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Some Properties of a Class of Vector Potentials with Continuous Tangential Density

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Abstract. The paper demonstrates the boundedness and compactness of operators generated by vector potentials with continuous tangential density in generalized Hölder spaces.

Key Words and Phrases: vector Helmholtz equation, Maxwell's system of equations, vector potential, tangential density, generalized Hölder space.

2010 Mathematics Subject Classifications: 45E05, 31B10

1. Introduction

It is known (see [2, pp. 153–154]) that electric and magnetic boundary value problems for the vector Helmholtz equation, as well as certain boundary value problems for Maxwell's system of equations, can be reduced to a system of integral equations involving vector potentials.

$$(Af)(x) = 2\int_{\Omega} [n(x), rot_x \{\Phi_k(x, y) f(y)\}] \, d\Omega_y, \quad x = (x_1, x_2, x_3) \in \Omega, \tag{1}$$

and

$$(A^* f)(x) = [n(x), (A[n, f])(x)] =$$
$$= 2\left[n(x), \int_{\Omega} [n(x), rot_x \{\Phi_k(x, y) [n(y), f(y)]\}] \, d\Omega_y\right], x \in \Omega, \tag{2}$$

with continuous tangential density f, i.e., the vector function $f(x) = (f_1(x), f_2(x), f_3(x))$ belongs to the space

$$C_{\perp}^3(\Omega) = \left\{ f \in C^3(\Omega) \mid (f(x), n(x)) = 0, \, x \in \Omega \right\},$$

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where $\Omega \subset R^3$ is the Lyapunov surface, $n(x) = (n_1(x), n_2(x), n_3(x))$ the outward unit normal at the point $x \in \Omega$, $[a, b]$ denotes the vector (cross) product of vectors a and b, (a, b) denotes the scalar (dot) product of vectors a and b, $C^3(\Omega) = C(\Omega) \times C(\Omega) \times C(\Omega)$, $C(\Omega)$ the space of all continuous functions on the surface Ω with the norm $\|\rho\|_\infty = \max\limits_{x\in\Omega} |\rho(x)|$,

$$\Phi_k(x, y) = \frac{\exp(ik|x-y|)}{4\pi|x-y|}, \; x, y \in R^3, \; x \neq y,$$

the fundamental solution of the Helmholtz equation $\Delta u + k^2 u = 0$, Δ the Laplace operator, and k the wave number, where $Im\, k \geq 0$. It should be noted (see [2, p. 74]) that the operators A and A^* are adjoint operators with respect to the dual system $\langle C^3(\Omega), C^3(\Omega)\rangle$, defined by the relation

$$\langle f, g\rangle = \int_\Omega (f(y), g(y))\, d\Omega,$$

i.e., $\langle Af, g\rangle = \langle f, A^* g\rangle$, $f, g \in C^3_\perp(\Omega)$.

The counterexample constructed by A.M. Lyapunov (see [4, pp. 89–90]) shows that, in general, the derivative does not exist for single-layer and double-layer potentials with continuous density. Therefore, the operators A and A^* are not defined in the space $C^3(\Omega)$. However, in [10], certain properties of the derivative of the logarithmic double-layer potential were studied; in [7], some properties of the derivative of the acoustic single-layer potential were investigated; in [6], properties of the normal derivative of the acoustic double-layer potential were examined; and in [11], certain properties of a class of vector potentials with weak singularity were analyzed. It should be noted that, using the results of work [6, 7, 10], in works [1, 8, 9] approximate methods for solving some classes of singular integral equations were investigated. In the present paper, the boundedness and compactness of the operators A and A^* with continuous tangential density are proved in generalized Hölder spaces.

