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M-Spectral Problem for Bessel Potential Function Type with Applications and Visuals

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Abstract. In this article, we deal with the solution of the $\mathcal{M}$-Sturm Liouville problem with Bessel potential function. We use the $\mathcal{M}$-Laplace transform. In this study, it is considered as an important advantage that the strong $\mathcal{M}$-derivative includes the truncated Mittag-Leffler function, which allows it to be evaluated as an extended version of the classical derivative and other local derivatives. This derivative is an extremely powerful tool for generalizing complex problems. The purpose of this paper is to spy on the behavior of the spectral structure of the Bessel type $\mathcal{M}$-Sturm-Liouville problem with visual analysis.

Key Words and Phrases: $\mathcal{M}$-derivative, potential function, Sturm-Liouville problem

2010 Mathematics Subject Classifications: 34B24, 44A10

1. Introduction

Spectral theory is an interdisciplinary field of study that plays a key role in the translation of abstract concepts into concrete applications and the study of complex systems, as a fundamental tool in a wide range of mathematics and applied sciences. This theory opens the way for scientific research by providing deep insights in both analysis and practical engineering applications. Research conducted by scientists has shown that determining equations that can be solved in accordance with the spectral expansions of differential operators and the solutions obtained through these equations play a critical role in understanding the behavior of various mathematical and physical systems. In this context, spectral analysis has found wide application areas such as modal analysis of systems, vibration analysis and quantum mechanics by detailing the properties of differential operators.

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Thus, this theoretical framework includes not only the abstract beauties of mathematical theories, but also the solutions brought to concrete problems in applied sciences such as engineering and physics. In mathematical analysis and applied sciences, the types and properties of differential operators constitute the cornerstones of the studies carried out in these areas. In this context, scientists have defined two different types of differential operators-regular and singular operators - and have extensively studied the spectral structures of these operators [1]. This distinction plays a critical role in deep understanding of the spectral properties of differential operators and in applying these properties to various mathematical and physical problems. Therefore, the interaction of spectral theory and differential operators presents itself as an indispensable component of contemporary mathematics and scientific research.

Today, the spectral analysis of second order regular differential operators is known as Sturm-Liouville theory. Later, Titchmarsh has deepened the understanding in this area by presenting an innovative approach to the spectral analysis of second order singular operators [2]. By focusing on the study of singular differential operators, Levitan has emphasized the important role of differential operators in spectral theory [1]. There have been studies that show the difference between two potential functions is small enough [3]. In the following period, many scientists have contributed to the work in this field by addressing smooth problems related to spectral analysis for Bessel operator [4, 5]. Derakhshan et al. have made significant contributions to the knowledge in this field by conducting extensive research on fractional Sturm-Liouville problems [6]. Discrete conformable Sturm-Liouville problems with spectral parameters are addressed in a wide range of mathematical analysis from both theoretical and applied perspectives with particular attention being paid to the structure of boundary conditions and the effects of these structures on spectral properties [7]. Sousa and Oliveira have formulated a $\mathcal{M}$-derivative that preserves the characteristic features of integer order calculus, including an innovative parameter and the Mittag-Leffler function [8]. This work has contributed to a better understanding of the mechanisms underlying integer order computations and opened new perspectives for mathematical analysis. The developed $\mathcal{M}$-derivative has proved its usability in both theoretical and practical applications with the harmonious combination of the relevant parameters and the Mittag-Leffler function [9]. This innovative approach has made a significant contribution to the literature in the relevant field and has established a solid foundation for future work. The Laplace transform plays a critical role in solving Sturm-Liouville problems. This transform accelerates the analytical solution processes of the problems and ensures that results are suitable for boundary conditions. These advantages of the Laplace transform enable it to be used as an effective tool in solving mathematical analysis and engineering problems [10].

In recent years, a new mathematical tool has been developed known as the $\mathcal{M}$-Laplace transform [11]. The main purpose of the $\mathcal{M}$-Laplace transform is to overcome some of the limitations of the classical Laplace transform [11]. This transform offers a more flexible and powerful tool for solving problems frequently encountered in mathematical analysis [12]. Another significant advantage of the $\mathcal{M}$-Laplace transform is its additives to the solution of eigenvalue problems. The transform performs a major role in obtaining solutions suitable for the boundary conditions of such problems and supplies faster analytical solutions.

In this article, we highlight the Bessel type $\mathcal{M}$-Sturm-Liouville problem and its applications. Moreover, we extend the theoretical framework of the Bessel type Sturm-Liouville problem through $\mathcal{M}$-derivative. In this way, the scientific and practical aspects of the relevant problem type are evaluated comprehensively and a contribution is made to the literature.

