Azerbaijan Journal of Mathematics
V. 16, No 2, 2026, July
ISSN 2218-6816
https://doi.org/10.59849/2218-6816.2026.2.169

Some Compact-Type and Connected-Type Properties of the Uniform Space of G-Permutation Degree

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Abstract. In this paper, we study the behavior of various compact-type properties (like pre-compactness, local compactness) and connected-type properties (like uniform linkedness, point finiteness star-finiteness) of uniform spaces under influence of the functor $\mathsf{SP}^{\mathsf{n}}_{\mathsf{G}}$ of G-permutation degree. Especially, it is proved that this functor preserves pre-compactness, local compactness, and uniform linkedness. Besides, it is also proved that the functor $\mathsf{SP}^{\mathsf{n}}_{\mathsf{G}}$ preserves perfect uniformly continuous mappings.

Key Words and Phrases: uniform space, permutation degree, pre-compact, uniformly linked, star-finite.

2010 Mathematics Subject Classifications: 18F60, 54B30, 54D99, 54E15

1. Introduction and preliminaries

In recent years many authors have investigated influence of different functors to various topological properties. In particular, such studies have been done for the functor of permutation degree (see, for instance, [2, 3, 10, 11, 12, 19, 21]). For example, most important network-type properties of the space of permutation degree of a topological space were considered in [20]. The papers [12] and [18] contain several similar interesting results on various tightness-type properties, and the paper [17] presents several results about cardinal invariants in the space of permutation degree.

On the other hand, different aspects of the theory of uniform spaces have been investigated in many works and interesting results were obtained (see [4, 6, 7, 8, 15, 16]). However, there are no many research papers investigating influence

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of functors to uniform spaces, in particular, influence of the functor of permutation degree and duality between uniform spaces and their spaces of permutation degree. In this paper, we study interrelations between certain important properties of uniform spaces (pre-compactness, local compactness, uniform linkedness, star-finiteness) and the functor $\mathsf{SP}^{\mathsf{n}}_{\mathsf{G}}$ of G-permutation degree. Especially, it has been proved that this functor preserves all the properties. It is also proved that the functor $\mathsf{SP}^{\mathsf{n}}_{\mathsf{G}}$ preserves perfectly uniformly continuous mappings. Our results complement and extend knowledge about the functor $\mathsf{SP}^{\mathsf{n}}_{\mathsf{G}}$.

The paper is organized as follows. In Introduction we recall some basic definitions which are useful for the rest of the article. In Section 2 we present results on compact-type and connected-type properties of the space of $G-$permutation degree with uniformity. Section 3 gives an overview of this article and an idea and motivation for future investigation on related issues.

1.1. The space of G-permutation degree

As usual, let S_n denote the permutation group of the set $\{1, 2, \dots, n\}$ of first n natural numbers, and let G be a subgroup of S_n. Let X^n be the n-th power of a space X. On X^n one defines a binary relation ρ by the rule

$$\mathbf{x} = (x_1, x_2, \dots, x_n) \ \rho \ \mathbf{y} = (y_1, y_2, \dots, y_n) \Leftrightarrow \ \exists \ p \in G \text{ such that } y_i = x_{p(i)}.$$

It is easy to see that ρ is an equivalence relation called the *symmetric G-equivalence relation*. Denote by $[\mathbf{x}]_G = [(x_1, x_2, \dots, x_n)]_G$ the equivalence class of the element $\mathbf{x} = (x_1, x_2, \dots, x_n) \in X^n$. Let $\mathsf{SP}^{\mathsf{n}}_{\mathsf{G}}X$ denote the quotient set X^n/ρ, and let $\pi^s_{n,G} : X^n \to \mathsf{SP}^{\mathsf{n}}_{\mathsf{G}}X$ be the quotient mapping defined as

$$\pi^s_{n,G}((x_1, x_2, \dots, x_n)) = [(x_1, x_2, \dots, x_n)]_G.$$

The set $\mathsf{SP}^{\mathsf{n}}_{\mathsf{G}}X$ endowed with the quotient topology is called the *space of G-permutation degree* (of the space X) and denoted also by $\mathsf{SP}^{\mathsf{n}}_{\mathsf{G}}X$.

Let $f : X \to Y$ be a continuous mapping. For an equivalence class $[(x_1, x_2, \dots, x_n)] \in \mathsf{SP}^{\mathsf{n}}_{\mathsf{G}}X$ we put

$$\mathsf{SP}^{\mathsf{n}}_{\mathsf{G}}f[(x_1, x_2, \dots, x_n)]_G = [(f(x_1), f(x_2), \dots, f(x_n))]_G.$$

One obtains a mapping $\mathsf{SP}^{\mathsf{n}}_{\mathsf{G}}f : \mathsf{SP}^{\mathsf{n}}_{\mathsf{G}}X \to \mathsf{SP}^{\mathsf{n}}_{\mathsf{G}}Y$. It is easy to check that the operation $\mathsf{SP}^{\mathsf{n}}_{\mathsf{G}}$ so constructed is a functor (which is a normal functor in the category of compacta). This functor is called the *functor of G-permutation degree.*

More information about functors the reader can find in [11, 10].