2. An estimate of the type of A. Zygmund's estimate for the operators generated by the vector potentials (1) and (2)

Let us introduce the modulus of continuity of the vector function $f \in C^3(\Omega)$:

$$\omega(f, \delta) = \delta \sup_{\tau \geq \delta} \frac{\bar{\omega}(f, \tau)}{\tau}, \quad \delta > 0,$$

where

$$\bar{\omega}(f,\delta)=\max_{\substack{|x-y|\le\delta,\\ x,y\in\Omega}}\sqrt{\sum_{m=1}^{3}\left(f_m(x)-f_m(y)\right)^2},\quad \delta>0.$$

Theorem 1. *Let $\Omega\subset R^3$ be a Lyapunov surface with exponent $0<\alpha\le 1$ and $f\in C_{\perp}^{3}(\Omega)$. Then the integral (1) converges as an improper integral, and for any points $x',x''\in\Omega$, the following estimate holds:*

$$\left|(Af)(x')-(Af)(x'')\right|\le$$

$$\le M^{\dagger}\left(\|f\|_3\,h^{\alpha}\,|\ln h|+\int_0^h\frac{\omega(f,t)}{t^{1-\alpha}}dt+h^{\alpha}\int_h^{diam\Omega}\frac{\omega(f,t)}{t}dt\right),$$

where $h=|\,x'-x''\,|$ and $\|f\|_3=\max\left\{\|f_1\|_\infty,\|f_2\|_\infty,\|f_3\|_\infty\right\}$.

Proof. Taking into account the condition $f\in C_{\perp}^{3}(\Omega)$, it is not difficult to compute that

$$[n(x),rot_x\{\Phi_k(x,\,y)\,f(y)\}]=$$

$$=(n(x)-n(y),f(y))\,grad_x\Phi_k(x,\,y)-f(y)\,\frac{\partial\Phi_k(x,\,y)}{\partial n(x)},$$

where

$$\frac{\partial\Phi_k(x,\,y)}{\partial x_m}=\frac{(ik\,|x-y|-1)\,(x_m-y_m)\exp(ik\,|x-y|)}{4\pi\,|x-y|^3},m=1,2,3,$$

and

$$\frac{\partial\Phi_k(x,\,y)}{\partial n(x)}=\frac{(ik\,|x-y|-1)\,(x-y,n(x))\exp(ik\,|x-y|)}{4\pi\,|x-y|^3}.$$

By the definition of a Lyapunov surface with exponent $0<\alpha\le 1$,

$$|n(x)-n(y)|\le M\,|x-y|^{\alpha},\ \ x,y\in\Omega.\tag{3}$$

Moreover, it is known that (see [12, p. 403])

$$|\cos\gamma\,(x-y,n(x))|\le M\,|x-y|^{\alpha},\ \ x,y\in\Omega,$$

and therefore,

$$|(x-y,n(x))|=|x-y|\,|\cos\gamma\,(x-y,n(x))|\le M\,|x-y|^{1+\alpha},\ \ x,y\in\Omega,\tag{4}$$

†Here and throughout, M will denote positive constants, which may vary from one inequality to another.

where $\gamma\left(a\,,b\right)$ denotes the angle between the vectors a and b. Then it is clear that for any points $x,y\in\Omega$, $x\neq y$,

$$\left|\left(n\left(x\right)-n\left(y\right),f\left(y\right)\right)\frac{\partial\Phi_k\left(x,\,y\right)}{\partial x_1}-f_1\left(y\right)\frac{\partial\Phi_k\left(x,\,y\right)}{\partial n\left(x\right)}\right|\leq$$

$$\leq M\left\|f\right\|_3\frac{1}{\left|x-y\right|^{2-\alpha}},\quad m=1,2,3.$$

Therefore, integral (1) is a weakly singular integral, i.e., it exists as an improper integral. As can be seen,

$$\left(Af\right)\left(x\right)=e_1\left(A_1f\right)\left(x\right)+e_2\left(A_2f\right)\left(x\right)+e_3\left(A_3f\right)\left(x\right),$$

where $e_1=(1,0,0)$, $e_2=(0,1,0)$, $e_3=(0,0,1)$,

$$\left(A_1f\right)\left(x\right)=2\int_{\Omega}\left(\left(n\left(x\right)-n\left(y\right),f\left(y\right)\right)\frac{\partial\Phi_k\left(x,\,y\right)}{\partial x_1}-f_1\left(y\right)\frac{\partial\Phi_k\left(x,\,y\right)}{\partial n\left(x\right)}\right)d\Omega_y,$$

