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Solvability of a New Fractional Differential Equation in a Double Sequence Space bv_q^2

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Abstract. We define a new Caputo fractional differential equation of higher-order on bounded interval and we introduce the new double sequence space bv_q^2 and we define a Hausdorff MNC on bv_q^2. Further, by using this MNC we discuss the solvability of infinite systems of new fractional deferential equations in bv_q^2. Also, we give one example to show the usefulness of our result.

Key Words and Phrases: double sequence space, fractional differential equations, Hausdorff measure of noncompactness, Meir-Keeler condensing operator.

2010 Mathematics Subject Classifications: 47H09, 47H10, 46B45, 40A05, 34A08

1. Introduction and preliminaries

The fractional calculus appears has tremendous applications in the field of engineering, chemistry, physics, economics and many other fields ([2, 7, 12, 19, 20, 21, 22, 23]).

The measure of noncompactness (MNC) introduced by Kuratowski [25] is a powerful tool for studying the solvability of differential and integral equations. In 1957, Goldenstein et al. [15] defined the Hausdorff MNC χ and further studied in [16]. In recent time, the regular measures of noncompactness for certain Banach spaces, Fréchet spaces and sequence spaces have been emerged as an active area of research [4, 5, 6, 9, 11, 14, 17, 18, 27, 28, 30].

Recently, the solvability of n-order fractional differential equations was studied in the double sequence space $m^2(\Delta_v^u, \phi, p)$ [3]. In the present work, we define a new double sequence space of bounded variation bv_q^2 and construct a Hausdorff MNC on this space. Also, we study the solvability of a new Caputo fractional differential equation in bv_q^2. Finally, we give an example illustrating the working rule of our result.

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Let Λ be a real Banach space, and $\emptyset \neq \mathfrak{L} \subset \Lambda$. Then

• $\overline{\mathfrak{L}}$ is closure of $\mathfrak{L}$ and Conv$\mathfrak{L}$ is closed convex hull.

• $D(\nu, \sigma)$ is a closed ball in Λ.

• $\emptyset \neq \mathfrak{N}_\Lambda \subseteq \Lambda$ is the family of relatively compact, $\emptyset \neq \mathfrak{M}_\Lambda \subseteq \Lambda$ is the family of bounded.

Definition 1 ([8])**.** *The mappings $\mu : \mathfrak{M}_\Lambda \to \mathbb{R}_+$, is a measures of noncompactness (MNC) in Λ if*

(1) $\mathfrak{N}_\Lambda \supseteq \ker \mu = \{\mathcal{W} \in \mathfrak{M}_\Lambda : \mu(\mathcal{W}) = 0\} \neq \emptyset$.

(2) *If* $\mathcal{W} \subset \mathcal{Q}, \ \Rightarrow \ \mu(\mathcal{W}) \leq \mu(\mathcal{Q})$.

(3) $\mu(\mathcal{W}) = \mu(\overline{\mathcal{W}}) = \mu(Conv\mathcal{W})$.

(4) $\mu(\digamma\mathcal{W} + (1-\digamma)\mathcal{Q}) \leq \digamma\mu(\mathcal{W}) + (1-\digamma)\mu(\mathcal{Q})$ *for each* $0 \leq \digamma \leq 1$.

(5) *If for each* $n \in \mathbb{N}$, $\overline{\mathcal{W}_n} = \mathcal{W}_n \subseteq \mathfrak{M}_\Lambda$, $\mathcal{W}_{n+1} \subset \mathcal{W}_n$ *If* $\lim_{n\to\infty} \mu(\mathcal{W}_n) = 0$, $\Rightarrow$ $\emptyset \neq \mathcal{W}_\infty = \bigcap_{n=1}^{\infty} \mathcal{W}_n$.

Definition 2 ([8])**.** *Let (X, d) be a metric space and $\mathcal{R} \in \mathfrak{M}_X$. The Kuratowski MNC of $\mathcal{R}$ is denoted*

$$\chi(\mathcal{R}) = \inf\Big\{0 < \varepsilon : \mathcal{R} \subset \bigcup_{\imath=1}^{n} \mathcal{U}_\imath, \mathcal{U}_\imath \subset \mathcal{U}, diam(\mathcal{U}_\imath) < \varepsilon \ (\imath = 1, \ldots, n); \ n \in \mathbb{N}\Big\},$$

where $diam(\mathcal{U}_\imath) = \sup\{d(\nu, \varsigma) : \nu, \varsigma \in \mathcal{U}_\imath\}$,

and the Hausdorff MNC is defined by

$$\chi(\mathcal{R}) = \inf\Big\{0 < \varepsilon : \mathcal{R} \subset \bigcup_{\imath=1}^{n} D(y_\imath, \upsilon_\imath), y_\imath \in X, \upsilon_\imath < \varepsilon \ (\imath = 1, \ldots, n); \ n \in \mathbb{N}\Big\}.$$

Definition 3 ([26])**.** *Let $\tilde{\mu}$ be an arbitrary MNC on Λ and $\emptyset \neq \mathcal{B} \subseteq \Lambda$, $\mathcal{H} : \mathcal{B} \to \mathcal{B}$ is a Meir–Keeler condensing operator if for each $0 < \varepsilon$, and for any bounded $\mathcal{U} \subseteq \mathcal{B}$, $\exists\ \delta > 0$ so that*

$$\varepsilon \leq \tilde{\mu}(\mathcal{U}) < \varepsilon + \delta \quad implies \quad \tilde{\mu}(\mathcal{H}(\mathcal{U})) < \varepsilon$$

Theorem 1 ([1])**.** *Let $\emptyset \neq \mathcal{B} = \overline{\mathcal{B}} \subseteq \Lambda$ be bounded and convex. And $\tilde{\mu}$ be an arbitrary MNC on Λ and $\mathcal{H} : \mathcal{B} \to \mathcal{B}$ be a continuous Meir–Keeler condensing operator. Then $\mathcal{H}$ has at least one fixed point.*

Lemma 1 ([8])**.** *Let $\Upsilon \subseteq C(J, \Lambda)$ be bounded and equicontinuous. Then $\tilde{\mu}(\Upsilon(.))$ is continuous on J and*

$$\tilde{\mu}(\Upsilon) = \sup_{\tau\in J} \tilde{\mu}(\Upsilon(\wp)), \quad \tilde{\mu}\Big(\int_0^\wp \Upsilon(\varrho) d\varrho\Big) \leq \int_0^\wp \tilde{\mu}(\Upsilon(\varrho)) d\varrho.$$

2. Double sequence space bv_q^2

Basar and Altay [10] defined the new space of sequences bv_q as following:

$$bv_q = \{\jmath = (\jmath_k) \in \omega : \sum_{k=1}^{\infty} |\jmath_k - \jmath_{k-1}|^q < \infty\} \quad (1 \leq q < \infty),$$

where ω is the set of all complex sequences.
A double sequence $z = \langle z_{kl} \rangle$ of complex or real numbers is bounded if

$$\|z\|_{(\infty,2)} = \sup_{k,l} |z_{kl}| < \infty.$$

Definition 4 ([13]). *The double sequence $\{f_{kl}\}$ is pointwise convergent or Pringsheim limit function to f on $S \subset \mathbb{R}$ if, for each point $\jmath \in S$ and for each $0 < \varepsilon$, $\exists$ a integer $N(\jmath, \varepsilon) > 0$ so that $|f_{jk}(\jmath) - f(\jmath)| < \varepsilon$, $\forall\ k, l > N$. We write $\lim_{k,l\to\infty} f_{kl}(\jmath) = f(\jmath)$ or $f_{kl} \to f$ on S.*

We defined the double sequence space bv_q^2 as follows:

$$bv_q^2 = \{\jmath = (\jmath_{kl}) \in \omega^2 : \sum_{k,l=1}^{\infty} |\jmath_{kl} - \jmath_{k-1,l-1}|^q < \infty\} \quad (1 \leq q < \infty),$$

Theorem 2. *Double sequence bv_q^2 is a Banach space.*

Proof. Let $\xi > 0$ and $\jmath_{kl}^i$ be a Cauchy sequence in bv_q^2, choose $m_0 \in \mathbb{N}$ so that

$$\|\jmath_{kl}^i - \jmath_{kl}^j\|_{bv_q^2} < \xi,$$

$\forall\ k, l \in \mathbb{N}$ and $i, j \geq m_0$. So

$$\sum_{k,l=1}^{\infty} |(\jmath_{kl}^i - \jmath_{k-1,l-1}^i) - (\jmath_{kl}^j - \jmath_{k-1,l-1}^j)|^q < \xi,$$

$\forall\ i, j \geq m_0$ and for each fixed $(k, l) \in \mathbb{N} \times \mathbb{N}$. So $\jmath_{kl}^i$ is a Cauchy sequence in $\mathbb{C}$. So $\exists\ \jmath_{kl} \in \mathbb{C}$, so that $\jmath_{kl}^i \to \jmath_{kl}$, as $i \to \infty$, $\forall\ (n, k) \in \mathbb{N} \times \mathbb{N}$. So, $\forall\ i, j \geq m_0$. Thus we have

$$\sum_{k,l=1}^{\infty} |(\jmath_{kl}^i - \jmath_{k-1,l-1}^i) - (\jmath_{kl}^j - \jmath_{k-1,l-1}^j)|^q < \xi,$$