1.2. Uniform spaces

We say that a cover α of a set X is a *refinement* of a cover β (or α is *inscribed* in β), denoted by $\alpha \prec \beta$, if for any $A \in \alpha$ there exists a $B \in \beta$ such that $A \subset B$ [5, 9].

For a family α of subsets of X and $x \in X$, $M \subset X$, we write

$$\mathrm{St}(x, \alpha) = \cup\{A \in \alpha : x \in A\},$$
$$\mathrm{St}(M, \alpha) = \cup\{A \in \alpha : A \cap M \neq \emptyset\}.$$

The set $\mathrm{St}(M, \alpha)$ is called the *star* of M with respect to α.

Put $\alpha^* = \{\mathrm{St}(A, \alpha) : A \in \alpha\}$.

We say that a cover α is a *star refinement* of β (or α is *star inscribed* in β) if $\alpha^* \prec \beta$.

Let X be a non-empty set. A family $\mathbb{U}$ of covers of X is said to be a *uniformity* (in terms of covers) [5, 9] if the following conditions are satisfied:

(U_1) If $\alpha \in \mathbb{U}$ and β of X is a cover of X such that $\alpha \prec \beta$, then $\beta \in \mathbb{U}$;

(U_2) For any $\alpha_1, \alpha_2 \in \mathbb{U}$ there exists $\alpha \in \mathbb{U}$ which is a refinement of both α_1 and α_2;

(U_3) For any $\alpha \in \mathbb{U}$, there exists $\beta \in \mathbb{U}$ such that $\beta \prec^* \alpha$;

(U_4) For any pair of distinct points x and y of X, there exists $\alpha \in \mathbb{U}$ such that no element of α containing both x and y.

The pair $(X, \mathbb{U})$ is called a *uniform space*.

Notice that the condition (U_4) defines Hausdorff uniform spaces. Therefore, in this paper "uniform space" means "Hausdorff uniform space".

A family $\mathbb{B} \subset \mathbb{U}$ is called a *base* [5] of the uniformity $\mathbb{U}$, if for every cover $\alpha \in \mathbb{U}$ there exists $\beta \in \mathbb{B}$ such that $\beta \prec \alpha$.

Proposition 1 ([5]). *A family $\mathbb{B}$ of covers of a set X is a base of a uniformity $\mathbb{U}$ in X if and only if the following conditions hold:*

(B1) For every pair $\beta_1, \beta_2 \in \mathbb{B}$ there exists $\beta \in \mathbb{B}$, such that β is a refinement of both β_1 and β_2.

(B2) For every $\beta \in \mathbb{B}$ there exists $\gamma \in \mathbb{B}$, such that γ is a star refinement of β.

(B3) $\bigcap\{\mathrm{St}(x, \beta) : \beta \in \mathbb{B}\} = \{x\}$ for every point $x \in X$.

We refer to [5, 9, 14] for more details about uniform spaces.

2. Results

We now present some results concerning the preservation of certain compact-like and connected-like properties of uniform spaces under the functor of G-permutation degree.

2.1. Compact-like properties

Lemma 1. *If α is a cover of a space X, then*

$$\mathcal{S}(\alpha) := \{[U_1 \times U_2 \times \ldots \times U_n]_G : U_i \in \alpha,\ i = 1, 2, \ldots, n\}$$

is a cover of the space $\mathsf{SP}^{\mathsf{n}}_{\mathsf{G}}X$. If $\alpha, \beta \in \mathbb{U}$ and $\alpha \prec \beta$, then $\mathcal{S}(\alpha) \prec \mathcal{S}(\beta)$.

Proof. Suppose that $\alpha = \{U_i : i \in \mathcal{I}\}$ is a cover of X. Then $\bigcup_{i\in\mathcal{I}} U_i = X$. It follows that the set

$$\left(\bigcup_{i_1\in\mathcal{I}} U_{i_1}\right) \times \left(\bigcup_{i_2\in\mathcal{I}} U_{i_2}\right) \times \ldots \times \left(\bigcup_{i_n\in\mathcal{I}} U_{i_n}\right) = \bigcup_{(i_1,i_2,\ldots,i_n)\in\mathcal{I}^n} (U_{i_1} \times U_{i_2} \times \ldots \times U_{i_n})$$

covers X^n, where $U_{i_1}, U_{i_2}, \ldots, U_{i_n} \in \alpha$. On the other hand, we have

$$\begin{aligned}\mathsf{SP}^{\mathsf{n}}_{\mathsf{G}}X = \pi^s_{n,G}(X^n) &= \pi^s_{n,G}\left(\bigcup_{(i_1,i_2,\ldots,i_n)\in\mathcal{I}^n}\right)(U_{i_1} \times U_{i_2} \times \ldots \times U_{i_n}) \\ &= \bigcup_{(i_1,i_2,\ldots,i_n)\in\mathcal{I}^n} \left(\pi^s_{n,G}(U_{i_1} \times U_{i_2} \times \ldots \times U_{i_n})\right) \\ &= \bigcup_{(i_1,i_2,\ldots,i_n)\in\mathcal{I}^n} ([U_{i_1} \times U_{i_2} \times \ldots \times U_{i_n}]_G) = \bigcup \mathcal{S}(\alpha).\end{aligned}$$

It means that $\mathcal{S}(\alpha)$ is a cover of the space $\mathsf{SP}^{\mathsf{n}}_{\mathsf{G}}X$.