In the second section, the basic definitions and theorems necessary to understand and discuss the findings presented in the main conclusion of our paper are explained in detail. In our study, the $\mathcal{M}$-Laplace transform stands out as a fundamental tool in our analysis process. In the third section, an analysis representation of the solution of the Bessel type $\mathcal{M}$-Sturm-Liouville problem is presented and this solution is obtained by using the $\mathcal{M}$-Laplace transform. In the fourth section, the behavior of solution is studied with a comprehensive graphical analysis for different values of the parameters ϑ, ς, s and H.

2. Preliminaries

In this section, important definitions and theorems used to obtain our results are presented in detail. These definitions and theorems, which form the scientific basis of our study, are compatible with the current information in the literature and support the scope and accuracy of our study.

Definition 1 ([13]). *Suppose that p:$[0,\infty)\to \Re$ is sone function. The $\mathcal{M}$-derivative for $0<\vartheta\leq 1$ is defined as follows:*

$$ {}_iD_{\mathcal{M}}^{\vartheta,\varsigma}p(t) = \lim_{\varepsilon\to 0} \frac{p\Big(t_i E_\varsigma(\varepsilon t^{-\vartheta})\Big) - p(t)}{\varepsilon}. \tag{1}$$

The $\mathcal{M}$-derivative is presented in many important functions [12].

Definition 2 ([11]). *Assume that* $p:[a,\infty)\rightarrow\Re$, $0<\vartheta\leq 1$ *and* $\varsigma>0$. *Laplace transform of* $\mathcal{M}$*-derivative is defined as follows:*

$$\mathcal{L}^{a}_{\vartheta,\varsigma}\{p(t)\}(s)=\Gamma(\varsigma+1)\int_{a}^{\infty}e^{-s\frac{\Gamma(\varsigma+1)(t-a)^{\vartheta}}{\vartheta}}p(t)d_{\vartheta}t, \tag{2}$$

where $d_{\vartheta}t=(t-a)^{\vartheta-1}dt$.

A detailed description of the Laplace transform in the context of the $\mathcal{M}$-derivative of several important functions is presented in [11].

Definition 3 ([11]). *Suppose that* $p(t)$ *and* $v(t)$ *are continuous functions. The convolution of* p *and* v *for the* $\mathcal{M}$*-derivative is defined by*

$$(p*v)(t)=\Gamma(\varsigma+1)\int_{a}^{t}p(\tau)v(a+((t-a)^{\vartheta}-(\tau-a)^{\vartheta})^{\frac{1}{\vartheta}})d_{\vartheta}\tau, \tag{3}$$

where $d_{\vartheta}\tau=(\tau-a)^{\vartheta-1}d\tau$.

Theorem 1 ([11]). *Let* $p,v:[a,\infty)\rightarrow\Re$, $0<\vartheta\leq 1$, $\varsigma>0$, $s>0$ *and there exist* $p(t)$ *and* $v(t)$ *such that* $P^{a}_{\vartheta,\varsigma}(s)=\mathcal{L}^{a}_{\vartheta,\varsigma}\{p(t)\}$ *and* $V^{a}_{\vartheta,\varsigma}(s)=\mathcal{L}^{a}_{\vartheta,\varsigma}\{v(t)\}$. *We have*

$$\mathcal{L}^{a}_{\vartheta,\varsigma}\{p*v\}(t)=P^{a}_{\vartheta,\varsigma}(s)V^{a}_{\vartheta,\varsigma}(s), \tag{4}$$

where $(p*v)$ *is the convolution of* p *and* v.

Theorem 2 ([11]). *Assume that* $p:[a,\infty)\rightarrow\Re$, $0<\vartheta\leq 1$ *for* $t\geq t_0$ *and there exist the constants* N,c,t_0 *such that* $|p(t)|\leq Ne^{c\Gamma(\varsigma+1)\frac{(t-a)^{\vartheta}}{\vartheta}}$. *Then*

$$\mathcal{L}_{\vartheta,\varsigma}\{D^{\vartheta,\varsigma}_{\mathcal{M}}p(t)\}=s\mathcal{L}^{a}_{\vartheta,\varsigma}\{p(t)\}-p(a) \tag{5}$$

and, in general structure,

$$\begin{aligned}\mathcal{L}_{\vartheta,\varsigma}\{D^{(n)\vartheta,\varsigma}_{\mathcal{M}}p(t)\}&=s^{n}\mathcal{L}^{a}_{\vartheta,\varsigma}\{p(t)\}-s^{n-1}p(a)\\&-s^{n-2}D^{\vartheta,\varsigma}_{\mathcal{M}}p(a)-...-sD^{(n-2)\vartheta,\varsigma}_{\mathcal{M}}p(a)-D^{(n-1)\vartheta,\varsigma}_{\mathcal{M}}p(a).\end{aligned} \tag{6}$$

3. Main results

Levitan is done important work in defining regular Sturm-Liouville operators [1]. Recently, innovative approaches develop for Sturm-Liouville problems with different potentials using conformable derivative. We submit in detail the representation of a more general solution of the $\mathcal{M}$-Sturm Liouville problem with the Bessel potential, using the Laplace transform of the $\mathcal{M}$-derivative.

