Azerbaijan Journal of Mathematics
V. 16, No 2, 2026, July
ISSN 2218-6816
https://doi.org/10.59849/2218-6816.2026.2.3

A Note on the Adams' Inequality

R.E. Castillo, H.C. Chaparro*

Abstract. In this paper, we present a novel proof of the well-known Adams' inequality. Our approach employs a different technique from those originally used by Adams. The primary constraints we impose on the parameters p and q are their conjugacy, expressed by the relation $\frac{1}{p} + \frac{1}{q} = 1$, and the condition $1 < p < n$.

Key Words and Phrases: Adams's inequality, Poincaré type inequality, Morrey Spaces.

2010 Mathematics Subject Classifications: 46E35, 43A20

1. Introduction and preliminaries

The study of function spaces and the inequalities governing their embeddings has been a central topic in analysis for decades. In this regard, we consider an inequality established by D.R. Adams in his 1971 paper "*Traces of potentials arising from translation invariant operators*" [1]. Adams' inequality provides an important estimate for potentials associated with certain integral operators and plays a crucial role in the study of Sobolev spaces and trace embeddings.

In this paper, we present a new proof of Adams' inequality. The motivation for revisiting this inequality stems from its wide range of applications in analysis and partial differential equations. Furthermore, alternative proofs often provide additional insights that may lead to generalizations and new connections with other branches of analysis, including nonlinear potential theory and spaces with variable smoothness and integrability. Through this work, we aim to contribute to the ongoing study of function space embeddings by offering a novel proof that may inspire further developments in the field.

In order to achieve our goal, we begin with a fairly brief review of Morrey spaces. Morrey spaces play a vital role in the study of regularity problems for solutions to PDEs. They provide a framework for characterizing the regularity of

*Corresponding author.

solutions and estimating their derivatives. Overall, Morrey spaces offer a versatile tool for studying the regularity properties of functions, particularly in the realm of PDEs.

Their introduction and development have enriched the field of analysis and provided valuable insights into the behavior of solutions to differential equations.

Definition 1 (Morrey Space)**.** *Let $1 \leq p < \infty$, $\lambda \geq 0$ and $B \subset \mathbb{R}^n$ be a ball centered at x with radius $r > 0$. The Morrey space $L^{p,\lambda}(\mathbb{R}^n)$ is defined by*

$$L^{p,\lambda}(\mathbb{R}^n) = \left\{ f \in L^p_{\text{loc}}(\mathbb{R}^n) : \sup_{x\in\mathbb{R}^n, r>0} \frac{1}{r^\lambda} \int\limits_{B(x,r)} |f(y)|^p \, dy < \infty \right\}$$

where

$$B(x,r) = \{y \in \mathbb{R}^n : |y - x| < r\}.$$

It is worth mentioning that every function belonging to a Morrey space also belongs to the classical Lebesgue space of locally integrable functions, i.e. $L^{p,\lambda}(\mathbb{R}^n) \subset L^p_{\text{loc}}(\mathbb{R}^n)$. Indeed, let $f \in L^{p,\lambda}(\mathbb{R}^n)$, then

$$\begin{aligned} \int\limits_{B(x,r)} |f(x)|^p \, dx =& r^\lambda \left(\frac{1}{r^\lambda} \int\limits_{B(x,r)} |f(x)|^p \, dx \right) \\ \leq& r^\lambda \cdot \sup_{x\in\mathbb{R}^n, r>0} \left(\frac{1}{r^\lambda} \int\limits_{B(x,r)} |f(y)|^p \, dy \right) < \infty. \end{aligned}$$

Hence, $f \in L^p_{\text{loc}}(\mathbb{R}^n)$.

The Morrey space is indeed a complete normed vector space. Although this is a well-known fact, for the reader's convenience, we include a proof.

Theorem 1. *The space $L^{p,\lambda}(\mathbb{R}^n)$ is complete with respect to the norm*

$$\|f\|_{L^{p,\lambda}(\mathbb{R}^n)} = \sup_{x\in\mathbb{R}^n, r>0} \left(\frac{1}{r^\lambda} \int\limits_{B(x,r)} |f(y)|^p \, dy \right)^{\frac{1}{p}}.$$