Let $\alpha \prec \beta$ and let $[U_1 \times U_2 \times ... \times U_n]_G$ be any element in $\mathcal{S}(\alpha)$. As $\alpha \prec \beta$, for every $i \le n$ there is $V_i \in \beta$ such that $U_i \subset V_i$. Then, as it is easy to check, $[U_1 \times U_2 \times ... \times U_n]_G \subset [V_1 \times V_2 \times ... \times V_n]_G$, that is, $\mathcal{S}(\alpha) \prec \mathcal{S}(\beta)$. ◄

Theorem 1. *[22, Theorem 2.1] If $\mathbb{B}$ is a base of a uniform space $(X, \mathbb{U})$, then $S(\mathbb{B}) = \{S(\beta) : \beta \in \mathbb{B}\}$ is a base of the uniform space $(\mathsf{SP}^{\mathsf{n}}_{\mathsf{G}}X, \mathsf{SP}^{\mathsf{n}}_{\mathsf{G}}\mathbb{U})$.*

A uniform space $(X, \mathbb{U})$ is called *pre-compact* if every $\alpha \in \mathbb{U}$ has a finite refinement $\beta \in \mathbb{U}$ [9, 8.3.D.a].

Theorem 2. *Let $(X, \mathbb{U})$ be a uniform space. The space $(X, \mathbb{U})$ is pre-compact if and only if $(\mathsf{SP}^{\mathsf{n}}_{\mathsf{G}}\mathsf{X}, \mathsf{SP}^{\mathsf{n}}_{\mathsf{G}}\mathbb{U})$ is pre-compact.*

Proof. (⇒) Let $(X, \mathbb{U})$ be a pre-compact space and $\mathcal{S}(\alpha) = \{[U_1 \times U_2 \times ... \times U_n]_G : U_i \in \alpha,\ i = 1, 2, ..., n\}$ be a cover of $(\mathsf{SP^n_G X}, \mathsf{SP^n_G}\mathbb{U})$ for some $\alpha \in \mathbb{U}$. Since $(X, \mathbb{U})$ is pre-compact, α has a finite refinement $\alpha_0 = \{A_1, A_2, A_3, ..., A_k\}$. We can construct the family

$$\mathcal{S}(\alpha_0) = \{[A_{j1} \times A_{j2} \times ... \times A_{jn}]_G :\ A_{ji} \in \alpha_0\}$$

from the elements of α_0. By Lemma 1, $\mathcal{S}(\alpha_0)$ is a cover of the space $(\mathsf{SP^n_G}, \mathsf{SP^n_G}\mathbb{U})$ and refinement of $\mathcal{S}(\alpha)$. On the other hand, $\mathcal{S}(\alpha_0)$ has at most k^n elements, i.e. it is finite. It means that $(\mathsf{SP^n_G}X, \mathsf{SP^n_G}\mathbb{U})$ is pre-compact.

(⇐) Let $(\mathsf{SP^n_G}X, \mathrm{SP^n_G}\mathbb{U})$ be a pre-compact space. Now we show that $(X, \mathbb{U})$ is pre-compact. Let $\alpha \in \mathbb{U}$ be a cover of $(X, \mathbb{U})$. Clearly, by Lemma 1, $\mathcal{S}(\alpha) = \{[U_1 \times U_2 \times \ldots \times U_n]_G : U_i \in \alpha,\ i = 1, 2, \ldots, n\}$ is a cover of $(\mathsf{SP^n_G}X, \mathsf{SP^n_G}\mathbb{U})$. Since the space $(\mathsf{SP^n_G}X, \mathsf{SP^n_G}\mathbb{U})$ is pre-compact, there exists a finite refinement

$$\left\{[V_1^1 \times V_2^1 \times \ldots \times V_n^1]_G, [V_1^2 \times V_2^2 \times \ldots \times V_n^2]_G, \ldots, [V_1^k \times V_2^k \times \ldots \times V_n^k]_G\right\}$$

of $\mathcal{S}(\alpha)$. Put

$$\beta = \left\{V_1^1, V_2^1, \ldots, V_n^1, V_1^2, V_2^2, \ldots, V_n^2, \ldots, V_1^k, V_2^k, \ldots, V_n^k\right\}.$$

It is easy to check that the finite family β is a refinement of α. It means that $(X, \mathbb{U})$ is a pre-compact space. ◄

Remark 1. *Several authors have constructed (under some set-theoretic assumptions such as the continuum hypothesis CH or Martin's axiom MA) pre-compact topological groups H whose square H^2 is not pre-compact. (In 2019, in [13] it was announced, but never published, such a result in ZFC.) However, by the previous theorem, any such group H is such that $\mathsf{SP^2_G}H$ is a pre-compact uniform space.*

Let $(X, \mathbb{U})$ and $(Y, \mathbb{V})$ be two uniform spaces. A mapping $f : (X, \mathbb{U}) \to (Y, \mathbb{V})$ is called *uniformly continuous* [5, 9], if for every $\beta \in \mathbb{V}$ there exists $\alpha \in \mathbb{U}$ such that $\alpha \prec f^{\leftarrow}\beta$, where $f^{\leftarrow}\beta = \{f^{\leftarrow}(B) : B \in \beta\}$.