$\forall\ i, j \geq m_0$. This implies that $\jmath_{kl}^i - \jmath_{kl} \in bv_q^2$ for all $i, j \geq m_0$. ◄

Theorem 3 ([29]). *Let $C \subseteq l_q$ $(1 \leq q \leq \infty)$ be bounded. If $q_n : l_q \to l_q$ be the projector defined by $q_n(\jmath) = \jmath^{[n]} = (\jmath_0, \jmath_1, \ldots, \jmath_n, 0, 0, \ldots)$ $\forall\ \jmath \in l_q$. Then,*

$$\mathcal{X}(C) = \lim_{n\to\infty} \left(\sup_{\jmath \in C} \|(I - q_n)(\jmath)\|_{l_q} \right).$$

If $C \in \mathfrak{M}_{l_q}$, then

$$\mathcal{X}(C) = \lim_{n\to\infty} \left(\sup_{\jmath \in C} \sum_{k \geq n} |\jmath_k|^q \right).$$

Theorem 4. *Let $\mathcal{R} \subseteq bv_q^2$ be bounded. If $q_{kl} : bv_q^2 \to bv_q^2$ be projector defined by $q_{kl}(\jmath) = (\jmath_{k1}, \jmath_{k2}, \ldots, \jmath_{kl}, \jmath_{k1}, \jmath_{k2}, \ldots, \jmath_{kl},, 2\overline{\theta}, 2\overline{\theta}, \ldots)$ where $\jmath = \jmath_{kl} \in bv_q^2\ \forall\ k, l \in \mathbb{N}$. Then,*

$$\mathcal{X}(\mathcal{R}) = \lim_{k,l\to\infty} \Big(\sup_{\jmath\in\mathcal{R}} \|(I - q_{kl})(\jmath)\|_{bv_q^2} \Big).$$

If $\mathcal{R} \in \mathfrak{M}_{bv_q^2}$, then

$$\mathcal{X}(\mathcal{R}) = \lim_{k,l\to\infty} \big(\sum_{m\geq k, n\geq l} |\ell_{mn} - \ell_{m-1,n-1}|^q \big)^{\frac{1}{q}}. \tag{1}$$

Proof. Let $\mathcal{R} \subseteq bv_q^2$ be bounded and q_{kl} a projection mapping then,

$$\mathcal{R} \subset q_{kl}\mathcal{R} + (1 - q_{kl})\mathcal{R}. \tag{2}$$

By (2) and properties of χ, we get

$$\begin{aligned} \chi(\mathcal{R}) &\leq \chi(q_{kl}\mathcal{R}) + \chi\big((I - q_{kl})\mathcal{R}\big) = \chi\big((I - q_{kl})\mathcal{R}\big) \\ &\leq \dim\big((I - q_{kl})\mathcal{R}\big) = \sup_{\jmath\in\mathcal{R}} \|(I - q_{kl})\jmath\|_{bv_q^2}. \end{aligned}$$

So we have

$$\chi(\mathcal{R}) \leq \lim_{k,l\to\infty} \sup_{\jmath\in\mathcal{R}} \|(I - q_{kl})\jmath\|_{bv_q^2}. \tag{3}$$

Moreover, let $\{L^1, L^2, \ldots, L^k\}$ be a $[\mathcal{X}(\mathcal{R}) + \varepsilon]$-net of $\mathcal{R}$. Then

$$\mathcal{R} \subset \{L^1, L^2, \ldots, L^k\} + [\chi(\mathcal{R}) + \varepsilon]\mathbb{B}\big(bv_q^2\big),$$

where $\mathbb{B}_{\left(bv_q^2\right)}(2\overline{\theta}, 1)$ is unit ball of bv_q^2. So

$$\sup_{\jmath\in\mathcal{R}} \|(I - q_{kl})\jmath\|_{bv_q^2} \leq \sup_{1\leq i\leq k} \|(I - q_{kl})L^i\|_{bv_q^2} + [\chi(\mathcal{R}) + \varepsilon].$$

Hence

$$\lim_{k,l\to\infty} \sup_{\jmath\in\mathcal{R}} \|(I - q_{kl})\jmath\|_{bv_q^2} \leq \chi(\mathcal{R}) + \varepsilon. \tag{4}$$

By combining Eq. (3) and (4), we obtain that

$$\chi(\mathcal{R}) = \lim_{k,l\to\infty} \sup_{\jmath\in\mathcal{R}} \|(I - q_{kl})(\jmath)\|_{bv_q^2},$$

or

$$\chi(\mathcal{R}) = \lim_{k,l\to\infty} \Big\{ \sup_{\jmath\in\mathcal{R}} \Big\{ \big(\sum_{m\geq k, n\geq l} |\ell_{mn} - \ell_{m-1,n-1}|^q \big)^{\frac{1}{q}} \Big\} \Big\}.$$

◀

3. New fractional differential equations and application

Definition 5 ([31]). *The fractional integral of order $\Im$ is defined*

$$I^{\Im} f(\wp) = \frac{1}{\Gamma(\Im)} \int_0^{\wp} \frac{f(\gimel)}{(\wp - \gimel)^{1-\Im}} d\gimel \ \Im > 0.$$

Definition 6 ([31]). *Let $f : \mathbb{R}_+ \to \mathbb{R}$, then the Caputo fractional derivative of order $\Im > 0$ is defined by*

$${}^{c}D^{\Im} f(\wp) = \frac{1}{\Gamma(n-\Im)} \int_0^{\wp} \frac{f^{(n)}(\gimel)}{(\wp - \gimel)^{\Im-n+1}} d\gimel,$$

where $n - 1 = [\Im]$.

Lemma 2 ([24]). *Let $u \in C([0,\infty)) \cap L^1([0,\infty))$ with the Caputo fractional derivative of order $\Im$ that belongs to $C([0,\infty)) \cap L^1([0,\infty))$. So*

$$I^{\Im}\ {}^{c}D^{\Im} u(\wp) = u(\wp) + c_1 + c_2\wp + c_3\wp^2 + \ldots + c_n\wp^{n-1},$$

where $c_i \in \mathbb{R}$, $i = 1, 2, \ldots, n$ and $n = [\Im]$.

Lemma 3. *Let $f(\wp) \in L^1(\mathbb{R}_+)$ be continuous. Then the boundary value problem of FDEs*

$$\begin{cases} {}^{c}D^{\Im}\jmath(\wp) = f(\wp), \ \Im \in (n, n+1), \ \wp \in [0, \aleph], \\ \jmath(1) = \int_0^{\xi} \frac{(\xi-\gimel)^{\Im-1}}{\Gamma(\Im)} f(\gimel) d\gimel, \quad \jmath'''(0) = \jmath(\eta), \\ \jmath'(0) + \jmath''(0) = 0, \quad 0 < \xi < \eta < \aleph, \quad \eta \in \mathbb{R} \\ \jmath(0) = \jmath^{(4)}(0) = \jmath^{(5)}(0) = \ldots = \jmath^{(n)}(0) = 0, \end{cases} \tag{5}$$

has a unique solution

$$\begin{aligned} \jmath(\wp) \quad = \quad & \int_0^{\wp} \frac{(\wp-\gimel)^{\Im-1}}{\Gamma(\Im)} f(\gimel) d\gimel + \frac{\wp(2\eta^3 - 12)}{\eta^3 + \eta^2 - 2\eta - 6} \Big(\int_0^{\xi} \frac{(\xi-\gimel)^{\Im-1}}{\Gamma(\Im)} f(\gimel) d\gimel - \int_0^{1} \frac{(1-\gimel)^{\Im-1}}{\Gamma(\Im)} f(\gimel) d\gimel \\ & + \frac{1}{\eta^3 - 6} \int_0^{\eta} \frac{(\eta-\gimel)^{\Im-1}}{\Gamma(\Im)} d\gimel \Big) + \frac{\wp^2(6 - \eta^3)}{\eta^3 + \eta^2 - 2\eta - 6} \Big(\int_0^{\xi} \frac{(\xi-\gimel)^{\Im-1}}{\Gamma(\Im)} f(\gimel) d\gimel - \int_0^{1} \frac{(1-\gimel)^{\Im-1}}{\Gamma(\Im)} f(\gimel) d\gimel \\ & + \frac{1}{\eta^3 - 6} \int_0^{\eta} \frac{(\eta-\gimel)^{\Im-1}}{\Gamma(\Im)} f(\gimel) d\gimel \Big) + \wp^3 \Big(\frac{1}{6 - \eta^3} \int_0^{\eta} \frac{(\eta-\gimel)^{\Im-1}}{\Gamma(\Im)} f(\wp) d\wp + \frac{2\eta - \eta^2}{\eta^3 + \eta^2 - 2\eta - 6} \\ & \Big(\int_0^{\xi} \frac{(\xi-\gimel)^{\Im-1}}{\Gamma(\Im)} f(\gimel) d\gimel - \int_0^{1} \frac{(1-\gimel)^{\Im-1}}{\Gamma(\Im)} f(\gimel) d\gimel + \frac{1}{\eta^3 - 6} \int_0^{\eta} \frac{(\eta-\gimel)^{\Im-1}}{\Gamma(\Im)} f(\gimel) d\gimel \Big) \Big). \end{aligned}$$