3.1. Analysis of the Bessel Type $\mathcal{M}$-Sturm Liouville Problem via the $\mathcal{M}$-Laplace Transform

We address the $\mathcal{M}$-Sturm Liouville problem using the Bessel type potential function

$$-D_{\mathcal{M}}^{(2)\vartheta,\varsigma}u+\left[q(x)+\frac{l(l+1)}{x^2}\right]u=\lambda u, \tag{7}$$

where $x\in(0,\pi]$, $0<\vartheta\leq 1$, $\varsigma>0$, $|2l|<1$ and the initial conditions are

$$u(0,\lambda)=0, D_{\mathcal{M}}^{\vartheta,\varsigma}u(0,\lambda)=H \tag{8}$$

with $H=\cot\beta$.

Theorem 3. *The solution of the problem* (7)-(8) *has the following form:*

$$\begin{aligned}u(x,\lambda)=&\frac{H}{s}\sin\left(s\Gamma(\varsigma+1)\frac{x^\vartheta}{\vartheta}\right)\\&+\frac{1}{s}\int_0^x\sin\left(s\Gamma(\varsigma+1)\left(\frac{x^\vartheta}{\vartheta}-\frac{\tau^\vartheta}{\vartheta}\right)\right)\left[q(\tau)+\frac{l(l+1)}{\tau^2}\right]u(\tau,\lambda)d_\vartheta\tau.\end{aligned} \tag{9}$$

Proof. In light of the $\mathcal{M}$-Laplace transform, we examine the detailed representation of the solution. If we apply the $\mathcal{M}$-Laplace transform to both sides of (7), then

$$-\mathcal{L}_{\vartheta,\varsigma}\{D_{\mathcal{M}}^{(2)\vartheta,\varsigma}u\}+\mathcal{L}_{\vartheta,\varsigma}\left\{\left[q(x)+\frac{l(l+1)}{x^2}\right]u\right\}=\mathcal{L}_{\vartheta,\varsigma}\{\lambda u\}. \tag{10}$$

$$-\mathcal{L}_{\vartheta,\varsigma}\{D_{\mathcal{M}}^{(2)\vartheta,\varsigma}u\}=-\left[s^2U_{\vartheta,\varsigma}(s)-su(0,\lambda)-D_{\mathcal{M}}^{\vartheta,\varsigma}u(0,\lambda)\right]. \tag{11}$$

Taking into account the initial conditions in (8), we obtain

$$-\mathcal{L}_{\vartheta,\varsigma}\{D_{\mathcal{M}}^{(2)\vartheta,\varsigma}u\}=-s^2U_{\vartheta,\varsigma}(s)+H. \tag{12}$$

After some calculations, we obtain

$$U_{\vartheta,\varsigma}(s)=\frac{H}{s^2+\lambda}+\frac{1}{s^2+\lambda}\int_0^\infty e^{-s\frac{\Gamma(\varsigma+1)x^\vartheta}{\vartheta}}\left[q(x)+\frac{l(l+1)}{x^2}\right]ud_\vartheta x. \tag{13}$$

Applying the inverse Laplace transform of the $\mathcal{M}$-derivative to both sides of equation (13) yields

$$\mathcal{L}^{-1}_{\vartheta,\varsigma}\big[U_{\vartheta,\varsigma}(s)\big] = \mathcal{L}^{-1}_{\vartheta,\varsigma}\left[\frac{H}{s^2+\lambda}\right] + \mathcal{L}^{-1}_{\vartheta,\varsigma}\left[\frac{1}{s^2+\lambda}\int_0^\infty e^{-s\frac{\Gamma(\varsigma+1)x^\vartheta}{\vartheta}}\left[q(x)+\frac{l(l+1)}{x^2}\right]u d_\vartheta x\right]. \quad (14)$$