Theorem 3. *[22, Theorem 2.2] If $f : (X, \mathbb{U}) \to (Y, \mathbb{V})$ is a uniformly continuous mapping, then the mapping $\mathsf{SP^n}f : (\mathsf{SP^n}X, \mathsf{SP^n}\mathbb{U}) \to (\mathsf{SP^n}Y, \mathsf{SP^n}\mathbb{V})$ is also uniformly continuous.*

Let $(X, \mathbb{U})$ and $(Y, \mathbb{V})$ be uniform spaces.

Proposition 2. *If $f : (X, \mathbb{U}) \to (Y, \mathbb{V})$ is a perfect uniformly continuous mapping, then so is $\mathsf{SP^n_G}f : (\mathsf{SP^n_G}X, \mathsf{SP^n_G}\mathbb{U}) \to (\mathsf{SP^n_G}Y, \mathsf{SP^n_G}\mathbb{V})$.*

Proof. Let $f : (X, \mathbb{U}) \to (Y, \mathbb{V})$ be a perfect uniformly continuous mapping. By Theorem 3 we have that $\mathsf{SP}^{\mathsf{n}}_{\mathsf{G}} f : (\mathsf{SP}^{\mathsf{n}}_{\mathsf{G}} X, \mathsf{SP}^{\mathsf{n}}_{\mathsf{G}} \mathbb{U}) \to (\mathsf{SP}^{\mathsf{n}}_{\mathsf{G}} Y, \mathsf{SP}^{\mathsf{n}}_{\mathsf{G}} \mathbb{V})$ is a uniformly continuous mapping. On the other hand, for every point $[(y_1, y_2, \ldots, y_n)]_G$ of $\mathsf{SP}^{\mathsf{n}}_{\mathsf{G}} Y$ the set $\mathsf{SP}^{\mathsf{n}}_{\mathsf{G}} f^{\leftarrow}([(y_1, y_2, \ldots, y_n)]_G)$ is finite and hence a compact subset of $\mathsf{SP}^{\mathsf{n}}_{\mathsf{G}} X$. So, the mapping $\mathsf{SP}^{\mathsf{n}}_{\mathsf{G}} f : (\mathsf{SP}^{\mathsf{n}}_{\mathsf{G}} X, \mathsf{SP}^{\mathsf{n}}_{\mathsf{G}} \mathbb{U}) \to (\mathsf{SP}^{\mathsf{n}}_{\mathsf{G}} Y, \mathsf{SP}^{\mathsf{n}}_{\mathsf{G}} \mathbb{V})$ is perfect. ◀

A uniform space $(X, \mathbb{U})$ is called *locally compact*, if there exists $\alpha \in \mathbb{U}$ consisting of compact subsets of X.

Theorem 4. *If a uniform space* $(X, \mathbb{U})$ *is locally compact, then so is* $(\mathsf{SP}^{\mathsf{n}}_{\mathsf{G}} X, \mathsf{SP}^{\mathsf{n}}_{\mathsf{G}} \mathbb{U})$.

Proof. Let $(X, \mathbb{U})$ be a locally compact uniform space. Now we will show that $(\mathsf{SP}^{\mathsf{n}}_{\mathsf{G}} X, \mathsf{SP}^{\mathsf{n}}_{\mathsf{G}} \mathbb{U})$ is locally compact. Since $(X, \mathbb{U})$ is locally compact, there is a cover $\alpha = \{U_s : \ s \in S\}$ of X, where U_s is a compact subset of X for every $s \in S$. By Lemma 1, the collection $\mathcal{S}(\alpha)$ covers $\mathsf{SP}^{\mathsf{n}}_{\mathsf{G}} X$. On the other hand, every element $[U_{s_1} \times U_{s_2} \times \ldots \times U_{s_n}]_G$ of $\mathcal{S}(\alpha)$ is a compact subset of $\mathsf{SP}^{\mathsf{n}}_{\mathsf{G}} X$. It shows that the uniform space $(\mathsf{SP}^{\mathsf{n}}_{\mathsf{G}} X, \mathsf{SP}^{\mathsf{n}}_{\mathsf{G}} \mathbb{U})$ is locally compact. ◀

2.2. Connected-like properties

In this subsection we consider some properties of uniform spaces close to connectedness which are preserved by the functor of G-permutation degree.

A finite sequence $\{A_1, A_2, \ldots, A_k\}$ of subsets of a space X is called *linked* if $A_i \cap A_{i+1} \neq \emptyset$ for each $i = 1, 2, \ldots, k-1$.