Proof. By Lem 2, E.q (5) is equivalent to the integral form

$$\jmath(\wp) = I^{\Im} f(\wp) + c_1 + c_2\wp + c_3\wp^2 + c_4\wp^3 + \ldots + c_{n+1}\wp^n,$$

for some $c_i \in \mathbb{R}$, $i = 1, 2, \ldots, n+1$.
By conditions of (5), we get

$$c_1 = c_5 = c_6 = c_7 = c_8 = \ldots = c_{n+1} = 0,$$

and

$$\jmath(\wp) = \int_0^{\wp} \frac{(\wp-\gimel)^{\Im-1}}{\Gamma(\Im)} f(\gimel) d\gimel + c_2\wp + c_3\wp^2 + c_4\wp^3. \tag{6}$$

So,

$$\begin{aligned}
\jmath'(\wp) &= \int_0^{\wp} \frac{(\wp-\beth)^{\Im-2}}{\Gamma(\Im-1)} f(\beth)d\beth + c_2 + 2c_3\wp + 3c_4\wp^2,\\
\jmath''(\wp) &= \int_0^{\wp} \frac{(\wp-\beth)^{\Im-3}}{\Gamma(\Im-2)} f(\beth)d\beth + 2c_3 + 6c_4\wp,\\
\jmath'''(\wp) &= \int_0^{\wp} \frac{(\wp-\beth)^{\Im-4}}{\Gamma(\Im-3)} f(\beth)d\beth + 6c_4.
\end{aligned}$$

Applying, $\jmath'(0)+\jmath''(0)=0$ we get $c_2=-2c_3$. By $\jmath'''(0)=\jmath(\eta)$ we get

$$6c_4 = \int_0^{\eta} \frac{(\eta-\beth)^{\Im-1}}{\Gamma(\Im)} f(\beth)d\beth + c_2\eta + c_3\eta^2 + c_4\eta^3.$$

Consequently,

$$c_4 = \frac{1}{6-\eta^3}\Big(\int_0^{\eta} \frac{(\eta-\beth)^{\Im-1}}{\Gamma(\Im)} f(\beth)d\beth + (\eta^2-2\eta)c_3\Big).$$

By $\jmath(1)=\int_0^{\xi}\frac{(\xi-\beth)^{\Im-1}}{\Gamma(\Im)}f(\beth)d\beth$ we have

$$\int_0^1 \frac{(1-\beth)^{\Im-1}}{\Gamma(\Im)} f(\beth)d\beth + c_2 + c_3 + c_4 = \int_0^{\xi} \frac{(\xi-\beth)^{\Im-1}}{\Gamma(\Im)} f(\beth)d\beth. \tag{7}$$

Substituting the value c_4, $c_2=-2c_3$ in (7) we get

$$\int_0^1 \frac{(1-\beth)^{\Im-1}}{\Gamma(\Im)} f(\beth)d\beth - c_3 + \frac{1}{6-\eta^3}\int_0^{\eta} \frac{(\eta-\beth)^{\Im-1}}{\Gamma(\Im)} f(\beth)d\beth + \frac{\eta^2-2\eta}{6-\eta^3}c_3 = \int_0^{\xi} \frac{(\xi-\beth)^{\Im-1}}{\Gamma(\Im)} f(\beth)d\beth.$$

Then we have

$$\Big(\frac{\eta^3+\eta^2-2\eta-6}{6-\eta^3}\Big)c_3 = \int_0^{\xi} \frac{(\xi-\beth)^{\Im-1}}{\Gamma(\Im)} f(\beth)d\beth - \int_0^1 \frac{(1-\beth)^{\Im-1}}{\Gamma(\Im)} f(\beth)d\beth + \frac{1}{\eta^3-6}\int_0^{\eta} \frac{(\eta-\beth)^{\Im-1}}{\Gamma(\Im)} f(\beth)d\beth.$$

Consequently,

$$c_3 = \frac{6-\eta^3}{\eta^3+\eta^2-2\eta-6}\Big(\int_0^{\xi} \frac{(\xi-\beth)^{\Im-1}}{\Gamma(\Im)} f(\beth)d\beth - \int_0^1 \frac{(1-\beth)^{\Im-1}}{\Gamma(\Im)} f(\beth)d\beth + \frac{1}{\eta^3-6}\int_0^{\eta} \frac{(\eta-\beth)^{\Im-1}}{\Gamma(\Im)} f(\beth)d\beth\Big),$$

and by $c_2=-2c_3$

$$c_2 = \frac{2\eta^3-12}{\eta^3+\eta^2-2\eta-6}\Big(\int_0^{\xi} \frac{(\xi-\beth)^{\Im-1}}{\Gamma(\Im)} f(\beth)d\beth - \int_0^1 \frac{(1-\beth)^{\Im-1}}{\Gamma(\Im)} f(\beth)d\beth + \frac{1}{\eta^3-6}\int_0^{\eta} \frac{(\eta-\beth)^{\Im-1}}{\Gamma(\Im)} f(\beth)d\beth\Big).$$

Also,

$$\begin{aligned}
c_4 = \frac{1}{6-\eta^3}\int_0^{\eta} \frac{(\eta-\beth)^{\Im-1}}{\Gamma(\Im)} f(\beth)d\beth &+ \frac{2\eta-\eta^2}{\eta^3+\eta^2-2\eta-6}\Big(\int_0^{\xi} \frac{(\xi-\beth)^{\Im-1}}{\Gamma(\Im)} f(\beth)d\beth\\
&- \int_0^1 \frac{(1-\beth)^{\Im-1}}{\Gamma(\Im)} f(\beth)d\beth + \frac{1}{\eta^3-6}\int_0^{\eta} \frac{(\eta-\beth)^{\Im-1}}{\Gamma(\Im)} f(\beth)d\beth\Big).
\end{aligned}$$

Substituting the value of c_2, c_3 and c_3 in (6), it yields

$$\jmath(t) = \int_0^{\wp} \frac{(\wp-\beth)^{\Im-1}}{\Gamma(\Im)} f(\beth)d\beth + \frac{\wp(2\eta^3-12)}{\eta^3+\eta^2-2\eta-6}\Big(\int_0^{\xi} \frac{(\xi-\beth)^{\Im-1}}{\Gamma(\Im)} f(\beth)d\beth - \int_0^1 \frac{(1-\beth)^{\Im-1}}{\Gamma(\Im)} f(\beth)d\beth$$

$$+\frac{1}{\eta^3-6}\int_0^\eta \frac{(\eta-\beth)^{\Im-1}}{\Gamma(\Im)}f(\beth)d\beth)+\frac{\wp^2(6-\eta^3)}{\eta^3+\eta^2-2\eta-6}\Big(\int_0^\xi \frac{(\xi-\beth)^{\Im-1}}{\Gamma(\Im)}f(\beth)d\beth-\int_0^1 \frac{(1-\beth)^{\Im-1}}{\Gamma(\Im)}f(\beth)d\beth$$
$$+\frac{1}{\eta^3-6}\int_0^\eta \frac{(\eta-\beth)^{\Im-1}}{\Gamma(\Im)}f(\beth)d\beth)+t^3\Big(\frac{1}{6-\eta^3}\int_0^\eta \frac{(\eta-\beth)^{\Im-1}}{\Gamma(\Im)}f(\beth)d\beth+\frac{2\eta-\eta^2}{\eta^3+\eta^2-2\eta-6}$$
$$\Big(\int_0^\xi \frac{(\xi-\beth)^{\Im-1}}{\Gamma(\Im)}f(\beth)d\beth-\int_0^1 \frac{(1-\beth)^{\Im-1}}{\Gamma(\Im)}f(\beth)d\beth+\frac{1}{\eta^3-6}\int_0^\eta \frac{(\eta-\beth)^{\Im-1}}{\Gamma(\Im)}f(\beth)d\beth)\Big).$$

◄

Consider:

(A_1) For $k,l\in\mathbb{N}$, $f_{kl}\in C\big(I\times\mathbb{R}^\infty,\mathbb{R}\big)$, the continuous operator $f: I\times bv_q^2\to bv_q^2$ is defined by $(f\jmath)(\wp)=\langle(\wp,\jmath(\wp))=\langle f_{kl}(\wp,\jmath(\wp))\rangle$, where $I=[0,\aleph]$ and the family of functions $\{(f\jmath)(\wp)\}_{\wp\in I}$ is equicontinuous in bv_q^2.