$$\left(A_2f\right)\left(x\right)=2\int_{\Omega}\left(\left(n\left(x\right)-n\left(y\right),f\left(y\right)\right)\frac{\partial\Phi_k\left(x,\,y\right)}{\partial x_2}-f_2\left(y\right)\frac{\partial\Phi_k\left(x,\,y\right)}{\partial n\left(x\right)}\right)d\Omega_y,$$

and

$$\left(A_3f\right)\left(x\right)=2\int_{\Omega}\left(\left(n\left(x\right)-n\left(y\right),f\left(y\right)\right)\frac{\partial\Phi_k\left(x,\,y\right)}{\partial x_3}-f_3\left(y\right)\frac{\partial\Phi_k\left(x,\,y\right)}{\partial n\left(x\right)}\right)d\Omega_y.$$

Let us denote by $d>0$ the radius of the standard sphere for Ω (see [12, p. 400]), and let $\Omega_\varepsilon\left(x\right)=\{y\in\Omega:\ |x-y|<\varepsilon\}$, where $x\in\Omega$ and $\varepsilon>0$. It is known that for each $x\in\Omega$, the set $\Omega_d\left(x\right)$ is uniquely projected onto the set $\Pi_d(x)$, which lies in the tangent plane $\Gamma(x)$ to Ω at the point x. On the patch $\Omega_d(x)$, we choose a local rectangular coordinate system (u,v,w) with the origin at the point x, where the w-axis is directed along the normal $n\left(x\right)$, and the u and v axes lie in the tangent plane $\Gamma\left(x\right)$. Then, in this coordinate system, the neighborhood $\Omega_d(x)$ can be described by the equation $w=\psi\left(u,\,v\right)$, $\left(u,\,v\right)\in\Pi_d(x)$, where $\psi\in H_{1,\alpha}\left(\Pi_d\left(x\right)\right)$ and $\psi\left(0,\,0\right)=0$, $\frac{\partial\psi(0,0)}{\partial u}=0$, $\frac{\partial\psi(0,0)}{\partial v}=0$. Here, $H_{1,\alpha}\left(\Pi_d\left(x\right)\right)$ denotes the linear space of all continuously differentiable functions on $\Pi_d\left(x\right)$, for which $grad\psi$ satisfies the Hölder condition with exponent $0<\alpha\leq1$, i.e.,

$$\left|\,grad\psi\left(u_1,v_1\right)-grad\,\psi\left(u_2,v_2\right)\,\right|\leq$$

$$\leq M_\psi\left(\sqrt{\left(u_1-u_2\right)^2+\left(v_1-v_2\right)^2}\right)^\alpha,\quad\left(u_1,v_1\right),\left(u_2,v_2\right)\in\Pi_d(x),$$

where M_ψ is a positive constant depending on ψ, but not on (u_1,v_1) and (u_2,v_2).

Let us take any points $x',\ x''\in\Omega$ such that the quantity h is less than $\frac{d}{2}$. It is not difficult to show that the following representation holds:

$$(A_1 f)\left(x'\right)-(A_1 f)\left(x''\right)=\sum_{m=1}^{8} R_m\left(x',x''\right),$$

where

$$R_1\left(x',x''\right)=2\int\limits_{\Omega\backslash\Omega_d(x')}\left(\left(n\left(x'\right)-n\left(y\right),f\left(y\right)\right)\frac{\partial\Phi_k\left(x',\,y\right)}{\partial x_1}-f_1\left(y\right)\frac{\partial\Phi_k\left(x',\,y\right)}{\partial n\left(x'\right)}\right)d\Omega_y-$$