A uniform space $(X, \mathbb{U})$ is called *uniformly linked* if for any cover $\alpha \in \mathbb{U}$ there exists $m \in \mathbb{N}$ such that for any points $x, y \in X$ there is a linked sequence $\{A_1, A_2, ..., A_k\} \subset \alpha$ such that $k \leq m$, $x \in A_1$ and $y \in A_k$.

Theorem 5. *Let* $(X, \mathbb{U})$ *be a uniform space. The space* $(X, \mathbb{U})$ *is uniformly linked if and only if* $(\mathsf{SP}^{\mathsf{n}}_{\mathsf{G}} X, \mathsf{SP}^{\mathsf{n}}_{\mathsf{G}} \mathbb{U})$ *is uniformly linked.*

Proof. ($\Rightarrow$) Suppose that $(X, \mathbb{U})$ is a uniformly linked space. Let $\mathcal{S}(\alpha) = \{[U_1 \times U_2 \times \ldots \times U_n]_G : U_i \in \alpha, \ i = 1, 2, \ldots, n\} \in \mathsf{SP}^{\mathsf{n}}_{\mathsf{G}} \mathbb{U}$ be an arbitrary cover of $\mathsf{SP}^{\mathsf{n}}_{\mathsf{G}} X$ and $[\mathbf{x}]_G, [\mathbf{y}]_G \in \mathsf{SP}^{\mathsf{n}}_{\mathsf{G}} X$. Put $(\pi^s_{n,G})^{\leftarrow}([\mathbf{x}]_G) = \{\mathbf{x}^1, \mathbf{x}^2, \ldots, \mathbf{x}^{l_1}\}$, $(\pi^s_{n,G})^{\leftarrow}([\mathbf{y}]_G) = \{\mathbf{y}^1, \mathbf{y}^2, \ldots, \mathbf{y}^{l_2}\} \subset X^n$, where $l_1, l_2 \leq n$. Since $(X^n, \mathbb{U}^n)$ is uniformly linked, for every two points $\mathbf{x}^i$, $\mathbf{y}^j$ in X there exist $m \in \mathbb{N}$ and a linked sequence $\{A_1, A_2, \ldots, A_k\} \subset \alpha^n$ such that $k \leq m$, $\mathbf{x}^i \in A_1$ and $\mathbf{y}^j \in A_k$. Clearly, $\{\mathsf{SP}^{\mathsf{n}}_{\mathsf{G}} A_1, \mathsf{SP}^{\mathsf{n}}_{\mathsf{G}} A_2, \ldots, \mathsf{SP}^{\mathsf{n}}_{\mathsf{G}} A_k\} \subset \mathcal{S}(\alpha)$ and $[\mathbf{x}^i]_G \in \mathsf{SP}^{\mathsf{n}}_{\mathsf{G}} A_1$, $[\mathbf{y}^j]_G \in \mathsf{SP}^{\mathsf{n}}_{\mathsf{G}} A_k$. On the other hand, for every $t = 1, 2, \ldots, k-1$ we have that $A_t \cap A_{t+1} \neq \emptyset$. Therefore, $\pi^s_{n,G}(A_t \cap A_{t+1})$ is non-empty and $\pi^s_{n,G}(A_t \cap A_{t+1}) \subset \mathsf{SP}^{\mathsf{n}}_{\mathsf{G}} A_t \cap \mathsf{SP}^{\mathsf{n}}_{\mathsf{G}} A_{t+1}$. It shows that $\mathsf{SP}^{\mathsf{n}}_{\mathsf{G}} A_t \cap \mathsf{SP}^{\mathsf{n}}_{\mathsf{G}} A_{t+1} \neq \emptyset$ and it is shown that $\{\mathsf{SP}^{\mathsf{n}}_{\mathsf{G}} A_1, \mathsf{SP}^{\mathsf{n}}_{\mathsf{G}} A_2, \ldots, \mathsf{SP}^{\mathsf{n}}_{\mathsf{G}} A_k\} \subset$

$\mathcal{S}(\alpha)$ is a linked sequence, where $k \le m$. It means that the uniform space $(\mathsf{SP}^{\mathsf{n}}_{\mathsf{G}}X, \mathsf{SP}^{\mathsf{n}}_{\mathsf{G}}\mathbb{U})$ is uniformly linked.