Proof. Let $\{f_n\}_{n\in\mathbb{N}}$ be a Cauchy sequence in $L^{p,\lambda}$ with respect to the norm $\|\cdot\|_{L^{p,\lambda}}$. Then for all $\epsilon > 0$, there exists $N \in \mathbb{N}$ such that, for all $m, n \geq N$:

$$\|f_m - f_n\|_{L^{p,\lambda}} < \epsilon.$$

This implies that, for every ball $B(x,r) \subset \mathbb{R}^n$

$$\left(\frac{1}{r^\lambda} \int\limits_{B(x,r)} |f_m(y) - f_n(y)|^p \, dy \right)^{\frac{1}{p}} < \epsilon.$$

Hence,

$$\|f_m - f_n\|_{L^p(B(x,r))} < \epsilon r^{\frac{\lambda}{p}}.$$

Therefore, $\{f_k\}$ is a Cauchy sequence in $L^p(B(x,r))$. Since $L^p(B(x,r))$ is complete (see [3, p. 5]), there exists a function $f_{x,r} \in L^p(B(x,r))$ such that

$$f_k \to f_{x,r} \quad \text{in } L^p(B(x,r)).$$

Since this holds for every ball $B(x,r)$, there exists a function $f \in L^p_{loc}(\mathbb{R}^n)$ such that $f_k \to f$ in $L^p(B(x,r))$ for every ball.

Now, we prove that $f \in L^{p,\lambda}(\mathbb{R}^n)$. To this end, we use the convergence

$$f_k \to f \quad \text{in } L^p_{loc}(\mathbb{R}^n)$$

and the lower semicontinuity of the Morrey norm with respect to convergence in $L^p_{loc}(\mathbb{R}^n)$.

$$\left(\frac{1}{r^\lambda} \int\limits_{B(x,r)} |f(y)|^p \, dy \right)^{\frac{1}{p}} \le \liminf_{k\to\infty} \left(\frac{1}{r^\lambda} \int\limits_{B(x,r)} |f_k(y)|^p \, dy \right)^{\frac{1}{p}}.$$

Taking the supremum over all $x \in \mathbb{R}^n, r > 0$, we have

$$\|f\|_{L^{p,\lambda}} \le \liminf_{k\to\infty} \|f_k\|_{L^{p,\lambda}} < \infty,$$

(since $\{f_k\}$ is Cauchy, the norms are bounded). Therefore, $f \in L^{p,\lambda}(\mathbb{R}^n)$. It remains to prove that

$$\|f_k - f\|_{L^{p,\lambda}} \to 0.$$

Indeed, for a given $\epsilon > 0$, choose N such that $\|f_k - f_n\|_{L^{p,\lambda}} < \epsilon$ for all $k, n \ge N$. Since $f_n \to f$ in $L_p(B(x,r))$ and by previous semicontinuous argument, we get

$$\|f_k - f\|_{L^{p,\lambda}} \le \liminf_{k\to\infty} \|f_k - f_n\|_{L^{p,\lambda}} < \epsilon.$$

Thus, $f_k \to f$ in the Morrey norm. Hence, we have proved that $L^{p,\lambda}(\mathbb{R}^n)$ is complete with respect to its norm. ◀

2. A new proof of Adams' Inequality

In his paper [1], D.R. Adams proved the following inequality

$$\left(\int_{\mathbb{R}^n} |u(x)|^q V(x)\, dx\right)^{\frac{1}{q}} \leq C(p,\lambda,n)\|V\|^{\frac{1}{q}}_{L^{p,\lambda}(\mathbb{R}^n)}\|\nabla u\|_{L_p(\mathbb{R}^n)}, \tag{1}$$

for $V \in L^{p,\lambda}(\mathbb{R}^n)$, for all $u \in C_0^\infty(\mathbb{R}^n)$, $q = \frac{p\lambda}{n-p}$, $1 < p < n$. As announced at the beginning of this note, we shall use a different technique to prove (1).

Theorem 2 (Adam's inequality)**.** *Let $V \in L^{,\lambda}$, $1 < p < n$, $q = \frac{p\lambda}{n-p}$ and $u \in C_0^\infty(\mathbb{R}^n)$. Then there is a constant $C = C(n,q,\lambda)$ such that*

$$\left(\int_{\mathbb{R}^n} |u(x)|^q |V(x)|\, dx\right)^{\frac{1}{q}} \leq C\|V\|^{\frac{1}{q}}_{L^{p,\lambda}(\mathbb{R}^n)}\|\nabla u\|_{L^p(\mathbb{R}^n)}, \tag{2}$$

where $\frac{1}{p} + \frac{1}{q} = 1$.