$$-2\int\limits_{\Omega\backslash\Omega_d(x')}\left(\left(n\left(x''\right)-n\left(y\right),f\left(y\right)\right)\frac{\partial\Phi_k\left(x'',\,y\right)}{\partial x_1}-f_1\left(y\right)\frac{\partial\Phi_k\left(x'',\,y\right)}{\partial n\left(x''\right)}\right)d\Omega_y,$$

$$R_2\left(x',x''\right)=2\int\limits_{\Omega_{h/2}(x')}\Big(\left(n\left(x'\right)-n\left(y\right),f\left(y\right)\right)\times$$

$$\times\frac{\partial\Phi_k\left(x',\,y\right)}{\partial x_1}-f_1\left(y\right)\frac{\partial\Phi_k\left(x',\,y\right)}{\partial n\left(x'\right)}\bigg)d\Omega_y,$$

$$R_3\left(x',x''\right)=-2\int\limits_{\Omega_{h/2}(x'')}\Big(\left(n\left(x''\right)-n\left(y\right),f\left(y\right)\right)\times$$

$$\times\frac{\partial\Phi_k\left(x'',\,y\right)}{\partial x_1}-f_1\left(y\right)\frac{\partial\Phi_k\left(x'',\,y\right)}{\partial n\left(x''\right)}\bigg)d\Omega_y,$$

$$R_4\left(x',x''\right)=-2\int\limits_{\Omega_{h/2}(x')}\Big(\left(n\left(x''\right)-n\left(y\right),f\left(y\right)\right)\times$$

$$\times\frac{\partial\Phi_k\left(x'',\,y\right)}{\partial x_1}-f_1\left(y\right)\frac{\partial\Phi_k\left(x'',\,y\right)}{\partial n\left(x''\right)}\bigg)d\Omega_y,$$

$$R_5\left(x',x''\right)=2\int\limits_{\Omega_{h/2}(x'')}\Big(\left(n\left(x'\right)-n\left(y\right),f\left(y\right)\right)\times$$

$$\times\frac{\partial\Phi_k\left(x',\,y\right)}{\partial x_1}-f_1\left(y\right)\frac{\partial\Phi_k\left(x',\,y\right)}{\partial n\left(x'\right)}\bigg)d\Omega_y,$$

$$R_6\left(x',x''\right)=2\int\limits_{\Omega_d(x')\backslash\left(\Omega_{h/2}(x')\bigcup\Omega_{h/2}(x'')\right)}\left(n\left(x'\right)-n\left(x''\right),f\left(y\right)\right)\frac{\partial\Phi_k\left(x',\,y\right)}{\partial x_1}d\Omega_y,$$

$$R_7\left(x',x''\right)=2\int\limits_{\Omega_d(x')\backslash\left(\Omega_{h/2}(x')\bigcup\Omega_{h/2}(x'')\right)}\left(\frac{\partial\Phi_k\left(x'',\,y\right)}{\partial n\left(x''\right)}-\frac{\partial\Phi_k\left(x',\,y\right)}{\partial n\left(x'\right)}\right)f_1\left(y\right)d\Omega_y$$

and

$$R_8\left(x',x''\right)=2\int\limits_{\Omega_d(x')\backslash\left(\Omega_{h/2}(x')\bigcup\Omega_{h/2}(x'')\right)}\left(n\left(x''\right)-n\left(y\right),f\left(y\right)\right)\times$$

$$\times\left(\frac{\partial\Phi_k\left(x',\,y\right)}{\partial x_1}-\frac{\partial\Phi_k\left(x'',\,y\right)}{\partial x_1}\right)d\Omega_y.$$