($\Leftarrow$) Suppose that $(\mathsf{SP}^{\mathsf{n}}_{\mathsf{G}}X, \mathsf{SP}^{\mathsf{n}}_{\mathsf{G}}\mathbb{U})$ is a uniformly linked space. We will show that $(X,\mathbb{U})$ is a uniformly linked space. Let x, y be arbitrary points of the space $(X,\mathbb{U})$. Put $[\mathbf{x}]_G = [(x,x,\dots,x)]_G = (x,x,\dots,x)$, $[\mathbf{y}]_G = [(y,y,\dots,y)]_G = (y,y,\dots,y) \in \mathsf{SP}^{\mathsf{n}}_{\mathsf{G}}X$. Since the space $(\mathsf{SP}^{\mathsf{n}}_{\mathsf{G}}X, \mathsf{SP}^{\mathsf{n}}_{\mathsf{G}}\mathbb{U})$ is uniformly linked, there exists $m \in \mathbb{N}$ such that one can be chosen a linked sequence $\{\mathsf{SP}^{\mathsf{n}}_{\mathsf{G}}A_1, \mathsf{SP}^{\mathsf{n}}_{\mathsf{G}}A_2, \dots, \mathsf{SP}^{\mathsf{n}}_{\mathsf{G}}A_k\} \subset \mathcal{S}(\alpha)$ such that $k \le m$, $[\mathbf{x}]_G \in \mathsf{SP}^{\mathsf{n}}_{\mathsf{G}}A_1$ and $[\mathbf{y}]_G \in \mathsf{SP}^{\mathsf{n}}_{\mathsf{G}}A_k$. Let $\mathsf{SP}^{\mathsf{n}}_{\mathsf{G}}A_1 = [A_1^1 \times A_2^1 \times \dots \times A_n^1]_G$, $\mathsf{SP}^{\mathsf{n}}_{\mathsf{G}}A_2 = [A_1^2 \times A_2^2 \times \dots \times A_n^2]_G$, $\dots$, $\mathsf{SP}^{\mathsf{n}}_{\mathsf{G}}A_k = [A_1^k \times A_2^k \times \dots . \times A_n^k]_G$. from $[x]_G \in \mathsf{SP}^{\mathsf{n}}_{\mathsf{G}}A_1$ and $[y]_G \in \mathsf{SP}^{\mathsf{n}}_{\mathsf{G}}A_k$ it follows that $x \in A_i^1$, $y \in A_i^k$ and $k \le m$ for every $i = 1,2,\dots,n$. Since the sequence $\{\mathsf{SP}^{\mathsf{n}}_{\mathsf{G}}A_1, \mathsf{SP}^{\mathsf{n}}_{\mathsf{G}}A_2, \dots, \mathsf{SP}^{\mathsf{n}}_{\mathsf{G}}A_k\} \subset \mathcal{S}(\alpha)$ is uniformly linked, $\mathsf{SP}^{\mathsf{n}}_{\mathsf{G}}A_t \bigcap \mathsf{SP}^{\mathsf{n}}_{\mathsf{G}}A_{t+1} \ne \emptyset$ for any $t = 1,2,\dots,k-1$. Let $[\mathbf{x}^t]_G \in \mathsf{SP}^{\mathsf{n}}_{\mathsf{G}}A_t \bigcap \mathsf{SP}^{\mathsf{n}}_{\mathsf{G}}A_{t+1}$, where $[\mathbf{x}^1]_G = [\mathbf{x}]_G = [(x,x,\dots,x)]_G$, $[\mathbf{x}^k]_G = [\mathbf{y}]_G = [(y,y,\dots,y)]_G$. It means that $[\mathbf{x}^t]_G = [(x_1^t, x_2^t, \dots, x_n^t)]_G \in [A_1^t \times A_2^t \times \dots \times A_n^t]_G \bigcap [A_1^{t+1} \times A_2^{t+1} \times \dots \times A_n^{t+1}]_G$. Then there exist A_i^t and A_j^{t+1} such that $x_l^t \in A_i^t \cap A_j^{t+1}$ for every $t = 1,2,\dots.,k-1$ and $l = 1,2,\dots,n$. Especially, $x_i^t \in A_i^t \cap A_i^{t+1}$ for every $t = 1,2,\dots,k-1$ and $i = 1,2,\dots,n$. Consequently,

$$\begin{gathered}
x = x_i^1 \in A_i^1 \cap A_i^2, \ \Rightarrow \ A_i^1 \cap A_i^2 \ne \emptyset, \\
x_i^2 \in A_i^2 \cap A_i^3, \ \Rightarrow \ A_i^2 \cap A_i^3 \ne \emptyset, \\
\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots \\
x_i^{k-1} \in A_i^{k-1} \cap A_i^k, \ \Rightarrow \ A_i^{k-1} \cap A_i^k \ne \emptyset.
\end{gathered}$$

Therefore, we have constructed a linked sequence $\{A_i^1, A_i^2, A_i^3, \dots, A_i^{k-1}, A_i^k\}$ such that $x \in A_i^1$ and $y \in A_i^k$. It shows that the space $(X,\mathbb{U})$ is uniformly linked. ◀

A uniform space $(X,\mathbb{U})$ is called *uniformly connected* [5], and uniformity $\mathbb{U}$ is called *connected*, if every uniformly continuous mapping $f : (X,\mathbb{U}) \to (D,\mathbb{U}_D)$ from $(X,\mathbb{U})$ to every discrete uniform space $(D,\mathbb{U}_D)$ is constant.

In [5], it was shown that every uniformly linked space is uniformly connected, but in general, the converse is not true. In the pre-compact spaces uniform linkedness coincides with uniform connectedness. Therefore, we have the following result.