Proof. For any $u \in C_0^\infty$, let us consider a ball $B = B(x,r)$ such that $u \in C_0^\infty(B)$. By using the well known inequality

$$|u(x)| \leq C \int_{B(x,r)} \frac{|\nabla u(y)|}{|x-y|^{n-1}}\, dy,$$

(see [2, Lemma 2.1]) and the Hölder inequality, we have

$$\begin{aligned}
|u(x)|^q \leq & \left(C \int_{B(x,r)} \frac{|\nabla u(y)|}{|x-y|^{n-1}}\, dy\right)^q \\
\leq & C\left(\int_{B(x,r)} \frac{dy}{|x-y|^{q(n-1)}}\, dy\right)\left(\int_{B(x,r)} |\nabla u(y)|^p\, dy\right)^{\frac{q}{p}} \\
\leq & C\left(\omega_n \int_0^\infty \frac{t^{n-1}\chi_{B(0,r)}(t)}{t^{q(n-1)}}\, dt\right)\left(\int_{\mathbb{R}^n} |\nabla u(y)|^p\, dy\right)^{\frac{q}{p}} \\
= & C\left(\omega_n \int_0^r t^{n-1-q(n-1)}\, dt\right)\|\nabla u\|^q_{L^p(\mathbb{R}^n)}
\end{aligned}$$

$$
\begin{aligned}
&=C\left[\frac{\omega_n}{n-q(n-1)}\, r^{n-q(n-1)}\right] \|\nabla u\|^q_{L^p(\mathbb{R}^n)}\\
&=C\left[\frac{\omega_n}{n-q(n-1)}\, r^{q\left(1-\frac{n}{p}\right)}\right] \|\nabla u\|^q_{L^p(\mathbb{R}^n)}\\
&=C\left[\frac{\omega_n}{n-q(n-1)}\, r^{-q\left(\frac{n-p}{p}\right)}\right] \|\nabla u\|^q_{L^p(\mathbb{R}^n)}\\
&=C\left[\frac{\omega_n}{n-q(n-1)}\, r^{-\frac{p\lambda}{n-p}\left(\frac{n-p}{p}\right)}\right] \|\nabla u\|^q_{L^p(\mathbb{R}^n)}\\
&=C\left[\frac{\omega_n}{n-q(n-1)}\, r^{-\lambda}\right] \|\nabla u\|^q_{L^p(\mathbb{R}^n)}.
\end{aligned}
$$

Hence

$$
|u(x)|^q \le C\left[\frac{\omega_n}{n-q(n-1)}\, r^{-\lambda}\right] \|\nabla u\|^q_{L^p(\mathbb{R}^n)}. \tag{3}
$$

Next, multiplying both sides of (3) by $|V(x)|$ and integrating over $B(x_0,r)$ with respect to x, and applying Hölder's inequality once more, we obtain

$$
\begin{aligned}
\int_{B(x_0,r)} |V(x)||u(x)|^q\,dx &\le \left[\frac{\omega_n}{n-q(n-1)}\, r^{-\lambda}\right]\left(\int_{B(x_0,r)} |V(x)|\,dx\right)\|\nabla u\|^q_{L^p(\mathbb{R}^n)}\\
&\le \left[\frac{\omega_n}{n-q(n-1)}\, r^{-\lambda}\right]\left(\int_{B(x_0,r)} |V(x)|^p\,dx\right)^{\frac{1}{p}} (\mathrm{m}_n(B(x_0,r)))^{\frac{1}{q}}\,\|\nabla u\|^q_{L^p(\mathbb{R}^n)}\\
&= \left[\frac{\omega_n}{n-q(n-1)}\, r^{-\lambda}\,(\omega_n r^n)^{\frac{1}{q}}\right]\left(\int_{B(x_0,r)} |V(x)|^p\,dx\right)^{\frac{1}{p}} \|\nabla u\|^q_{L^p(\mathbb{R}^n)}\\
&= \left[\frac{\omega_n}{n-q(n-1)}\, \omega_n^{\frac{1}{q}} r^{-\frac{\lambda}{p}} r^{-\frac{\lambda}{q}+\frac{n}{q}}\right]\left(\int_{B(x_0,r)} |V(x)|^p\,dx\right)^{\frac{1}{p}} \|\nabla u\|^q_{L^p(\mathbb{R}^n)}\\
&= \left[\frac{\omega_n}{n-q(n-1)}\, \omega_n^{\frac{1}{q}} r^{-\frac{\lambda}{q}+\frac{n}{q}}\right]\left(\frac{1}{r^\lambda}\int_{B(x_0,r)} |V(x)|^p\,dx\right)^{\frac{1}{p}} \|\nabla u\|^q_{L^p(\mathbb{R}^n)}\\
&= \left[\frac{\omega_n^{1+\frac{1}{q}}}{n-q(n-1)}\, \frac{(2r)^{\frac{n-\lambda}{q}}}{2^{\frac{n-\lambda}{q}}}\right]\left(\frac{1}{r^\lambda}\int_{B(x_0,r)} |V(x)|^p\,dx\right)^{\frac{1}{p}} \|\nabla u\|^q_{L^p(\mathbb{R}^n)}
\end{aligned}
$$