Since the function

$$W\left(x\right)=2\int\limits_{\Omega\backslash\Omega_d(x)}\left(\left(n\left(x\right)-n\left(y\right),f\left(y\right)\right)\frac{\partial\Phi_k\left(x,\,y\right)}{\partial x_1}-f_1\left(y\right)\frac{\partial\Phi_k\left(x,\,y\right)}{\partial n\left(x\right)}\right)d\Omega_y$$

is continuously differentiable on the surface Ω, then

$$\left|R_1\left(x',x''\right)\right|\le M\left\|f\right\|_3h.$$

Let us estimate the expression $R_2\left(x',x''\right)$. It is obvious that

$$R_2\left(x',x''\right)=2\int\limits_{\Omega_{h/2}(x')}\left(\left(n\left(x'\right)-n\left(y\right),f\left(x'\right)\right)\times\right.$$

$$\left.\times\frac{\partial\Phi_k\left(x',\,y\right)}{\partial x_1}-f_1\left(x'\right)\frac{\partial\Phi_k\left(x',\,y\right)}{\partial n\left(x'\right)}\right)d\Omega_y+$$

$$+2\int\limits_{\Omega_{h/2}(x')}\left(\left(n\left(x'\right)-n\left(y\right),f\left(y\right)-f\left(x'\right)\right)\times\right.$$

$$\left.\times\frac{\partial\Phi_k\left(x',\,y\right)}{\partial x_1}-\left(f_1\left(y\right)-f_1\left(x'\right)\right)\frac{\partial\Phi_k\left(x',\,y\right)}{\partial n\left(x'\right)}\right)d\Omega_y.$$

Then, taking into account inequalities (3) and (4), and using the formula for reducing a surface integral to a double integral (see [3, p. 276]), we obtain:

$$\left|R_2\left(x',x''\right)\right|\le M\int_{\Omega_{h/2}(x')}\frac{\left\|f\right\|_3+\omega\left(f,\left|x'-y\right|\right)}{\left|x'-y\right|^{2-\alpha}}d\Omega_y=$$

$$=M\int_{\Pi_{h/2}(x')}\frac{\left\|f\right\|_3+\omega\left(f,\sqrt{u^2+v^2}\right)}{\left(\sqrt{u^2+v^2}\right)^{2-\alpha}}\times$$

$$\times\sqrt{1+\left(\frac{\partial\psi\left(u,\,v\right)}{\partial u}\right)^{2}+\left(\frac{\partial\psi\left(u,\,v\right)}{\partial v}\right)^{2}}dudv\leq$$

$$\leq M\int_{0}^{2\pi}\int_{0}^{h/2}\frac{\|f\|_{3}+\omega\left(f,t\right)}{t^{2-\alpha}}tdtd\tau\leq M\left(\|f\|_{3}\,h^{\alpha}+\int_{0}^{h}\frac{\omega\left(f,t\right)}{t^{1-\alpha}}dt\right).$$

In a similar manner, it can be shown that

$$\left|R_{3}\left(x',x''\right)\right|\leq M\left(\|f\|_{3}\,h^{\alpha}+\int_{0}^{h}\frac{\omega\left(f,t\right)}{t^{1-\alpha}}dt\right).$$

Since

$$R_{4}\left(x',x''\right)=2\int_{\Omega_{h/2}(x')}\left(\left(n\left(y\right)-n\left(x''\right),f\left(x''\right)\right)\times\right.$$

$$\left.\times\frac{\partial\Phi_{k}\left(x'',\,y\right)}{\partial x_{1}}+f_{1}\left(x''\right)\times\frac{\partial\Phi_{k}\left(x'',\,y\right)}{\partial n\left(x''\right)}\right)d\Omega_{y}+$$

$$+2\int_{\Omega_{h/2}(x')}\left(\left(n\left(y\right)-n\left(x''\right),f\left(y\right)-f\left(x''\right)\right)\frac{\partial\Phi_{k}\left(x'',\,y\right)}{\partial x_{1}}+\right.$$

$$\left.+\left(f_{1}\left(y\right)-f_{1}\left(x''\right)\right)\frac{\partial\Phi_{k}\left(x'',\,y\right)}{\partial n\left(x''\right)}\right)d\Omega_{y},$$

then, taking into account inequalities (3), (4), and

$$h/2\leq\left|y-x''\right|\leq 3h/2,y\in\Omega_{h/2}\left(x'\right),$$

we have:

$$\left|R_{4}\left(x',x''\right)\right|\leq M\int_{\Omega_{h/2}(x')}\frac{\|f\|_{3}+\omega\left(f,|y-x''|\right)}{|y-x''|^{2-\alpha}}d\Omega_{y}\leq$$

$$\leq Mh^{\alpha-2}\left(\|f\|_{3}+\omega\left(f,3h/2\right)\right)mes\Omega_{h/2}\left(x'\right)\leq Mh^{\alpha}\left(\|f\|_{3}+\omega\left(f,h\right)\right).$$