(A_2) Assume that

$$0<\Bigg[|\frac{T^\Im}{\Im\Gamma(\Im)}|^q+|\frac{\aleph(2\eta^3-12)}{\eta^3+\eta^2-2\eta-6}|^q\Big(|\frac{\aleph^\Im}{\Im\Gamma(\Im)}|^q+|\frac{1}{\Gamma(\Im)}|^q+|\frac{1}{\eta^3-6}|^q(|\frac{\aleph^\Im}{\Im\Gamma(\Im)}|^q)\Big)$$
$$+|\frac{6\aleph^2}{\eta^3+\eta^2-2\eta-6}|^q\big(|\frac{\aleph^\Im}{\Im\Gamma(\Im)}|^q+|\frac{1}{\Gamma(\Im)}|^q+|\frac{1}{\eta^3-6}|^q(|\frac{\aleph^\Im}{\Im\Gamma(\Im)}|^q)$$
$$+|\aleph^3|^q\Bigg(|\frac{1}{6-\eta^3}|^q(|\frac{\aleph^\Im}{\Im\Gamma(\Im)}|^q+|\frac{2\eta-\eta^2}{\eta^3+\eta^2-2\eta-6}|^q\big(|\frac{\aleph^\Im}{\Im\Gamma(\Im)}|^q+|\frac{1}{\Gamma(\Im)}|^q+|\frac{1}{\eta^3-6}|^q(|\frac{\aleph^\Im}{\Im\Gamma(\Im)}|^q)\big)\Bigg)\Bigg]=A.$$

(A_3) For each $\wp\in I$ and $\jmath\in bv_q^2$ the following inequalities holds:

$$|f_{kl}(\wp,\jmath(\wp))-f_{k-1,l-1}(\wp,\jmath(\wp))|^q\le|q_{k,l}(\wp)|^q|\jmath_{k,l}(\wp)-\jmath_{k-1,l-1}(\wp)|^q,$$

where $q_{kl}(\wp)$ is real functions continuous such that $(q_{kl}(\wp))$ is uniformly equibounded on I, Put

$$\sup_{\wp\in I}\sup_{k,l}|q_{kl}(\wp)|^q=Q.$$

Theorem 5. *Let $(A_1)-(A_3)$ hold, if $2^{10q}QA<1$, then (5) for each $\wp\in I$ has at least one solution $\jmath=\langle\jmath_{kl}(\wp)\rangle\in C\big(I,bv_q^2\big)$.*

Proof. Let $\jmath=\langle\jmath_{kl}\rangle$ be a double sequence function which fulfils the E.q (5) and $\forall\ k,l\in\mathbb{N}$, $\jmath_{kl}$ be continuous on I. Define $\mathcal{H}: C(I,bv_q^2)\to C(I,bv_q^2)$ by
$(\mathcal{H}\jmath)(\wp)$

$$=\int_0^\wp \frac{(\wp-\beth)^{\Im-1}}{\Gamma(\Im)}f_{kl}(\beth,\jmath(\beth))d\beth+\frac{\wp(2\eta^3-12)}{\eta^3+\eta^2-2\eta-6}\Big(\int_0^\xi \frac{(\xi-\beth)^{\Im-1}}{\Gamma(\Im)}f_{kl}(\beth,\jmath(\beth))d\beth$$
$$-\int_0^1 \frac{(1-\beth)^{\Im-1}}{\Gamma(\Im)}f_{kl}(\beth,\jmath(\beth))d\beth+\frac{1}{\eta^3-6}\int_0^\eta \frac{(\eta-\beth)^{\Im-1}}{\Gamma(\Im)}f_{kl}(\beth,\jmath(\beth))d\beth\Big)$$
$$+\frac{\wp^2(6-\eta^3)}{\eta^3+\eta^2-2\eta-6}\Big(\int_0^\xi \frac{(\xi-\beth)^{\Im-1}}{\Gamma(\Im)}f_{kl}(\beth,\jmath(\beth))d\beth-\int_0^1 \frac{(1-\beth)^{\Im-1}}{\Gamma(\Im)}f_{kl}(\beth,\jmath(\beth))d\beth$$

$$+\frac{1}{\eta^3-6}\int_0^\eta \frac{(\eta-\beth)^{\Im-1}}{\Gamma(\Im)} f_{kl}(\beth,\jmath(\beth))d\beth\big) + \wp^3\big(\frac{1}{6-\eta^3}\int_0^\eta \frac{(\eta-\beth)^{\Im-1}}{\Gamma(\Im)} f_{kl}(\beth,\jmath(\beth))d\beth + \frac{2\eta-\eta^2}{\eta^3+\eta^2-2\eta-6}$$
$$\big(\int_0^\xi \frac{(\xi-\beth)^{\Im-1}}{\Gamma(\Im)} f_{kl}(\beth,\jmath(\beth))d\beth - \int_0^1 \frac{(1-\beth)^{\Im-1}}{\Gamma(\Im)} f_{kl}(\beth,\jmath(\beth))d\beth + \frac{1}{\eta^3-6}\int_0^\eta \frac{(\eta-\beth)^{\Im-1}}{\Gamma(\Im)} f_{kl}(\beth,\jmath(\beth))d\beth\big)\big).$$

By our assumptions, we get

$$\begin{aligned}
\|(\mathcal{H}\jmath)(\wp)\|^q_{bv_q^2} &= \sum_{k,l=1}^\infty \Bigg(\Big|\int_0^\wp \frac{(\wp-\beth)^{\Im-1}}{\Gamma(\Im)}\big(f_{kl}(\beth,\jmath(\beth)) - f_{k-1,l-1}(\beth,\jmath(\beth))\big)d\beth \\
&+\frac{\wp(2\eta^3-12)}{\eta^3+\eta^2-2\eta-6}\big(\int_0^\xi \frac{(\xi-\beth)^{\Im-1}}{\Gamma(\Im)}\big(f_{kl}(\beth,\jmath(\beth)) - f_{k-1,l-1}(\beth,\jmath(\beth))\big)d\beth \\
&-\int_0^1 \frac{(1-\beth)^{\Im-1}}{\Gamma(\Im)}\big(f_{kl}(\beth,\jmath(\beth)) - f_{k-1,l-1}(\beth,\jmath(\beth))\big)d\beth \\
&+\frac{1}{\eta^3-6}\int_0^\eta \frac{(\eta-\beth)^{\Im-1}}{\Gamma(\Im)}\big(f_{kl}(\beth,\jmath(\beth)) - f_{k-1,l-1}(\beth,\jmath(\beth))\big)d\beth\big) \\
&+\frac{\wp^2(6-\eta^3)}{\eta^3+\eta^2-2\eta-6}\big(\int_0^\xi \frac{(\xi-\beth)^{\Im-1}}{\Gamma(\Im)}\big(f_{kl}(\beth,\jmath(\beth)) - f_{k-1,l-1}(\beth,\jmath(\beth))\big)d\beth \\
&-\int_0^1 \frac{(1-\beth)^{\Im-1}}{\Gamma(\Im)}\big(f_{kl}(\beth,\jmath(\beth)) - f_{k-1,l-1}(\beth,\jmath(\beth))\big)d\beth \\
&+\frac{1}{\eta^3-6}\int_0^\eta \frac{(\eta-\beth)^{\Im-1}}{\Gamma(\Im)}\big(f_{kl}(\beth,\jmath(\beth)) - f_{k-1,l-1}(\beth,\jmath(\beth))\big)d\beth\big) \\
&+\wp^3\big(\frac{1}{6-\eta^3}\int_0^\eta \frac{(\eta-\beth)^{\Im-1}}{\Gamma(\Im)}\big(f_{kl}(\beth,\jmath(\beth)) - f_{k-1,l-1}(\beth,\jmath(\beth))\big)d\beth \\
&+\frac{2\eta-\eta^2}{\eta^3+\eta^2-2\eta-6}\big(\int_0^\xi \frac{(\xi-\beth)^{\Im-1}}{\Gamma(\Im)}\big(f_{kl}(\beth,\jmath(\beth)) - f_{k-1,l-1}(\beth,\jmath(\beth))\big)d\beth \\
&-\int_0^1 \frac{(1-\beth)^{\Im-1}}{\Gamma(\Im)}\big(f_{kl}(\beth,\jmath(\beth)) - f_{k-1,l-1}(\beth,\jmath(\beth))\big)d\beth \\
&+\frac{1}{\eta^3-6}\int_0^\eta \frac{(\eta-\beth)^{\Im-1}}{\Gamma(\Im)}\big(f_{kl}(\beth,\jmath(\beth)) - f_{k-1,l-1}(\beth,\jmath(\beth))\big)d\beth\Big|^q\big)\big)\Bigg) \\
&\leq 2^{10q}\Bigg[\sum_{k,l=1}^\infty \Bigg(\Big|\int_0^\wp \frac{(\wp-\beth)^{\Im-1}}{\Gamma(\Im)}\big(f_{kl}(\beth,\jmath(\beth)) - f_{k-1,l-1}(\beth,\jmath(\beth))\big)d\beth\Big|^q \\
&+\Big|\frac{\wp(2\eta^3-12)}{\eta^3+\eta^2-2\eta-6}\Big|^p \sum_{k,l=1}^\infty \big(\Big|\int_0^\xi \frac{(\xi-\beth)^{\Im-1}}{\Gamma(\Im)}\big(f_{kl}(\beth,\jmath(\beth)) - f_{k-1,l-1}(\beth,\jmath(\beth))\big)d\beth\Big|^q \\
&+\sum_{k,l=1}^\infty \Big|\int_0^1 \frac{(1-\beth)^{\Im-1}}{\Gamma(\Im)}\big(f_{kl}(\beth,\jmath(\beth)) - f_{k-1,l-1}(\beth,\jmath(\beth))\big)d\beth\Big|^q \\
&+\Big|\frac{1}{\eta^3-6}\Big|^p \sum_{k,l=1}^\infty \Big|\int_0^\eta \frac{(\eta-\beth)^{\Im-1}}{\Gamma(\Im)}\big(f_{kl}(\beth,\jmath(\beth)) - f_{k-1,l-1}(\beth,\jmath(\beth))\big)d\beth\Big|^q\big) \\
&+\Big|\frac{\wp^2(6-\eta^3)}{\eta^3+\eta^2-2\eta-6}\Big|^q\big(\sum_{k,l=1}^\infty \Big|\int_0^\xi \frac{(\xi-\beth)^{\Im-1}}{\Gamma(\Im)}\big(f_{kl}(\beth,\jmath(\beth)) - f_{k-1,l-1}(\beth,\jmath(\beth))\big)d\beth\Big|^q
\end{aligned}$$