$$= \left[\frac{\omega_n^{1+\frac{1}{q}}}{2^{\frac{n-\lambda}{q}}(n-q(n-1))}[\mathrm{diam}(B_r)]^{\frac{n-\lambda}{q}}\right]\left(\frac{1}{r^\lambda}\int\limits_{B(x_0,r)}|V(x)|^p\,dx\right)^{\frac{1}{p}}\|\nabla u\|^q_{L^p(\mathbb{R}^n)}$$

$$\leq \left[\frac{\omega_n^{1+\frac{1}{q}}}{2^{\frac{n-\lambda}{q}}(n-q(n-1))}[\mathrm{diam}(B_r)]^{\frac{n-\lambda}{q}}\right]\|V\|_{L^{p,\lambda}}\|\nabla u\|^q_{L^p(\mathbb{R}^n)}.$$

Finally

$$\int\limits_{\mathbb{R}^n}|V(x)||u(x)|^q\,dx \leq C(q,\lambda,n)[\mathrm{diam}(B_r)]^{\frac{n-\lambda}{q}}\|V\|_{L^{p,\lambda}}\|\nabla u\|^q_{L^p(\mathbb{R}^n)}.$$

Hence

$$\left(\int\limits_{\mathbb{R}^n}|V(x)||u(x)|^q\,dx\right)^{\frac{1}{q}} \leq C(q,\lambda,n)\|V\|^{\frac{1}{q}}_{L^{p,\lambda}}\|\nabla u\|_{L^p(\mathbb{R}^n)}$$

where

$$C(q,\lambda,n) = C\left[\frac{\omega_n^{1+\frac{1}{q}}}{2^{\frac{n-\lambda}{q}}(n-q(n-1))}\right][\mathrm{diam}(B_r)]^{n-\lambda},$$

as was announced. ◄

We conclude this note by presenting a Poincaré-type inequality, which may be proved in a similar way to (1).

Theorem 3. *Let $V \in L^{p,\lambda}(\mathbb{R}^n)$, $1 < p < n$, $u \in C_0^\infty(\mathbb{R}^n)$. Then*

$$\left(\int\limits_{B_R}|V(x)||u(x)-u_{B_R}|^q\,dx\right)^{\frac{1}{q}} \leq C(p,q,\lambda,n)\|V\|^{\frac{1}{q}}_{L^{p,\lambda}}\|\nabla u\|_{L_p},$$

where $u_{B_R} = \frac{1}{\mathrm{m}_n(B_R)}\int\limits_{B_R}u(y)\,dy$, and m_n stands for the Lebesgue measure on $\mathbb{R}^n$.

Proof. First, choose an arbitrary ball $B(x,R) \subset \mathbb{R}^n$. We claim that there exists a constant C, depending on n, such that

$$|u(x)-u_{B_R}| \leq C\int\limits_{B(x,R)}\frac{|\nabla u(y)|}{|x-y|^{n-1}}.$$

To this end, fix an arbitrary point $\omega \in \partial B(0,1)$. Then, for $0 < s < R$ we have

$$\begin{aligned}|u(x+s\omega)-u(x)| &= \left|\int_0^s \frac{d}{dt}u(x+t\omega)\,dt\right| \\ &= \left|\int_0^s \nabla u(x+t\omega)\cdot\omega\,dt\right| \\ &\le \int_0^s |\nabla u(x+t\omega)|\,dt.\end{aligned}$$

Hence

$$\begin{aligned}\int\limits_{\partial B(0,1)} |u(x+s\omega)-u(x)|\,ds &\le \int\limits_0^s \int\limits_{\partial B(0,1)} |\nabla u(x+t\omega)|\,dsdt \\ &= \int_0^s \int\limits_{\partial B(0,1)} |\nabla u(x+t\omega)|\frac{t^{n-1}}{t^{n-1}}\,dsdt.\end{aligned}$$