As a result, taking into account the inequality

$$\int_{0}^{h}\frac{\omega\left(f,t\right)}{t^{1-\alpha}}dt=\int_{0}^{h}\frac{\omega\left(f,t\right)}{t}t^{\alpha}dt\geq\frac{\omega\left(f,h\right)}{h}\int_{0}^{h}t^{\alpha}dt=$$

$$=\frac{\omega\left(f,h\right)}{h}\frac{h^{1+\alpha}}{1+\alpha}=\frac{1}{1+\alpha}\omega\left(f,h\right)h^{\alpha},$$

we obtain

$$\left|R_{4}\left(x',x''\right)\right|\leq M\left(\|f\|_{3}\,h^{\alpha}+\int_{0}^{h}\frac{\omega\left(f,t\right)}{t^{1-\alpha}}dt\right).$$

It can also be proven that

$$\left|R_5\left(x', x''\right)\right| \leq M\left(\|f\|_3 h^\alpha + \int_0^h \frac{\omega(f,t)}{t^{1-\alpha}} dt\right).$$

Let us estimate the expression $R_6(x', x'')$. Since

$$R_6\left(x', x''\right) = 2\left(n\left(x'\right) - n\left(x''\right), f\left(x'\right)\right) \int\limits_{\Omega_d(x')\backslash\left(\Omega_{h/2}(x')\bigcup\Omega_{h/2}(x'')\right)} \frac{\partial\Phi_k\left(x', y\right)}{\partial x_1} d\Omega_y +$$

$$+2 \int\limits_{\Omega_d(x')\backslash\left(\Omega_{h/2}(x')\bigcup\Omega_{h/2}(x'')\right)} \left(n\left(x'\right) - n\left(x''\right), f(y) - f\left(x'\right)\right) \frac{\partial\Phi_k\left(x', y\right)}{\partial x_1} d\Omega_y,$$

then

$$\left|R_6\left(x', x''\right)\right| \leq$$

$$\leq Mh^\alpha \left(\|f\|_3 \int\limits_{\Omega_d(x')\backslash\Omega_{h/2}(x')} \frac{d\Omega_y}{\left|x'-y\right|^2} + \int\limits_{\Omega_d(x')\backslash\Omega_{h/2}(x')} \frac{\omega\left(f, \left|x'-y\right|\right) d\Omega_y}{\left|x'-y\right|^2}\right) \leq$$

$$\leq Mh^\alpha \left(\|f\|_3 \int_h^d \frac{dt}{t} + \int_h^d \frac{\omega(f,t)\, dt}{t}\right) \leq$$

$$\leq M\left(\|f\|_3 h^\alpha \left|\ln h\right| + h^\alpha \int_h^d \frac{\omega(f,t)\, dt}{t}\right).$$

Now let us estimate the expression $R_7(x', x'')$. It is clear that

$$\frac{\partial\Phi_k\left(x', y\right)}{\partial n\left(x'\right)} - \frac{\partial\Phi_k\left(x'', y\right)}{\partial n\left(x''\right)} =$$

$$= \frac{\left(ik\left|x'-y\right| - ik\left|x''-y\right|\right)\left(x'-y, n\left(x'\right)\right)\exp\left(ik\left|x'-y\right|\right)}{4\pi\left|x'-y\right|^3} +$$

$$+ \frac{\left(ik\left|x''-y\right| - 1\right)\left(\left(x'-y, n\left(x'\right)\right) - \left(x''-y, n\left(x''\right)\right)\right)\exp\left(ik\left|x'-y\right|\right)}{4\pi\left|x'-y\right|^3} +$$