$$+\sum_{k,l=1}^{\infty}\Big|\int_0^1\frac{(1-\daleth)^{\Im-1}}{\Gamma(\Im)}\big(f_{kl}(\daleth,\jmath(\daleth))-f_{k-1,l-1}(\daleth,\jmath(\daleth))\big)d\daleth\Big|^q$$
$$+|\frac{1}{\eta^3-6}|^q\sum_{k,l=1}^{\infty}\Big|\int_0^\eta\frac{(\eta-\daleth)^{\Im-1}}{\Gamma(\Im)}\big(f_{kl}(\daleth,\jmath(\daleth))-f_{k-1,l-1}(\daleth,\jmath(\daleth))\big)d\daleth\Big|^q\big)$$
$$+|\wp^3|^q\big(|\frac{1}{6-\eta^3}|^q\sum_{k,l=1}^{\infty}\Big|\int_0^\eta\frac{(\eta-\daleth)^{\Im-1}}{\Gamma(\Im)}\big(f_{kl}(\daleth,\jmath(\daleth))-f_{k-1,l-1}(\daleth,\jmath(\daleth))\big)d\daleth\Big|^q$$
$$+|\frac{2\eta-\eta^2}{\eta^3+\eta^2-2\eta-6}|^q\sum_{k,l=1}^{\infty}\big(\Big|\int_0^\xi\frac{(\xi-\daleth)^{\Im-1}}{\Gamma(\Im)}\big(f_{kl}(\daleth,\jmath(\daleth))-f_{k-1,l-1}(\daleth,\jmath(\daleth))\big)d\daleth\Big|^q$$
$$+\sum_{k,l=1}^{\infty}\Big|\int_0^1\frac{(1-\daleth)^{\Im-1}}{\Gamma(\Im)}\big(f_{kl}(\daleth,\jmath(\daleth))-f_{k-1,l-1}(\daleth,\jmath(\daleth))\big)d\daleth\Big|^q$$
$$+\sum_{k,l=1}^{\infty}\Big|\frac{1}{\eta^3-6}\int_0^\eta\frac{(\eta-\daleth)^{\Im-1}}{\Gamma(\Im)}\big(f_{kl}(\daleth,\jmath(\daleth))-f_{k-1,l-1}(\daleth,\jmath(\daleth))\big)d\daleth\Big|^q\big)\big)\Big].$$

Then,
$\|(\mathcal{H}\jmath)(\wp)\|^q_{bv_q^2}$

$$\begin{aligned}
\leq\;&\sum_{k,l=1}^{\infty}|q_{kl}(\wp)|^q|\jmath_{kl}(\wp)-\jmath_{k-1,l-1}(\wp)|^q2^{10q}\Big[\big(\int_0^\wp\frac{(\wp-\daleth)^{\Im-1}}{\Gamma(\Im)}ds\big)^q+|\frac{\wp(2\eta^3-12)}{\eta^3+\eta^2-2\eta-6}|^q\big(\big(\int_0^\xi\frac{(\xi-\daleth)^{\Im-1}}{\Gamma(\Im)}d\daleth\big)^q\\
&+\big(\int_0^1\frac{(1-\daleth)^{\Im-1}}{\Gamma(\Im)}d\daleth\big)^q+|\frac{1}{\eta^3-6}|^q\big(\int_0^\eta\frac{(\eta-\daleth)^{\Im-1}}{\Gamma(\Im)}d\daleth\big)^q\big)+|\frac{\wp^2(6-\eta^3)}{\eta^3+\eta^2-2\eta-6}|^q\big(\big(\int_0^\xi\frac{(\xi-\daleth)^{\Im-1}}{\Gamma(\Im)}d\daleth\big)^q\\
&+\big(\int_0^1\frac{(1-\daleth)^{\Im-1}}{\Gamma(\Im)}d\daleth\big)^q+|\frac{1}{\eta^3-6}|^q\big(\int_0^\eta\frac{(\eta-\daleth)^{\Im-1}}{\Gamma(\Im)}d\daleth\big)^q\big)+|\wp^3|^q\Big(|\frac{1}{6-\eta^3}|^q\big(\big(\int_0^\eta\frac{(\eta-\daleth)^{\Im-1}}{\Gamma(\Im)}d\daleth\big)^q\\
&+|\frac{2\eta-\eta^2}{\eta^3+\eta^2-2\eta-6}|^q\big(\big(\int_0^\xi\frac{(\xi-\daleth)^{\Im-1}}{\Gamma(\Im)}d\daleth\big)^q+\big(\int_0^1\frac{(1-\daleth)^{\Im-1}}{\Gamma(\Im)}d\daleth\big)^q+|\frac{1}{6-\eta^3}|^q\big(\big(\int_0^\eta\frac{(\eta-\daleth)^{\Im-1}}{\Gamma(\Im)}d\daleth\big)^q\big)\big)\big)\Big)\Big]\\
\leq\;&Q\|\jmath_{kl}|^q_{bv_q^2}2^{10q}\Big[|\frac{\wp^\Im}{\Im\Gamma(\Im)}|^q+|\frac{\wp(2\eta^3-12)}{\eta^3+\eta^2-2\eta-6}|^q\Big(|\frac{\wp^\Im}{\Im\Gamma(\Im)}|^q+|\frac{1}{\Gamma(\Im)}|^q+|\frac{1}{\eta^3-6}|^q\big(|\frac{\wp^\Im}{\Im\Gamma(\Im)}|^q\big)\Big)\\
&+|\frac{6\wp^2}{\eta^3+\eta^2-2\eta-6}|^p\big(|\frac{\wp^\Im}{\Im\Gamma(\Im)}|^q+|\frac{1}{\Gamma(\Im)}|^q+|\frac{1}{\eta^3-6}|^q\big(|\frac{\wp^\Im}{\Im\Gamma(\Im)}|^q\big)+|\wp^3|^p\Big(|\frac{1}{6-\eta^3}|^p\big(|\frac{\wp^\Im}{\Im\Gamma(\Im)}|^q\\
&+|\frac{2\eta-\eta^2}{\eta^3+\eta^2-2\eta-6}|^q\big(|\frac{\wp^\Im}{\Im\Gamma(\Im)}|^q+|\frac{1}{\Gamma(\Im)}|^q+|\frac{1}{\eta^3-6}|^q\big(|\frac{\wp^\Im}{\Im\Gamma(\Im)}|^q\big)\big)\Big)\Big].
\end{aligned}$$

Also, $C(I,bv_q^2)$ have the norm

$$\|\jmath\|_{C(I,bv_q^2)}=\sup_{\wp\in I}\|\jmath(\wp)\|_{bv_q^2}.$$

So, we have

$$\|(\mathcal{H}\jmath)\|^q_{C(I,bv_q^2)}\leq(QA2^{10q})\|\jmath\|^q_{C(I,bv_q^2)}.$$

Then

$$r\leq 2^{10q}QAr. \tag{8}$$

Let r_0 be the optimal solution of (8). Now, we consider the set

$$B_{bv_q^2} = B_{bv_q^2}(\jmath, r_0) = \{\jmath \in C(I, bv_q^2) : \ \|\jmath\|_{C(I,bv_q^2)} \leq r_0\}.$$

Clearly, $B_{bv_q^2} = \overline{B}_{bv_q^2}$ is convex and bounded and $\mathcal{H}$ is bounded on B. Let $0 < \varepsilon$ by (A_1), $\exists$, $\delta > 0$ so that if $\ell \in B_{bv_p^2}$ and $\|\jmath - \ell\|_{C(I,bv_q^2)} \leq \delta$, then $\|\{\jmath - \{\ell\|^q_{C(I,bv_q^2)} \leq \frac{\varepsilon^q}{2^{10q}A}$. Thus, for each $\wp \in I$, We get