Setting $y = x + t\omega$, we obtain $t\omega = y - x$ and $t = |x-y|$. Since $|\omega| = 1$, it follows that

$$\begin{aligned}\int\limits_{\partial B(0,1)} |u(x+s\omega)-u(x)|\,ds &\le \int_0^s \int\limits_{\partial B(0,1)} \frac{|\nabla u(y)|}{|x-y|^{n-1}}t^{n-1}\,dsdt \\ &= \int\limits_{\partial B(0,1)} \int_0^s \frac{|\nabla u(y)|}{|x-y|^{n-1}}t^{n-1}\,dsdt \\ &= \int\limits_{\partial B(x,s)} \frac{|\nabla u(y)|}{|x-y|^{n-1}}\,dy \\ &\le \int\limits_{\partial B(x,R)} \frac{|\nabla u(y)|}{|x-y|^{n-1}}\,dy.\end{aligned}$$

In this way, we obtain the inequality

$$\int\limits_{\partial B(0,1)} |u(x+s\omega)-u(x)|\,ds \le \int\limits_{\partial B(x,R)} \frac{|\nabla u(y)|}{|x-y|^{n-1}}\,dy. \tag{4}$$

Multiplying both sides of (4) by s^{n-1}, and integrating from 0 to R with respect to s, we obtain

$$\int\limits_{\partial B(0,1)} |u(x+s\omega)-u(x)|s^{n-1}\,ds \le s^{n-1} \int\limits_{B(0,R)} \frac{|\nabla u(y)|}{|x-y|^{n-1}}\,dy.$$

Then

$$\int_0^R \int_{\partial B(0,1)} |u(x+s\omega)-u(x)|s^{n-1}\,dsds \leq \int_0^R s^{n-1} \int_{B(x,R)} \frac{|\nabla u(y)|}{|x-y|^{n-1}}\,dyds,$$

thus

$$\int_{\partial B(0,1)} \int_0^R |u(x+s\omega)-u(x)|s^{n-1}\,dsds \leq \frac{R^n}{n} \int_{B(x,R)} \frac{|\nabla u(y)|}{|x-y|^{n-1}}\,dy,$$

and hence

$$\int_{B(x,R)} |u(y)-u(x)|\,dy \leq \frac{R^n}{n} \int_{B(x,R)} \frac{|\nabla u(y)|}{|x-y|^{n-1}}\,dy,$$

and so

$$\frac{1}{\mathrm{m}_n(B(0,1))R^n} \int_{B(x,R)} |u(y)-u(x)|\,dy \leq \frac{1}{\mathrm{m}_n(B(0,1))n} \int_{\partial B(x,R)} \frac{|\nabla u(y)|}{|x-y|^{n-1}}\,dy.$$

Finally

$$\begin{aligned}
|u(x)-u_{B_R}| &= \left| \frac{1}{\mathrm{m}_n(B(x,R))} \int_{B(x,R)} (u(x)-u(y))\,dy \right| \\
&\leq \frac{1}{\mathrm{m}_n(B(x,R))} \int_{B(x,R)} |u(x)-u(y)|\,dy \\
&\leq C(n) \int_{B(x,R)} \frac{|\nabla u(y)|}{|x-y|^{n-1}}\,dy.
\end{aligned}$$

Thus,

$$|u(x)-u_{B_R}| \leq C(n) \int_{B(x,R)} \frac{|\nabla u(y)|}{|x-y|^{n-1}}\,dy,$$

which proves the claim.

The rest of the proof follows line by line as in the proof of Theorem 2. ◄

Acknowledgments

We would like to express our special gratitude to the anonymous reviewer. His/her comments, suggestions, and corrections have greatly enhanced the quality of this article.

Declarations

The authors have no conflicts of interest to declare that are relevant to the content of this article. No funds, grants, or other support was received.

References

[1] Adams, D.R. (1971) *Traces of potentials arising from translation invariant operators*, Ann. Scuola Norm. Sup. Pisa, **25**, 203–217.

[2] Castillo, R.E. (2024) *On Fefferman's inequality. A simple proof*, Acta Univ. Sapientiae, Mathematica, **16(1)**, 93–97.

[3] Maz'ya, V.G., Poborchi, S.V. (1997) *Differentiable Functions on Bad Domains*, World Scientific, Singapore–New Jersey–London–Hong Kong.

René Erlín Castillo
Department of Mathematics Universidad Nacional de Colombia, Bogotá, Colombia
E-mail: recastillo@unal.edu.co

Héctor Camilo Chaparro
Program of Mathematics Universidad de Cartagena, Cartagena de Indias, Colombia
E-mail: hchaparrog@unicartagena.edu.co

Received 27 March 2025
Accepted 12 April 2026