Corollary 1. *Let $(X,\mathbb{U})$ be a uniform pre-compact space. The space $(X,\mathbb{U})$ is uniformly connected if and only if $(\mathsf{SP}^{\mathsf{n}}_{\mathsf{G}}X, \mathsf{SP}^{\mathsf{n}}_{\mathsf{G}}\mathbb{U})$ is uniformly connected.*

A cover of X is called *point-finite*, if every point of X lies in only finitely many members of the cover. A cover of X is called *star-finite*, if every member of the cover intersects only finitely many members of the cover.

A uniform space is called *point-finite* (resp. *star-finite*) [5] if it has a base for its uniform covers consisting of point-finite (resp. star-finite) covers.

Theorem 6. *A uniform space* $(X,\mathbb{U})$ *is point-finite if and only if* $(\mathsf{SP}^{\mathsf{n}}_{\mathsf{G}}X, \mathsf{SP}^{\mathsf{n}}_{\mathsf{G}}\mathbb{U})$ *is point-finite.*

Proof. $(\Rightarrow)$ Suppose that $(X,\mathbb{U})$ is point-finite. Then it has a base $\mathbb{B}$ consisting of point-finite covers. Put $\mathcal{S}(\mathbb{B}) = \{\mathcal{S}(\alpha) : \alpha \in \mathbb{B}\}$. By Theorem 1, $\mathcal{S}(\mathbb{B})$ is a base for the uniform space $(\mathsf{SP}^{\mathsf{n}}_{\mathsf{G}}X, \mathsf{SP}^{\mathsf{n}}_{\mathsf{G}}\mathbb{U})$. Now we will show that $\mathcal{S}(\alpha)$ is a point-finite cover of $\mathsf{SP}^{\mathsf{n}}_{\mathsf{G}}X$ for every $\alpha \in \mathbb{B}$. Let $[\mathbf{x}]_G = [(x_1, x_2, ..., x_n)]_G$ be an arbitrary point of $\mathsf{SP}^{\mathsf{n}}_{\mathsf{G}}X$. Since $\mathbb{B}$ consists of point-finite covers, for every x_i we have finitely many members $U_i^1, U_i^2, \ldots, U_i^{k_i}$ of α such that $x_i \in \bigcap_{j=1}^{k_i} U_i^{j_i}$. Then $\mathcal{S}(\alpha) = \{[U_1^{j_1} \times U_2^{j_2} \times \ldots \times U_n^{j_n}]_G : U_i^{j_i} \in \alpha\}$, where $j_1 = 1, 2, \ldots, k_1$, $j_2 = 1, 2, \ldots, k_2, \ldots$, $j_n = 1, 2, \ldots, k_n$ and $k_1, k_2, \ldots, k_n$ are finite natural numbers. It means that $\mathcal{S}(\alpha)$ is a point-finite cover of $\mathsf{SP}^{\mathsf{n}}_{\mathsf{G}}X$ with the cardinality $\leq k_1 \cdot k_2 \cdot \cdots \cdot k_n$. It shows that the uniform space $(\mathsf{SP}^{\mathsf{n}}_{\mathsf{G}}X, \mathsf{SP}^{\mathsf{n}}_{\mathsf{G}}\mathbb{U})$ is point-finite.

$(\Leftarrow)$ Suppose that the space $(\mathsf{SP}^{\mathsf{n}}_{\mathsf{G}}X, \mathsf{SP}^{\mathsf{n}}_{\mathsf{G}}\mathbb{U})$ is point-finite. Then it has a base $\mathcal{S}(\mathbb{B})$ that consists of point-finite covers, where $\mathcal{S}(\mathbb{B}) = \{\mathcal{S}(\alpha) : \alpha \in \mathbb{B}\}$. Now we will prove that $\mathbb{B}$ consists of point-finite covers. Let x be an arbitrary point of X. We take a point $[\mathbf{x}]_G = [(x, x, \ldots, x)]_G$ form the space $\mathrm{SP}^{\mathrm{n}}_{\mathrm{G}}X$. Since elements of $\mathcal{S}(\mathbb{B})$ are point-finite covers, we have that the point $[\mathbf{x}]_G$ lies in only finitely many elements of $\mathcal{S}(\alpha)$ for every $\mathcal{S}(\alpha) \in \mathcal{S}(\mathbb{B})$. It shows that $x \in U_i$, $U_i \in \alpha$, $i = 1, 2, ..., n$, hence the point x lies in finitely many elements of α. It means that α is a point-finite cover. This fact is true for every $\alpha \in \mathbb{B}$. Therefore, the space $(X,\mathbb{U})$ is point-finite. ◀

Theorem 7. *Let* $(X,\mathbb{U})$ *be a uniform space. The space* $(X,\mathbb{U})$ *is star-finite if and only if* $(\mathrm{SP}^{\mathrm{n}}_{\mathrm{G}}X, \mathrm{SP}^{\mathrm{n}}_{\mathrm{G}}\mathbb{U})$ *is star-finite.*