$$\begin{aligned}
&\|(\mathcal{H}\jmath)(\wp) - (\mathcal{H}\ell)(\wp)\|^q_{bv_q^2} \\
\leq\ & 2^{10q}\Bigg[\sum_{k,l=1}^{\infty}\Bigg(\Big|\int_0^{\wp}\frac{(\wp-\beth)^{\Im-1}}{\Gamma(\Im)}\big(f_{kl}(\beth,\jmath(\beth)) - f_{kl}(\beth,\ell(\beth))) - (f_{k-1,l-1}(\beth,\jmath(\beth)) - f_{k-1,l-1}(\beth,\ell(\beth)))\big)d\beth\Big|^q \\
&+\Big|\frac{\wp(2\eta^3-12)}{\eta^3+\eta^2-2\eta-6}\Big|^p\sum_{k,l=1}^{\infty}\Big(\Big|\int_0^{\xi}\frac{(\xi-\beth)^{\Im-1}}{\Gamma(\Im)}\big(f_{kl}(\beth,\jmath(\beth)) - f_{kl}(\beth,y(\beth))) - (f_{k-1,l-1}(\beth,\jmath(\beth)) - f_{k-1,l-1}(\beth,\ell(\beth)))\big)d\beth\Big|^q \\
&+\sum_{k,l=1}^{\infty}\Big|\int_0^{1}\frac{(1-\beth)^{\Im-1}}{\Gamma(\Im)}\big(f_{kl}(\beth,\jmath(\beth)) - f_{kl}(\beth,y(\beth))) - (f_{k-1,l-1}(\beth,\jmath(\beth)) - f_{k-1,l-1}(\beth,\ell(\beth)))\big)d\beth\Big|^q \\
&+\Big|\frac{1}{\eta^3-6}\Big|^p\sum_{k,l=1}^{\infty}\Big|\int_0^{\eta}\frac{(\eta-\beth)^{\Im-1}}{\Gamma(\Im)}\big(f_{kl}(\beth,\jmath(\beth)) - f_{kl}(\beth,y(\beth))) - (f_{k-1,l-1}(\beth,\jmath(\beth)) - f_{k-1,l-1}(\beth,\ell(\beth)))\big)d\beth\Big|^q\Big) \\
&+\Big|\frac{\wp^2(6-\eta^3)}{\eta^3+\eta^2-2\eta-6}\Big|^p\Big(\sum_{k,l=1}^{\infty}\Big|\int_0^{\xi}\frac{(\xi-\beth)^{\Im-1}}{\Gamma(\Im)}\big(f_{kl}(\beth,\jmath(\beth)) - f_{kl}(\beth,y(\beth))) - (f_{k-1,l-1}(\beth,\jmath(\beth)) - f_{k-1,l-1}(\beth,\ell(\beth)))\big)d\beth\Big|^q \\
&+\sum_{k,l=1}^{\infty}\Big|\int_0^{1}\frac{(1-\beth)^{\Im-1}}{\Gamma(\Im)}\big(f_{kl}(\beth,\jmath(\beth)) - f_{kl}(\beth,\ell(\beth))) - (f_{k-1,l-1}(\beth,\jmath(\beth)) - f_{k-1,l-1}(\beth,\ell(\beth)))\big)d\beth\Big|^q \\
&+\Big|\frac{1}{\eta^3-6}\Big|^p\sum_{k,l=1}^{\infty}\Big|\int_0^{\eta}\frac{(\eta-\beth)^{\Im-1}}{\Gamma(\Im)}\big(f_{kl}(\beth,\jmath(\beth)) - f_{kl}(\beth,\ell(\beth))) - (f_{k-1,l-1}(\beth,\jmath(\beth)) - f_{k-1,l-1}(\beth,\ell(\beth)))\big)d\beth\Big|^q\Big) \\
&+|\wp^3|^p\Big(\Big|\frac{1}{6-\eta^3}\Big|^p\sum_{k,l=1}^{\infty}\Big|\int_0^{\eta}\frac{(\eta-\beth)^{\Im-1}}{\Gamma(\Im)}\big(f_{kl}(\beth,\jmath(\beth)) - f_{kl}(\beth,\ell(\beth))) - (f_{k-1,l-1}(\beth,\jmath(\beth)) - f_{k-1,l-1}(\beth,\ell(\beth)))\big)d\beth\Big|^q \\
&+\Big|\frac{2\eta-\eta^2}{\eta^3+\eta^2-2\eta-6}\Big|^q\sum_{k,l=1}^{\infty}\Big(\Big|\int_0^{\xi}\frac{(\xi-\beth)^{\Im-1}}{\Gamma(\Im)}\big(f_{kl}(\beth,\jmath(\beth)) - f_{kl}(\beth,\ell(\beth))) - (f_{k-1,l-1}(\beth,\jmath(\beth)) - f_{k-1,l-1}(\beth,\ell(\beth)))\big)d\beth\Big|^q \\
&+\sum_{k,l=1}^{\infty}\Big|\int_0^{1}\frac{(1-\beth)^{\Im-1}}{\Gamma(\Im)}\big(f_{kl}(\beth,\jmath(\beth)) - f_{kl}(\beth,\ell(\beth))) - (f_{k-1,l-1}(\beth,\jmath(\beth)) - f_{k-1,l-1}(\beth,\ell(\beth)))\big)d\beth\Big|^q \\
&+\sum_{k,l=1}^{\infty}\Big|\frac{1}{\eta^3-6}\int_0^{\eta}\frac{(\eta-\beth)^{\Im-1}}{\Gamma(\Im)}\big(f_{kl}(\beth,\jmath(\beth)) - f_{kl}(\beth,\ell(\beth))) - (f_{k-1,l-1}(\beth,\jmath(\beth)) - f_{k-1,l-1}(\beth,\ell(\beth)))\big)d\beth\Big|^q\Big)\Big)\Bigg] \\
\leq\ & \sum_{k,l=1}^{\infty}|f_{kl}(\beth,\jmath(\beth)) - f_{kl}(\beth,\ell(\beth))) - (f_{k-1,l-1}(\beth,\jmath(\beth)) - f_{k-1,l-1}(\beth,\ell(\beth)))|^q 2^{10q}A \\
=\ & \|(f\jmath) - (f\ell)\|^q_{C(I,bv_q^2)}2^{10q}A
\end{aligned}$$

Accordingly,

$$\|(\mathcal{H}\jmath) - (\mathcal{H}\ell)\|_{C(I,bv_q^2)} = \sup_{\wp \in I}\|(\mathcal{H}\jmath)(\wp) - (\mathcal{H}\ell)(\wp)\|_{bv_q^2)} \leq \varepsilon,$$

so F is continuous. Now, we cheek the continuously $(\mathcal{H}\jmath)$ on $(0,\wp)$. Let $\wp_1\in(0,\wp)$ and $\varepsilon>0$, if $|\wp-\wp_1|<\varepsilon$, then

$\|(\mathcal{H}\jmath)(\wp)-(\mathcal{H}\jmath)(\wp_1)\|^q_{bv_q^2}$

$$
\begin{aligned}
\leq\ & 2^{10q}\Bigg[\sum_{k,l=1}^{\infty}\Bigg(\Big|\int_0^{\wp}\frac{(\wp-\daleth)^{\Im-1}}{\Gamma(\Im)}\big(f_{kl}(\daleth,\jmath(\daleth))-f_{k-1,l-1}(\daleth,\jmath(\daleth))\big)d\daleth\\
&-\int_0^{\wp_1}\frac{(\wp_1-\daleth)^{\Im-1}}{\Gamma(\Im)}\big(f_{kl}(\daleth,\jmath(\daleth))-f_{k-1,l-1}(\daleth,\jmath(\daleth))\big)d\daleth\Big|^q\\
&+\Big|\frac{(\wp-\wp_1)(2\eta^3-12)}{\eta^3+\eta^2-2\eta-6}\Big|^q\sum_{k,l=1}^{\infty}\Big(\Big|\int_0^{\xi}\frac{(\xi-\daleth)^{\Im-1}}{\Gamma(\Im)}\big(f_{kl}(\daleth,\jmath(\daleth))-f_{k-1,l-1}(\daleth,\jmath(\daleth))\big)d\daleth\Big|^q\\
&+\sum_{k,l=1}^{\infty}\Big|\int_0^{1}\frac{(1-\daleth)^{\Im-1}}{\Gamma(\Im)}\big(f_{kl}(\daleth,\jmath(\daleth))-f_{k-1,l-1}(\daleth,\jmath(\daleth))\big)d\daleth\Big|^q\\
&+\Big|\frac{1}{\eta^3-6}\Big|^q\sum_{k,l=1}^{\infty}\Big|\int_0^{\eta}\frac{(\eta-\daleth)^{\Im-1}}{\Gamma(\Im)}\big(f_{kl}(\daleth,\jmath(\daleth))-f_{k-1,l-1}(\daleth,\jmath(\daleth))\big)d\daleth\Big|^q\Big)\\
&+\Big|\frac{(\wp^2-\wp_1^2)(6-\eta^3)}{\eta^3+\eta^2-2\eta-6}\Big|^p\Big(\sum_{k,l=1}^{\infty}\Big|\int_0^{\xi}\frac{(\xi-\daleth)^{\Im-1}}{\Gamma(\Im)}\big(f_{kl}(\daleth,\jmath(\daleth))-f_{k-1,l-1}(\daleth,\jmath(\daleth))\big)d\daleth\Big|^q\\
&+\sum_{k,l=1}^{\infty}\Big|\int_0^{1}\frac{(1-\daleth)^{\Im-1}}{\Gamma(\Im)}\big(f_{kl}(\daleth,\jmath(\daleth))-f_{k-1,l-1}(\daleth,\jmath(\daleth))\big)d\daleth\Big|^q\\
&+\Big|\frac{1}{\eta^3-6}\Big|^p\sum_{k,l=1}^{\infty}\Big|\int_0^{\eta}\frac{(\eta-\daleth)^{\Im-1}}{\Gamma(\Im)}\big(f_{kl}(\daleth,\jmath(\daleth))-f_{k-1,l-1}(\daleth,\jmath(\daleth))\big)d\daleth\Big|^q\Big)\\
&+|\wp^3-\wp_1^3|^q\Big(\Big|\frac{1}{6-\eta^3}\Big|^p\sum_{k,l=1}^{\infty}\Big|\int_0^{\eta}\frac{(\eta-\daleth)^{\Im-1}}{\Gamma(\Im)}\big(f_{kl}(\daleth,\jmath(\daleth))-f_{k-1,l-1}(\daleth,\jmath(\daleth))\big)d\daleth\Big|^q\\
&+\Big|\frac{2\eta-\eta^2}{\eta^3+\eta^2-2\eta-6}\Big|^q\sum_{k,l=1}^{\infty}\Big(\Big|\int_0^{\xi}\frac{(\xi-\daleth)^{\Im-1}}{\Gamma(\Im)}\big(f_{kl}(\daleth,\jmath(\daleth))-f_{k-1,l-1}(\daleth,\jmath(\daleth))\big)d\daleth\Big|^q\\
&+\sum_{k,l=1}^{\infty}\Big|\int_0^{1}\frac{(1-\daleth)^{\Im-1}}{\Gamma(\Im)}\big(f_{kl}(\daleth,\jmath(\daleth))-f_{k-1,l-1}(\daleth,\jmath(\daleth))\big)d\daleth\Big|^q\\
&+\sum_{k,l=1}^{\infty}\Big|\frac{1}{\eta^3-6}\int_0^{\eta}\frac{(\eta-\daleth)^{\Im-1}}{\Gamma(\Im)}\big(f_{kl}(\daleth,\jmath(\daleth))-f_{k-1,l-1}(\daleth,\jmath(\daleth))\big)d\daleth\Big|^q\Big)\Big)\Bigg].
\end{aligned}
$$