By using the $\mathcal{M}$-Laplace transform formulas in [11] and (3), we obtain the representation of the solution

$$u(x,\lambda) = \frac{H}{s}\sin\left(s\Gamma(\varsigma+1)\frac{x^\vartheta}{\vartheta}\right) + \frac{1}{s}\int_0^x \sin\left(\big(s\Gamma(\varsigma+1)\big)\left(\frac{x^\vartheta}{\vartheta}-\frac{\tau^\vartheta}{\vartheta}\right)\right)\left[q(\tau)+\frac{l(l+1)}{\tau^2}\right]u(\tau,\lambda)d_\vartheta\tau. \quad (15)$$

In equation (15), $\sqrt{\lambda}=s$ and $\Gamma(\varsigma+1)\tau^{\vartheta-1}d\tau = d_\vartheta\tau$ are taken. ◄

Application 1.

$$-D^{(2)\vartheta,\varsigma}_{\mathcal{M}}u + \left[\sin(x)+\frac{l(l+1)}{x^2}\right]u = \lambda u, \quad (16)$$

with the initial conditions

$$u(0,\lambda)=0, D^{\vartheta,\varsigma}_{\mathcal{M}}u(0,\lambda)=H. \quad (17)$$

Let us get the representation of the solution of the Bessel type $\mathcal{M}$-Sturm Liouville problem (16)-(17) using the $\mathcal{M}$-Laplace transform.

We perform the $\mathcal{M}$-Laplace transform on both sides of (16):

$$-\mathcal{L}_{\vartheta,\varsigma}\{D^{(2)\vartheta,\varsigma}_{\mathcal{M}}u\} + \mathcal{L}_{\vartheta,\varsigma}\left\{\left[\sin(x)+\frac{l(l+1)}{x^2}\right]u\right\} = \mathcal{L}_{\vartheta,\varsigma}\{\lambda u\}. \quad (18)$$

$$-\mathcal{L}_{\vartheta,\varsigma}\{D^{(2)\vartheta,\varsigma}_{\mathcal{M}}u\} = -\big[s^2U_{\vartheta,\varsigma}(s) - su(0,\lambda) - D^{\vartheta,\varsigma}_{\mathcal{M}}u(0,\lambda)\big]. \quad (19)$$

Taking into account the initial conditions (17), we obtain

$$-\mathcal{L}_{\vartheta,\varsigma}\{D^{(2)\vartheta,\varsigma}_{\mathcal{M}}u\} = -s^2U_{\vartheta,\varsigma}(s) + H. \quad (20)$$

By performing the required calculations, we acquire

$$U_{\vartheta,\varsigma}(s) = \frac{H}{s^2+\lambda} + \frac{1}{s^2+\lambda}\int_0^\infty e^{-s\frac{\Gamma(\varsigma+1)x^\vartheta}{\vartheta}}\Big[\sin(x) + \frac{l(l+1)}{x^2}\Big]u d_\vartheta x. \tag{21}$$

Applying the inverse Laplace transform of the $\mathcal{M}$-derivative to both sides of (21), we get

$$\mathcal{L}^{-1}_{\vartheta,\varsigma}\big[U_{\vartheta,\varsigma}(s)\big] = \mathcal{L}^{-1}_{\vartheta,\varsigma}\Bigg[\frac{H}{s^2+\lambda}\Bigg] + \mathcal{L}^{-1}_{\vartheta,\varsigma}\Bigg[\frac{1}{s^2+\lambda}\int_0^\infty e^{-s\frac{\Gamma(\varsigma+1)x^\vartheta}{\vartheta}}\Big[\sin(x) + \frac{l(l+1)}{x^2}\Big]u d_\vartheta x\Bigg]. \tag{22}$$

Using the $\mathcal{M}$-Laplace transform formulas in [11] and (3), we get the representation of the solution

$$u(x,\lambda) = \frac{H}{s}\sin\left(s\Gamma(\varsigma+1)\frac{x^\vartheta}{\vartheta}\right) + \frac{1}{s}\int_0^x \sin\left(\big(s\Gamma(\varsigma+1)\big)\left(\frac{x^\vartheta}{\vartheta} - \frac{\tau^\vartheta}{\vartheta}\right)\right)\Big[\sin(\tau) + \frac{l(l+1)}{\tau^2}\Big]u(\tau,\lambda)d_\vartheta\tau. \tag{23}$$

In equation (23), $\sqrt{\lambda} = s$ and $\Gamma(\varsigma+1)\tau^{\vartheta-1}d\tau = d_\vartheta\tau$.

