t_0,\tau)))^{-\frac{1}{p}} \|f\|_{L_{p,w^p}(\Gamma(t_0,r))} = \|f\|_{LM_{p,\varphi_1}^{\{t_0\}}(\Gamma,w)}$$

and for $p = 1$

$$\|\mathcal{I}^\alpha f\|_{WLM_{q,\varphi_2}^{\{t_0\}}(\Gamma,w)} \lesssim \sup_{r>0} \varphi_2(t_0,r)^{-1} \int_r^{\infty} \|f\|_{L_{1,w}(\Gamma(t_0,\tau))} (w^q(\Gamma(t_0,\tau)))^{-\frac{1}{q}} \frac{d\tau}{\tau}$$

$$\lesssim \sup_{r>0} \varphi_1(t_0,r)^{-1} (w(\Gamma(t_0,\tau)))^{-1} \|f\|_{L_1(\Gamma(t_0,r))} = \|f\|_{LM_{1,\varphi_1}^{\{t_0\}}(\Gamma,w)}.$$

◄

From Theorem 2 we get the following

Corollary 2. *Let Γ be a Carleson curve, $1 \leq p < q < \infty$, $0 < \alpha < \frac{1}{p}$, $\frac{1}{q} = \frac{1}{p} - \alpha$, $w \in A_{p,q}(\Gamma)$, and the pair (φ_1,φ_2) satisfy the condition*

$$\int_r^{\infty} \frac{\operatorname*{ess\,inf}_{\tau<s<\infty} \varphi_1(t,s)\,(w^p(\Gamma(t,s)))^{1/p}}{(w^q(\Gamma(t,\tau)))^{1/q}} \frac{d\tau}{\tau} \leq C\,\varphi_2(t,r), \qquad (20)$$

where C does not depend on t and r. Then for $p > 1$ the operator $\mathcal{I}^\alpha$ is bounded from $M_{p,\varphi_1}(\Gamma,w)$ to $M_{q,\varphi_2}(\Gamma,w)$ and for $p = 1$ the operator $\mathcal{I}^\alpha$ is bounded from $M_{1,\varphi_1}(\Gamma,w)$ to $WM_{q,\varphi_2}(\Gamma,w)$.

For proving our main results, we need the following estimate.

Lemma 6. *Let Γ be a Carleson curve and $\Gamma_0 := \Gamma(t_0, r_0)$. Then $r_0^\alpha \lesssim \mathcal{I}^\alpha \chi_{\Gamma_0}(t)$ for every $t \in \Gamma_0$.*

Proof. If $t, y \in \Gamma_0$, then $|t - y| \leq |t - t_0| + |t_0 - y| < 2r_0$. Since $0 < \alpha < 1$, we get $r_0^{\alpha-1} \leq 2^{1-\alpha}|t - y|^{\alpha-Q}$. Therefore,

$$\mathcal{I}^\alpha \chi_{\Gamma_0}(t) = \int_\Gamma \chi_{\Gamma_0}(y)|t - y|^{\alpha-1} d\nu(y) = \int_{\Gamma_0} |t - y|^{\alpha-1} d\nu(y) \geq c_0 2^{1-\alpha} r_0^\alpha.$$

◀

The following theorem is one of our main results.

Theorem 5. *Let Γ be a Carleson curve, $0 < \alpha < 1$, $t_0 \in \Gamma$, $w \in A_{p,q}(\Gamma)$ and $1 \leq p < q < \infty$.*

1. *If $1 \leq p < \frac{1}{\alpha}$ and $\frac{1}{q} = \frac{1}{p} - \alpha$, then the condition (20) is sufficient for the boundedness of the operator $\mathcal{I}^\alpha$ from $LM_{p,\varphi_1}^{\{t_0\}}(\Gamma, w)$ to $WLM_{q,\varphi_2}^{\{t_0\}}(\Gamma, w)$. Moreover, if $1 < p < \frac{1}{\alpha}$, the condition (20) is sufficient for the boundedness of the operator $\mathcal{I}^\alpha$ from $LM_{p,\varphi_1}^{\{t_0\}}(\Gamma, w)$ to $LM_{q,\varphi_2}^{\{t_0\}}(\Gamma, w)$.*