Since $|\wp-\wp_1|\leq\varepsilon$ we get

$\|(\mathcal{H}\jmath)(\wp)-(\mathcal{H}\jmath)(\wp_1)\|^q_{bv_q^2}$

$$
\begin{aligned}
\leq\ & 2^{10q}\Bigg[\sum_{k,l=1}^{\infty}\Big|\int_0^{\wp}\frac{(\wp-\daleth)^{\Im-1}}{\Gamma(\Im)}\big(f_{kl}(\daleth,\jmath(\daleth))-f_{k-1,l-1}(\daleth,\jmath(\daleth))\big)d\daleth\\
&-\int_0^{\wp_1}\frac{(\wp_1-\daleth)^{\Im-1}}{\Gamma(\Im)}\big(f_{kl}(\daleth,\jmath(\daleth))-f_{k-1,l-1}(\daleth,\jmath(\daleth))\big)d\daleth\Big|^q\Bigg]\\
\leq\ & 2^{10q}\Bigg[\sum_{k,l=1}^{\infty}\Big|\int_0^{\wp_1}\frac{((\wp-\daleth)^{\Im-1}-(\wp_1-\daleth)^{\Im-1})}{\Gamma(\Im)}\big(f_{kl}(\daleth,\jmath(\daleth))-f_{k-1,l-1}(\daleth,\jmath(\daleth))\big)d\daleth
\end{aligned}
$$

$$+\int_{\wp_1}^{\wp}\frac{(\wp-⅂)^{\Im-1}}{\Gamma(\Im)}\big(f_{kl}(⅂,\jmath(⅂))-f_{k-1,l-1}(⅂,\jmath(⅂))\big)d⅂\Big|^q\Big]$$
$$\leq\ 2^{10q}\Big|\frac{\wp^{\Im}-\wp_1^{\Im}}{\Im\Gamma(\Im)}\Big|^q Q\Big[\sum_{k,l=1}^{\infty}\big|\jmath_{kl}(⅂)-\jmath_{k-1,l-1}(⅂)\big|^q\Big]$$

Since $|\wp-\wp_0|<\varepsilon$ and $n<\Im\leq n+1$ so $|\wp^{\Im}-\wp_1^{\Im}|^q<\varepsilon$. So $(\mathcal{H}\jmath)$ is continuous for each $\wp\in I$. Now, we prove that $\mathcal{H}$ is a Meir–Keeler condensing operator w.r.t. the Hausdorff MNC χ on $C\big(I,bv_q^2\big)$. By Th 4 and Lem 1 we get

$$(\mathcal{X}_{C\left(I,bv_q^2\right)}(B_{bv_q^2}))^q=\sup_{\wp\in I}(\mathcal{X}_{bv_q^2}(B_{bv_q^2}(\wp)))^q. \tag{9}$$

Again by using (1), Lemma 1 and our assumption, we have

$$(\mathcal{X}_{bv_q^2}[(\mathcal{H}B_{bv_q^2})(\wp)])^q$$
$$\begin{aligned}
\leq\ & 2^{10q}\Bigg[\lim_{k,l\to\infty}\Bigg\{\sup_{\jmath\in bv_q^2}\Bigg\{\sum_{m\geq k,n\geq l}\Big|\int_0^{\wp}\frac{(\wp-⅂)^{\Im-1}}{\Gamma(\Im)}\big(f_{mn}(⅂,\jmath(⅂))-f_{m-1,n-1}(⅂,\jmath(⅂))\big)d⅂\Big|^q\Bigg\}\Bigg\}\\
&+\Big|\frac{\wp(2\eta^3-12)}{\eta^3+\eta^2-2\eta-6}\Big|^q\lim_{k,l\to\infty}\Bigg\{\sup_{\jmath\in bv_q^2}\Bigg\{\sum_{m\geq k,n\geq l}\Big(\Big|\int_0^{\xi}\frac{(\xi-⅂)^{\Im-1}}{\Gamma(\Im)}\big(f_{mn}(⅂,\jmath(⅂))-f_{m-1,n-1}(⅂,\jmath(⅂))\big)d⅂\Big|^q\Bigg\}\Bigg\}\\
&+\lim_{k,l\to\infty}\Bigg\{\sup_{\jmath\in bv_q^2}\Bigg\{\sum_{m\geq k,n\geq l}\Big|\int_0^{1}\frac{(1-⅂)^{\Im-1}}{\Gamma(\Im)}\big(f_{mn}(⅂,\jmath(⅂))-f_{m-1,n-1}(⅂,\jmath(⅂))\big)d⅂\Big|^q\Bigg\}\Bigg\}\\
&+\Big|\frac{1}{\eta^3-6}\Big|^q\lim_{k,l\to\infty}\Bigg\{\sup_{\jmath\in bv_q^2}\Bigg\{\sum_{m\geq k,n\geq l}\Big|\int_0^{\eta}\frac{(\eta-⅂)^{\Im-1}}{\Gamma(\Im)}\big(f_{mn}(⅂,\jmath(⅂))-f_{m-1,n-1}(⅂,\jmath(⅂))\big)d⅂\Big|^q\Big)\Bigg\}\Bigg\}\\
&+\Big|\frac{\wp^2(6-\eta^3)}{\eta^3+\eta^2-2\eta-6}\Big|^q\Big(\lim_{k,l\to\infty}\Bigg\{\sup_{\jmath\in bv_q^2}\Bigg\{\sum_{m\geq k,n\geq l}\Big|\int_0^{\xi}\frac{(\xi-⅂)^{\Im-1}}{\Gamma(\Im)}\big(f_{mn}(⅂,\jmath(⅂))-f_{m-1,n-1}(⅂,\jmath(⅂))\big)d⅂\Big|^q\Bigg\}\Bigg\}\\
&+\lim_{k,l\to\infty}\Bigg\{\sup_{\jmath\in bv_q^2}\Bigg\{\sum_{m\geq k,n\geq l}\Big|\int_0^{1}\frac{(1-⅂)^{\Im-1}}{\Gamma(\Im)}\big(f_{mn}(⅂,\jmath(⅂))-f_{m-1,n-1}(⅂,\jmath(⅂))\big)d⅂\Big|^q\Bigg\}\Bigg\}\\
&+\Big|\frac{1}{\eta^3-6}\Big|^q\lim_{k,l\to\infty}\Bigg\{\sup_{\jmath\in bv_q^2}\Bigg\{\sum_{m\geq k,n\geq l}\Big|\int_0^{\eta}\frac{(\eta-⅂)^{\Im-1}}{\Gamma(\Im)}\big(f_{mn}(⅂,\jmath(⅂))-f_{m-1,n-1}(⅂,\jmath(⅂))\big)d⅂\Big|^q\Big)\Bigg\}\Bigg\}\\
&+|\wp^3|^q\Big(\Big|\frac{1}{6-\eta^3}\Big|^q\lim_{k,l\to\infty}\Bigg\{\sup_{\jmath\in bv_q^2}\Bigg\{\sum_{m\geq k,n\geq l}\Big|\int_0^{\eta}\frac{(\eta-⅂)^{\Im-1}}{\Gamma(\Im)}\big(f_{mn}(⅂,\jmath(⅂))-f_{m-1,n-1}(⅂,\jmath(⅂))\big)d⅂\Big|^q\Bigg\}\Bigg\}\\
&+\Big|\frac{2\eta-\eta^2}{\eta^3+\eta^2-2\eta-6}\Big|^q\lim_{k,l\to\infty}\Bigg\{\sup_{\jmath\in bv_q^2}\Bigg\{\sum_{m\geq k,n\geq l}\Big(\Big|\int_0^{\xi}\frac{(\xi-⅂)^{\Im-1}}{\Gamma(\Im)}\big(f_{mn}(⅂,\jmath(⅂))-f_{m-1,n-1}(⅂,\jmath(⅂))\big)d⅂\Big|^q\Bigg\}\Bigg\}\\
&+\lim_{k,l\to\infty}\Bigg\{\sup_{\jmath\in bv_q^2}\Bigg\{\sum_{m\geq k,n\geq l}\Big|\int_0^{1}\frac{(1-⅂)^{\Im-1}}{\Gamma(\Im)}\big(f_{mn}(⅂,\jmath(⅂))-f_{m-1,n-1}(⅂,\jmath(⅂))\big)d⅂\Big|^q\Bigg\}\Bigg\}\\
&+\lim_{k,l\to\infty}\Bigg\{\sup_{\jmath\in bv_q^2}\Bigg\{\sum_{m\geq k,n\geq l}\Big|\frac{1}{\eta^3-6}\int_0^{\eta}\frac{(\eta-⅂)^{\Im-1}}{\Gamma(\Im)}\big(f_{mn}(⅂,\jmath(⅂))-f_{m-1,n-1}(⅂,\jmath(⅂))\big)d⅂\Big|^q\Big)\Big)\Bigg\}\Bigg\}\Bigg]
\end{aligned}$$

$$\leq \quad QA2^{10q} \lim_{k,l\to\infty} \left\{ \sup_{\jmath\in bv_q^2} \left\{ \sum_{m\geq k, n\geq l} |\jmath_{mn} - \jmath_{m-1,n-1}|^q \right\}\right\}$$