Application 2.

$$-D^{(2)\vartheta,\varsigma}_{\mathcal{M}}u + \left[x^2 + \frac{l(l+1)}{x^2}\right]u = \lambda u \tag{24}$$

with the initial conditions

$$u(0,\lambda) = 0, D^{\vartheta,\varsigma}_{\mathcal{M}}u(0,\lambda) = H. \tag{25}$$

Let us derive the representation for the solution of the Bessel type $\mathcal{M}$-Sturm Liouville problem specified in utilization (24)-(25) using $\mathcal{M}$-Laplace transform. If we apply the $\mathcal{M}$-Laplace transform to both sides of problem (24), then

$$-\mathcal{L}_{\vartheta,\varsigma}\{D^{(2)\vartheta,\varsigma}_{\mathcal{M}}u\} + \mathcal{L}_{\vartheta,\varsigma}\left\{\Big[x^2 + \frac{l(l+1)}{x^2}\Big]u\right\} = \mathcal{L}_{\vartheta,\varsigma}\{\lambda u\}. \tag{26}$$

$$-\mathcal{L}_{\vartheta,\varsigma}\big\{D^{(2)\vartheta,\varsigma}_{\mathcal{M}}u\big\} = -\big[s^2U_{\vartheta,\varsigma}(s) - su(0,\lambda) - D^{\vartheta,\varsigma}_{\mathcal{M}}u(0,\lambda)\big]. \tag{27}$$

Taking into account the initial conditions (25), we obtain

$$-\mathcal{L}_{\vartheta,\varsigma}\{D_{\mathcal{M}}^{(2)\vartheta,\varsigma}u\} = -s^2U_{\vartheta,\varsigma}(s) + H. \tag{28}$$

After some calculations, we obtain

$$U_{\vartheta,\varsigma}(s) = \frac{H}{s^2+\lambda} + \frac{1}{s^2+\lambda}\int_0^\infty e^{-s\frac{\Gamma(\varsigma+1)x^\vartheta}{\vartheta}}\Big[x^2+\frac{l(l+1)}{x^2}\Big]ud_\vartheta x. \tag{29}$$

Applying the inverse Laplace transform of the $\mathcal{M}$-derivative to both sides of (29) we get

$$\begin{aligned}\mathcal{L}_{\vartheta,\varsigma}^{-1}\big[U_{\vartheta,\varsigma}(s)\big] &= \mathcal{L}_{\vartheta,\varsigma}^{-1}\left[\frac{H}{s^2+\lambda}\right]\\ &+ \mathcal{L}_{\vartheta,\varsigma}^{-1}\left[\frac{1}{s^2+\lambda}\int_0^\infty e^{-s\frac{\Gamma(\varsigma+1)x^\vartheta}{\vartheta}}\Big[x^2+\frac{l(l+1)}{x^2}\Big]ud_\vartheta x\right].\end{aligned} \tag{30}$$

Using the $\mathcal{M}$-Laplace transform formulas in [11] and (3), we get the representation of the solution

$$\begin{aligned}u(x,\lambda) &= \frac{H}{s}\sin\left(s\Gamma(\varsigma+1)\frac{x^\vartheta}{\vartheta}\right)\\ &+ \frac{1}{s}\int_0^x \sin\left(\big(s\Gamma(\varsigma+1)\big)\left(\frac{x^\vartheta}{\vartheta}-\frac{\tau^\vartheta}{\vartheta}\right)\right)\Big[\tau^2+\frac{l(l+1)}{\tau^2}\Big]u(\tau,\lambda)d_\vartheta\tau.\end{aligned} \tag{31}$$

In equation (31), $\sqrt{\lambda} = s$ and $\Gamma(\varsigma+1)\tau^{\vartheta-1}d\tau = d_\vartheta\tau$.

Application 3.

$$-D_{\mathcal{M}}^{(2)\vartheta,\varsigma}u + \left[\cos x + \frac{l(l+1)}{x^2}\right]u = \lambda u \tag{32}$$

with the initial conditions

$$u(0,\lambda) = 0, D_{\mathcal{M}}^{\vartheta,\varsigma}u(0,\lambda) = H. \tag{33}$$

Let us derive the representation for the solution of the Bessel type $\mathcal{M}$-Sturm Liouville problem specified in (32)-(33) using the $\mathcal{M}$-Laplace transform.