2. *If $\varphi_1 \in \mathcal{G}_w^p$, then the condition*

$$r^\alpha \varphi_1(r) \leq C\varphi_2(r), \tag{21}$$

for all $r > 0$, where $C > 0$ does not depend on r, is necessary for the boundedness of the operator $\mathcal{I}^\alpha$ from $LM_{p,\varphi_1}^{\{t_0\}}(\Gamma, w)$ to $WLM_{q,\varphi_2}^{\{t_0\}}(\Gamma, w)$ and from $LM_{p,\varphi_1}^{\{t_0\}}(\Gamma, w)$ to $LM_{q,\varphi_2}^{\{t_0\}}(\Gamma, w)$.

3. *Let $1 \leq p < \frac{1}{\alpha}$ and $\frac{1}{q} = \frac{1}{p} - \alpha$. If $\varphi_1 \in \mathcal{G}_w^p$ satisfies the regularity condition*

$$\int_r^\infty s^{\alpha-1} \varphi_1(s)ds \leq Cr^\alpha \varphi_1(r), \tag{22}$$

for all $r > 0$, where $C > 0$ does not depend on r, then the condition (21) is necessary and sufficient for the boundedness of the operator $\mathcal{I}^\alpha$ from $LM_{p,\varphi_1}^{\{t_0\}}(\Gamma, w)$ to $WLM_{q,\varphi_2}^{\{t_0\}}(\Gamma, w)$. Moreover, if $1 < p < \frac{Q}{\alpha}$, then the condition (21) is necessary and sufficient for the boundedness of the operator $\mathcal{I}^\alpha$ from $LM_{p,\varphi_1}^{\{t_0\}}(\Gamma, w)$ to $LM_{q,\varphi_2}^{\{t_0\}}(\Gamma, w)$.

Proof. The first part of this theorem was proved in Theorem 2.

We shall now prove the second part. Let $\Gamma_0 = \Gamma(t_0, r_0)$ and $t \in \Gamma_0$. By Lemma 6 we have $r_0^\alpha \leq C\mathcal{I}^\alpha \chi_{\Gamma_0}(r)$. Therefore, by Lemma 4 and Lemma 6,

$$r_0^\alpha \lesssim (\nu(\Gamma_0))^{-\frac{1}{p}} \|\mathcal{I}^\alpha \chi_{\Gamma_0}\|_{L_q(\Gamma_0)} \lesssim \varphi_2(r_0)\|\mathcal{I}^\alpha \chi_{\Gamma_0}\|_{M_{q,\varphi_2}} \lesssim \varphi_2(r_0)\|\chi_{\Gamma_0}\|_{M_{p,\varphi_1}} \lesssim \frac{\varphi_2(r_0)}{\varphi_1(r_0)}$$

or

$$r_0^\alpha \lesssim \frac{\varphi_2(r_0)}{\varphi_1(r_0)} \text{ for all } r_0 > 0 \Longleftrightarrow r_0^\alpha \varphi_1(r_0) \lesssim \varphi_2(r_0) \text{ for all } r_0 > 0.$$

Since this is true for every $r_0 > 0$, we are done.

The third statement of the theorem follows from the first and the second parts of it. ◄

Remark 3. *If we take $w \equiv 1$, $\varphi_1(r) = r^{\frac{\lambda-1}{p}}$ and $\varphi_2(r) = r^{\frac{\mu-1}{q}}$ in Theorem 5, then conditions (22) and (21) are equivalent to $0 < \lambda < 1 - \alpha p$ and $\frac{\lambda}{p} = \frac{\mu}{q}$, respectively. Therefore, we get Theorem C from Theorem 5.*

Acknowledgements

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