Therefore, we deduce

$$\sup_{\wp\in I}(\mathcal{X}_{bv_q^2}[(\mathcal{H}B_{m^2(bv_q^2)})(\wp)])^q \leq Q2^{10q}A(\mathcal{X}_{C\left(I,bv_q^2\right)}(B_{bv_q^2}))^q.$$

By (9), we have

$$(\mathcal{X}_{C\left(I,bv_q^2\right)}(\mathcal{H}B_{bv_q^2}))^q \leq Q2^{10q}A(\mathcal{X}_{C\left(I,bv_q^2\right)}(B_{bv_q^2}))^q.$$

Hence

$$(\mathcal{X}_{C\left(I,bv_q^2\right)}(B_{bv_q^2})) < \frac{\varepsilon}{(Q2^{10q}A)^{\frac{1}{q}}}.$$

Take $\delta = \varepsilon(\frac{1}{(Q2^{10q}A)^{\frac{1}{q}}} - 1)$, we get $\mathcal{H}$ is a Meir-–Keeler condensing operator on $B_{bv_q^2} \subset m^2(bv_q^2)$.

So, Th 1 grantees that $\mathcal{H}$ has a fixed point in $B_{bv_p^2}$, so (5) has at least one solution in $C\big(I, bv_q^2\big)$. ◄

Example 1. *Consider the equations*

$$\begin{cases} {}^cD^{\frac{9}{2}}\jmath(\wp) = \frac{\sin(\jmath_{kl}(\wp))\cos(\jmath_{kl}(\wp))}{\wp^2+5e^{\wp^2+3}}, \ \Im\in(4,5), \ \wp\in[0,6], \\ \jmath(1) = \int_0^2 \frac{(2-\beth)^{\frac{9}{2}-1}}{\Gamma(\frac{9}{2})} f(\beth)d\beth, \quad \jmath'''(0) = \jmath(4), \\ \jmath'(0)+\jmath''(0) = 0, \quad 0<2<4<6, \\ \jmath(0) = \jmath^{(4)}(0) = \jmath^{(5)}(0) = 0, \end{cases} \tag{10}$$

Clearly the E.q (10) is particular case of (5) when $f_{kl}(\wp, \jmath(\wp)) = \frac{\cos(\jmath_{kl}(\wp))\sin(\jmath_{kl}(\wp))}{\wp^2+5e^{\wp^2+3}}$, $\xi = 2$, $\eta = 4$ $\Im = \frac{9}{2}$ *and* $\wp = 6$. *For,* $(A1)$ *of Th 5, let* $\jmath, \ell \in C(I, bv_q^2)$, $0 < \varepsilon$ *and* $\wp \in I$. *Take* $\delta = \varepsilon\sqrt[p]{2^{q-1}}$. *Now if* $\|\jmath(\wp) - \ell(\wp)\|_{bv_q^2} < \delta$, *then we have*

$$\|(f\jmath)(\wp) - (f\ell)(\wp)\|_{bv_q^2}^q$$

$$\begin{aligned}
\leq \quad & \frac{1}{5^q}\sum_{k,l=1}^{\infty}\Big|\big(\cos(\jmath_{k,l}(\wp))\sin(\jmath_{k,l}(\wp)) - \cos(\ell_{k,l}(\wp))\sin(\ell_{k,l}(\wp))\big) - \\
& \big(\cos(\jmath_{k-1,l-1}(\wp))\sin(\jmath_{k-1,l-1}(\wp)) - \cos(\ell_{k-1,l-1}(\wp))\sin(\ell_{k-1,l-1}(\wp))\big)\Big|^q \\
\leq \quad & \frac{1}{2^q}\bigg(\sum_{k,l=1}^{\infty}\Big|\cos(\jmath_{k,l}(\wp))\sin(\jmath_{k,l}(\wp)) - \cos(\ell_{k,l}(\wp))\sin(y_{k,l}(\wp))\Big|^q \\
& + \sum_{k,l=1}^{\infty}\Big|\cos(\jmath_{k-1,l-1}(\wp))\sin(\jmath_{k-1,l-1}(\wp)) - \cos(\ell_{k-1,l-1}(\wp))\sin(\ell_{k-1,l-1}(\wp)))\Big|^q\bigg) \\
\leq \quad & \frac{1}{2^q}\bigg(\sum_{k,l=1}^{\infty}\big(\Big|\cos(\jmath_{k,l}(\wp))\sin(\jmath_{k,l}(\wp)) - \cos(\jmath_{k,l}(\wp))\sin(\ell_{k,l}(\wp))\Big|^q
\end{aligned}$$

$$
\begin{aligned}
&+\Big|\cos(\jmath_{k,l}(\wp))\sin(\ell_{k,l}(\wp))-\cos(\ell_{k,l}(\wp))\sin(\ell_{k,l}(\wp))\Big|^{q}\big)\\
&+\sum_{k,l=1}^{\infty}\big(\Big|\cos(\jmath_{k-1,l-1}(\wp))\sin(\jmath_{k-1,l-1}(\wp))-\cos(\jmath_{k-1,l-1}(\wp))\sin(\ell_{k-1,l-1}(\wp))\Big|^{q}\\
&+\Big|\cos(\jmath_{k-1,l-1}(\wp))\sin(\ell_{k-1,l-1}(\wp))-\cos(\ell_{k-1,l-1}(\wp))\sin(\ell_{k-1,l-1}(\wp)))\Big|^{q}\big)\Big)\\
\leq\ & \frac{1}{2^{q}}\Big(\sum_{k,l=1}^{\infty}\big(|\cos(\jmath_{k,l}(\wp))|^{q}\Big|\sin(\jmath_{k,l}(\wp))-\sin(\ell_{k,l}(\wp))\Big|^{q}\\
&+|\sin(\ell_{k,l}(\wp))|^{q}\Big|\cos(\jmath_{k,l}(\wp))-\cos(\ell_{k,l}(\wp))\Big|^{q}\big)\\
&+\sum_{k,l=1}^{\infty}\big(|\cos(\jmath_{k-1,l-1}(\wp))|^{q}\Big|\sin(\jmath_{k-1,l-1}(\wp))-\sin(\ell_{k-1,l-1}(\wp))\Big|^{q}\\
&+|\sin(\ell_{k-1,l-1}(\wp))|^{q}\Big|\cos(\jmath_{k-1,l-1}(\wp))-\cos(\ell_{k-1,l-1}(\wp)))\Big|^{q}\big)\Big)\\
\leq\ & \frac{1}{2^{q}}\Big(\sum_{k,l=1}^{\infty}\big(\Big|\jmath_{k,l}(\wp)-\ell_{k,l}(\wp)\Big|^{q}+\Big|\jmath_{k,l}(\wp)-\ell_{k,l}(\wp)\Big|^{q}\big)\\
&+\sum_{k,l=1}^{\infty}\big(\Big|\jmath_{k-1,l-1}(\wp)-\ell_{k-1,l-1}(\wp)\Big|^{q}+\Big|\jmath_{k-1,l-1}(\wp)-\ell_{k-1,l-1}(\wp))\Big|^{q}\big)\Big)\\
\leq\ & \frac{1}{2^{q-1}}\|\jmath(\wp)-\ell(\wp)\|_{bv_q^2}^{q}<\varepsilon^{q}.
\end{aligned}
$$

Hence $((f\jmath)(\wp))_{\wp\in I}$ is equicontinuous. By, using an easy computation we get

$$|f_{k,l}(\wp,\jmath(\wp))-f_{k-1,l-1}(\wp,\jmath(\wp))|^{q}\leq\frac{1}{2^{q-1}}\big|\jmath_{k,l}(\wp)-\jmath_{k-1,l-1}(\wp)\big|^{q}.$$

For each natural number $k,l\in\mathbb{N}$, we take $q_{k,l}(\wp)=\frac{1}{2^{q-1}}$. Thus $(q_{k,l}(\wp))$ is equibounded on I, and $Q=\frac{1}{2^{q-1}}$. Obviously, the relation holds. So, all the conditions of Th 5 are satisfied. So E.q (10) has at least one solution in $C(I,bv_q^2)$ for all $q\geq 1$.

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