We apply the $\mathcal{M}$-Laplace transform to both sides of (32), then

$$-\mathcal{L}_{\vartheta,\varsigma}\{D_{\mathcal{M}}^{(2)\vartheta,\varsigma}u\}+\mathcal{L}_{\vartheta,\varsigma}\left\{\left[\cos x+\frac{l(l+1)}{x^2}\right]u\right\}=\mathcal{L}_{\vartheta,\varsigma}\{\lambda u\}, \tag{34}$$

$$-\mathcal{L}_{\vartheta,\varsigma}\{D_{\mathcal{M}}^{(2)\vartheta,\varsigma}u\}=-\left[s^2U_{\vartheta,\varsigma}(s)-su(0,\lambda)-D_{\mathcal{M}}^{\vartheta,\varsigma}u(0,\lambda)\right]. \tag{35}$$

Taking into account the initial conditions (33), we obtain

$$-\mathcal{L}_{\vartheta,\varsigma}\{D_{\mathcal{M}}^{(2)\vartheta,\varsigma}u\}=-s^2U_{\vartheta,\varsigma}(s)+H. \tag{36}$$

By performing the necessary operations, we acquire

$$U_{\vartheta,\varsigma}(s)=\frac{H}{s^2+\lambda}+\frac{1}{s^2+\lambda}\int_0^\infty e^{-s\frac{\Gamma(\varsigma+1)x^\vartheta}{\vartheta}}\left[\cos x+\frac{l(l+1)}{x^2}\right]ud_\vartheta x. \tag{37}$$

Applying the inverse Laplace transform of the $\mathcal{M}$-derivative to both sides of (37), we get

$$\begin{aligned}\mathcal{L}_{\vartheta,\varsigma}^{-1}\left[U_{\vartheta,\varsigma}(s)\right]&=\mathcal{L}_{\vartheta,\varsigma}^{-1}\left[\frac{H}{s^2+\lambda}\right]\\&\quad+\mathcal{L}_{\vartheta,\varsigma}^{-1}\left[\frac{1}{s^2+\lambda}\int_0^\infty e^{-s\frac{\Gamma(\varsigma+1)x^\vartheta}{\vartheta}}\left[\cos x+\frac{l(l+1)}{x^2}\right]ud_\vartheta x\right].\end{aligned} \tag{38}$$

Using the $\mathcal{M}$-Laplace transform formulas in [11] and (3), we get the representation of the solution

$$\begin{aligned}u(x,\lambda)&=\frac{H}{s}\sin\left(s\Gamma(\varsigma+1)\frac{x^\vartheta}{\vartheta}\right)\\&\quad+\frac{1}{s}\int_0^x\sin\left(\left(s\Gamma(\varsigma+1)\right)\left(\frac{x^\vartheta}{\vartheta}-\frac{\tau^\vartheta}{\vartheta}\right)\right)\left[\cos\tau+\frac{l(l+1)}{\tau^2}\right]u(\tau,\lambda)d_\vartheta\tau.\end{aligned} \tag{39}$$

In equation (39), $\sqrt{\lambda}=s$ and $\Gamma(\varsigma+1)\tau^{\vartheta-1}d\tau=d_\vartheta\tau$.

Application 4.

$$-D_{\mathcal{M}}^{(2)\vartheta,\varsigma}u+\left[e^x+\frac{l(l+1)}{x^2}\right]u=\lambda u \tag{40}$$

with the initial conditions

$$u(0,\lambda)=0, D_{\mathcal{M}}^{\vartheta,\varsigma}u(0,\lambda)=H. \tag{41}$$

Let us derive the representation for the solution of the Bessel type $\mathcal{M}$-Sturm Liouville problem specified in (40)-(41) using the $\mathcal{M}$-Laplace transform method.

If we apply the $\mathcal{M}$-Laplace transform to both sides of problem (40), then

$$-\mathcal{L}_{\vartheta,\varsigma}\{D_{\mathcal{M}}^{(2)\vartheta,\varsigma}u\} + \mathcal{L}_{\vartheta,\varsigma}\left\{\left[e^x + \frac{l(l+1)}{x^2}\right]u\right\} = \mathcal{L}_{\vartheta,\varsigma}\{\lambda u\}, \tag{42}$$

$$-\mathcal{L}_{\vartheta,\varsigma}\{D_{\mathcal{M}}^{(2)\vartheta,\varsigma}u\} = -\left[s^2 U_{\vartheta,\varsigma}(s) - su(0,\lambda) - D_{\mathcal{M}}^{\vartheta,\varsigma}u(0,\lambda)\right]. \tag{43}$$

taking into account the initial conditions (41), we obtain

$$-\mathcal{L}_{\vartheta,\varsigma}\{D_{\mathcal{M}}^{(2)\vartheta,\varsigma}u\} = -s^2 U_{\vartheta,\varsigma}(s) + H. \tag{44}$$

By taking the required operations, we receive

$$U_{\vartheta,\varsigma}(s) = \frac{H}{s^2+\lambda} + \frac{1}{s^2+\lambda}\int_0^\infty e^{-s\frac{\Gamma(\varsigma+1)x^\vartheta}{\vartheta}}\left[e^x + \frac{l(l+1)}{x^2}\right]u d_\vartheta x. \tag{45}$$