Proof. $(\Rightarrow)$ Let $(X,\mathbb{U})$ be a star-finite uniform space. Then it has the base $\mathbb{B}$ that consists of star-finite covers. Put $\mathcal{S}(\mathbb{B}) = \{\mathcal{S}(\alpha) : \alpha \in \mathbb{B}\}$. Clearly, $\mathcal{S}(\mathbb{B})$ is a base for the uniform space $(\mathsf{SP}^{\mathsf{n}}_{\mathsf{G}}X, \mathsf{SP}^{\mathsf{n}}_{\mathsf{G}}\mathbb{U})$ (by Theorem 1). Now we will show that $\mathcal{S}(\alpha)$ is a star-finite cover of the space $\mathsf{SP}^{\mathsf{n}}_{\mathsf{G}}X$ for every $\alpha \in \mathbb{B}$. Let $\mathsf{SP}^{\mathsf{n}}_{\mathsf{G}}U_0 = [U_1^0 \times U_2^0 \times \ldots \times U_n^0]_G$ be an arbitrary member of $\mathcal{S}(\alpha)$. Since α is a star-finite cover of $(X,\mathbb{U})$, for each U_i^0 there exist finitely many members $U_{i1}, U_{i2}, ..., U_{ik_i}$ of α such that $U_i^0 \cap U_{ij} \neq \emptyset$, where $j = 1, 2, \ldots, k_i$, $i = 1, 2, \ldots, n$ and $k_1, k_2, \ldots, k_n$ are finite numbers. It follows that $[U_1^0 \times U_2^0 \times \ldots \times U_n^0]_G \cap [U_{1j_1} \times U_{2j_2} \times \ldots \times U_{nj_n}]_G \neq \varnothing$, where $j_1 = 1, 2, \ldots, k_1$, $j_2 = 1, 2, \ldots, k_2$, $\ldots$, $j_n = 1, 2, \ldots, k_n$. The collection of sets $\{[U_{1j_1} \times U_{2j_2} \times \ldots \times U_{nj_n}]_G\}$ is finite (cardinality $\leq k_1 \cdot k_2 \cdot \ldots \cdot k_n$). It means that the set $\mathrm{SP}^{\mathrm{n}}_{\mathrm{G}}U_0$ intersects at most finitely many elements of $\mathcal{S}(\alpha)$. Therefore, $(\mathsf{SP}^{\mathsf{n}}_{\mathsf{G}}X, \mathsf{SP}^{\mathsf{n}}_{\mathsf{G}}\mathbb{U})$ is star-finite.

$(\Leftarrow)$ Suppose that the uniform space $(\mathsf{SP}^{\mathsf{n}}_{\mathsf{G}}X, \mathrm{SP}^{\mathrm{n}}_{\mathrm{G}}\mathbb{U})$ is star-finite. Let $\mathcal{S}(\mathbb{B})$ be a base that consists of star-finite covers, where $\mathcal{S}(\mathbb{B}) = \{\mathcal{S}(\alpha) : \alpha \in \mathbb{B}\}$.

Let U be an arbitrary member of a cover α. Put $[U \times U \times \ldots \times U]_G \in \mathcal{S}(\alpha)$. Since $\mathcal{S}(\alpha)$ is a star-finite cover, there are finitely many members $[U_1^1 \times U_2^1 \times \ldots \times U_n^1]_G$, $[U_1^2 \times U_2^2 \times \ldots \times U_n^2]_G$, $\ldots$, $[U_1^k \times U_2^k \times \ldots \times U_n^k]_G$ of $\mathcal{S}(\alpha)$ such that $[U \times U \times \ldots \times U]_G \cap [U_1^j \times U_2^j \times \ldots \times U_n^j]_G \neq \varnothing$ for every $j = 1, 2, \ldots, k$. It means that $U \cap U_i^j \neq \varnothing$, where $j = 1, 2, \ldots, k$ and $i = 1, 2, \ldots, n$. Note that U does not intersect another element of α; otherwise the set $[U \times U \times \ldots \times U]_G$ would intersect another member of $\mathcal{S}(\alpha)$ different of $[U_1^1 \times U_2^1 \times \ldots \times U_n^1]_G$, $[U_1^2 \times U_2^2 \times \ldots \times U_n^2]_G$, $\ldots$, $[U_1^k \times U_2^k \times \ldots \times U_n^k]_G$, which is a contradiction. It follows that the set U intersects only elements of the collection $\{U_i^j\}$, where $j = 1, 2, \ldots, k$ and $i = 1, 2, \ldots, n$. The collection $\{U_i^j\}$ has finitely many ($\leq k \cdot n$) elements of α. So, the space $(X, \mathbb{U})$ is star-finite. ◀

3. Conclusion

In this paper we proved that the functor of G-permutation degree preserves several compact-like and connected-like properties of uniform spaces. We hope that similar investigations can be extended to other functors. Especially, it would be interesting to study (non)-preservation of properties considered in this article under influence of other functors, for example, under influence of the functor I_τ of τ-smooth idempotent probability measures acting in the category of Tychonoff spaces, hence uniformizable spaces.

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Received 26 January 2026
Accepted 17 May 2026