Applying the inverse Laplace transform of the $\mathcal{M}$-derivative to both sides of (45), we get

$$\begin{aligned}\mathcal{L}_{\vartheta,\varsigma}^{-1}\left[U_{\vartheta,\varsigma}(s)\right] &= \mathcal{L}_{\vartheta,\varsigma}^{-1}\left[\frac{H}{s^2+\lambda}\right] \\ &+ \mathcal{L}_{\vartheta,\varsigma}^{-1}\left[\frac{1}{s^2+\lambda}\int_0^\infty e^{-s\frac{\Gamma(\varsigma+1)x^\vartheta}{\vartheta}}\left[e^x + \frac{l(l+1)}{x^2}\right]u d_\vartheta x\right].\end{aligned} \tag{46}$$

Using the $\mathcal{M}$-Laplace transform formulas in [11] and (3), we get the representation of the solution

$$\begin{aligned}u(x,\lambda) &= \frac{H}{s}\sin\left(s\Gamma(\varsigma+1)\frac{x^\vartheta}{\vartheta}\right) \\ &+ \frac{1}{s}\int_0^x \sin\left(\left(s\Gamma(\varsigma+1)\right)\left(\frac{x^\vartheta}{\vartheta} - \frac{\tau^\vartheta}{\vartheta}\right)\right)\left[e^\tau + \frac{l(l+1)}{\tau^2}\right]u(\tau,\lambda)d_\vartheta\tau.\end{aligned} \tag{47}$$

In equation (47), $\sqrt{\lambda} = s$ and $\Gamma(\varsigma+1)\tau^{\vartheta-1}d\tau = d_\vartheta\tau$.

4. Conclusions

In this work, based on Levitan's classical studies, the spectral properties of the Sturm-Liouville problem are examined with an original perspective and through the truncated $\mathcal{M}$-derivative, one of the most novel derivatives of today's mathematical approaches. Mathematical theory of the truncated $\mathcal{M}$-derivative is discussed in detail and the effects of the $\mathcal{M}$-derivative concept on the Bessel type Sturm-Liouville problem are analyzed in depth. It is shown that the truncated $\mathcal{M}$-derivative allows for a more precise and detailed examination of spectral properties when compared to the classical derivative, thus providing a new perspective on Levitan's studies. In this context, the study aims to make significant contributions in terms of both mathematical theory and applied mathematics. The important result this is obtained by taking into account the truncated M-derivative. It is examined in detail how the truncated $\mathcal{M}$-derivative can be applied to the spectral analysis of Sturm-Liouville problems using the Laplace transform. The Laplace transform method is used as a powerful tool to understand the role and effects of the truncated $\mathcal{M}$-derivative in spectral theory. Thus, a comprehensive theoretical framework regarding the truncated $\mathcal{M}$-derivative is established and new findings are obtained on how this derivative affects the solutions and spectral properties of the Sturm-Liouville problem. In light of our observations, we note that this is a general version of the classical Bessel type Sturm-Liouville problem. In line with this goal, the graphs presented below clearly demonstrate the behavior of the resulting solution. In this way, the accuracy and consistency of the results become more easily understandable through graphical representations. Figure 1 displays the solution of equation (15) at the values s=4, s=5, s=6, ς=2, H=1 and ϑ=0.3. Figure 2 demonstrates the solution of equation (23) at the values s=2, ς=5, ς=4, H=1 and ϑ=0.5. Figure 3 shows the solution of equation (31) at the values s=0, 2, ς=3, H=1, ϑ=0.62, ϑ=0.72, ϑ=0.82 and ϑ=0.92. Figure 4 shows the solution of equation (39) at the values s=7, ς=2, H=1, ϑ=0.31, ϑ=0.36 and ϑ=0.41. Figure 5 displays the solution of equation (47) at the values s=0.8, ς=3, H=1, ϑ=0.50, ϑ=0.54 and ϑ=0.58. Figure 6 shows the solution of equation (47) at the values ς=1, ϑ=0.25, H=1, s=7, s=8 and s=9. Figure 7 shows the solution of equation (39) at the values ς=6, ς=8, ϑ=0.90, H=1 and s=3. Figure 8, show representation of the solution of equation (15) at the values ς=5, ϑ=0.65, ϑ=0.75, H=1 and s=5. The obtained data have enabled the development of more in-depth understanding of the solution of the considered problem and have also formed a strong basis for further theoretical and applied research in related areas.

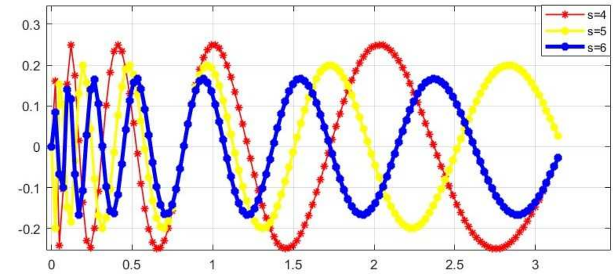

Figure 1: Visual representation of the solution (15)

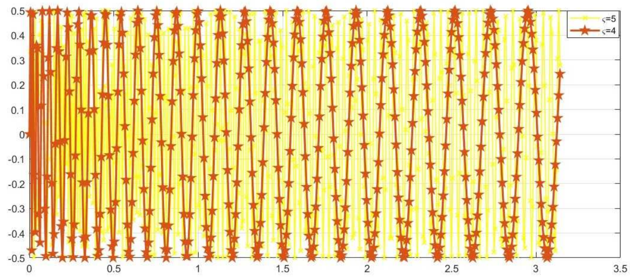

Figure 2: Visual representation of the solution (23)

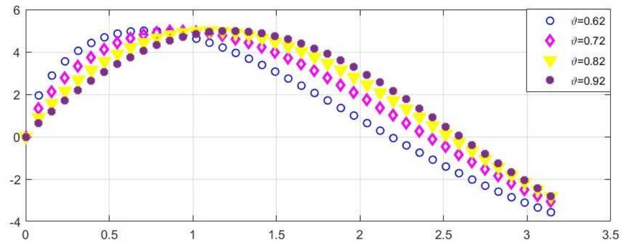

Figure 3: Visual representation of the solution (31)

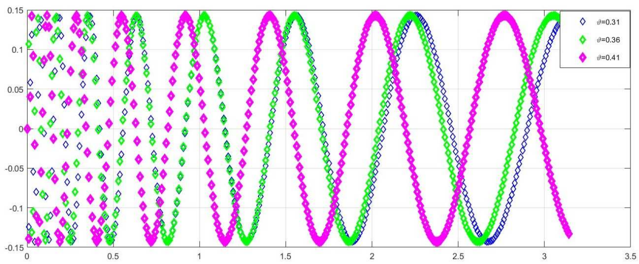

Figure 4: Visual representation of the solution (39)

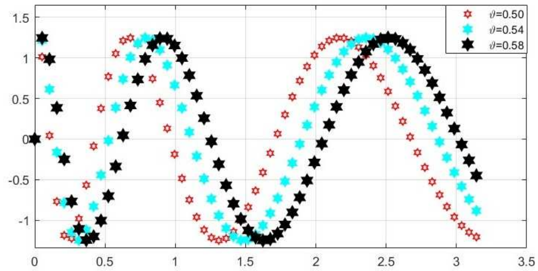

Figure 5: Visual representation of the solution (47)

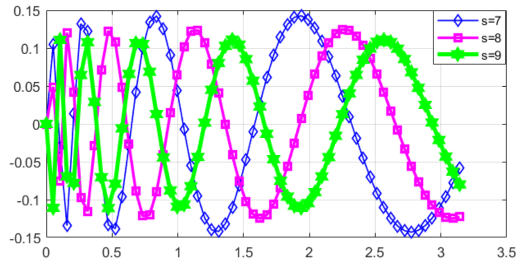

Figure 6: Visual representation of (47) for different values of s

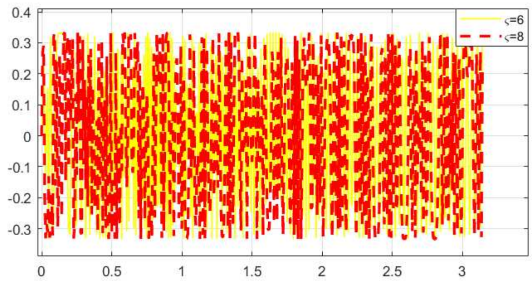

Figure 7: Visual representation of (39) for different values of ς

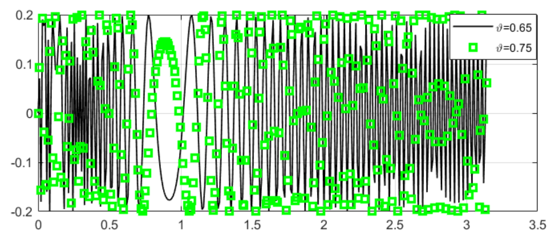

Figure 8: Visual representation of (15) for different values of ϑ

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