$$+ \frac{\left(ik\left|x''-y\right| - 1\right)\left(x''-y, n\left(x''\right)\right)\left(\exp\left(ik\left|x'-y\right|\right) - \exp\left(ik\left|x''-y\right|\right)\right)}{4\pi\left|x'-y\right|^3} +$$

$$+ \left(\frac{1}{4\pi\left|x'-y\right|^3} - \frac{1}{4\pi\left|x''-y\right|^3}\right) \times$$

$$\times \left(ik\left|x''-y\right| - 1\right)\left(x''-y, n\left(x''\right)\right)\exp\left(ik\left|x''-y\right|\right).$$

Taking into account the inequalities

$$\left|x''-y\right|\geq\frac{1}{2}\left|x'-x''\right|,y\in\Omega_d\left(x'\right)\setminus\left(\Omega_{h/2}\left(x'\right)\cup\Omega_{h/2}\left(x''\right)\right),$$

and

$$\left|x'-y\right|\geq\frac{1}{2}\left|x'-x''\right|,y\in\Omega_d\left(x'\right)\setminus\left(\Omega_{h/2}\left(x'\right)\cup\Omega_{h/2}\left(x''\right)\right),$$

we obtain the validity of the following inequalities:

$$\left|x'-y\right|\leq\left|x'-x''\right|+\left|x''-y\right|\leq 3\left|x''-y\right|,y\in\Omega_d\left(x'\right)\setminus\left(\Omega_{h/2}\left(x'\right)\cup\Omega_{h/2}\left(x''\right)\right),$$

and

$$\left|x''-y\right|\leq 3\left|x'-y\right|,y\in\Omega_d\left(x'\right)\setminus\left(\Omega_{h/2}\left(x'\right)\cup\Omega_{h/2}\left(x''\right)\right).$$

Moreover, from inequalities (3) and (4), we obtain that

$$\left|\left(x'-y,n\left(x'\right)\right)-\left(x''-y,n\left(x''\right)\right)\right|\leq$$

$$\leq\left|\left(x'-y,n\left(x'\right)-n\left(x''\right)\right)\right|+\left|\left(x'-x'',n\left(x''\right)\right)\right|\leq Mh^{\alpha}\left(\left|x'-y\right|+h\right).$$

Then

$$\left|\frac{\partial\Phi_k\left(x',\,y\right)}{\partial n\left(x'\right)}-\frac{\partial\Phi_k\left(x'',\,y\right)}{\partial n\left(x''\right)}\right|\leq M\left(\frac{h}{\left|x'-y\right|^{3-\alpha}}+\frac{h^{\alpha}}{\left|x'-y\right|^{2}}\right).\tag{5}$$

Therefore, representing the expression $R_7\left(x',x''\right)$ in the form

$$R_7\left(x',x''\right)=2f_1\left(x'\right)\int\limits_{\Omega_d(x')\setminus\left(\Omega_{h/2}(x')\bigcup\Omega_{h/2}(x'')\right)}\left(\frac{\partial\Phi_k\left(x'',\,y\right)}{\partial n\left(x''\right)}-\frac{\partial\Phi_k\left(x',\,y\right)}{\partial n\left(x'\right)}\right)d\Omega_y+$$

$$+2\int\limits_{\Omega_d(x')\setminus\left(\Omega_{h/2}(x')\bigcup\Omega_{h/2}(x'')\right)}\left(\frac{\partial\Phi_k\left(x'',\,y\right)}{\partial n\left(x''\right)}-\frac{\partial\Phi_k\left(x',\,y\right)}{\partial n\left(x'\right)}\right)\left(f_1\left(y\right)-f_1\left(x'\right)\right)d\Omega_y,$$

we obtain:

$$\left|R_7\left(x',x''\right)\right|\leq M\left\|f\right\|_3\int\limits_{\Omega_d(x')\setminus\Omega_{h/2}(x')}\left(\frac{h}{\left|x'-y\right|^{3-\alpha}}+\frac{h^{\alpha}}{\left|x'-y\right|^{2}}\right)d\Omega_y+$$

$$+M\int\limits_{\Omega_d(x')\setminus\Omega_{h/2}(x')}\left(\frac{h}{\left|x'-y\right|^{3-\alpha}}+\frac{h^{\alpha}}{\left|x'-y\right|^{2}}\right)\omega\left(f,\left|x'-y\right|\right)d\Omega_y\leq$$

$$\leq M\left(\left\|f\right\|_3\int_h^d\left(\frac{h}{t^{2-\alpha}}+\frac{h^{\alpha}}{t}\right)dt+\int_h^d\left(\frac{h}{t^{2-\alpha}}+\frac{h^{\alpha}}{t}\right)\omega\left(f,t\right)dt\right)\leq$$

$$\leq M\left(\|f\|_3\, h^{\alpha}\,|\ln h| + h\int_h^d \frac{\omega(f,t)}{t^{2-\alpha}}dt\right).$$

Proceeding in exactly the same way as in the proof of inequality (5), it can be shown that

$$\left|\frac{\partial\Phi_k(x',\,y)}{\partial x_1} - \frac{\partial\Phi_k(x'',\,y)}{\partial x_1}\right| \leq M\frac{h}{|x'-y|^3}.$$

Then, taking into account inequality (3), we obtain that

$$\left|R_8\left(x',x''\right)\right| \leq M\,\|f\|_3\,h \int\limits_{\Omega_d(x')\setminus\left(\Omega_{h/2}(x')\bigcup\Omega_{h/2}(x'')\right)} \frac{d\Omega_y}{|x'-y|^{3-\alpha}} \leq M\,\|f\|_3\,h^{\alpha}.$$

As a result, summing the obtained estimates for the expressions $R_m(x',x'')$, $m=\overline{1,8}$, and taking into account the

$$h\int_h^d \frac{\omega(f,t)}{t^{2-\alpha}}dt = h\int_h^d \frac{1}{t^{1-\alpha}}\frac{\omega(f,t)}{t}dt \leq$$

$$\leq h\frac{1}{h^{1-\alpha}}\int_h^d \frac{\omega(f,t)}{t}dt = h^{\alpha}\int_h^d \frac{\omega(f,t)}{t}dt,$$

we find:

$$\left|(A_1f)\left(x'\right) - (A_1f)\left(x''\right)\right| \leq$$

$$\leq M\left(\|f\|_3\,h^{\alpha}\,|\ln h| + \int_0^h \frac{\omega(f,t)}{t^{1-\alpha}}dt + h^{\alpha}\int_h^d \frac{\omega(f,t)}{t}dt\right).$$

It is not difficult to show that similar estimates also hold for the expressions $(A_2f)(x)$ and $(A_3f)(x)$. This completes the proof of the theorem. ◄

Theorem 2. *Let $\Omega\subset R^3$ be a Lyapunov surface with exponent $0<\alpha\leq 1$ and $f\in C^3_{\perp}(\Omega)$. Then $Af\in C^3_{\perp}(\Omega)$ and*

$$\omega(Af,h) \leq M_f\left(h^{\alpha}\,|\ln h| + \int_0^h \frac{\omega(f,t)}{t^{1-\alpha}}dt + h^{\alpha}\int_h^{diam\Omega} \frac{\omega(f,t)}{t}dt\right),$$

where $h\in(0, diam\,\Omega]$ and M_f is a positive constant depending only on Ω, k and f.

Proof. The continuity of the function $(Af)(x)$ on Ω follows directly from Theorem 1. Moreover, it is known that for any function $g\in C^3(\Omega)$, the following relation holds:

$$(n(x)\,,[n(x)\,,g(x)]) = 0,\; x\in\Omega.$$

Then

$$(n(x),(Af)(x)) = 2\int_{\Omega}(n(x),[n(x),rot_x\{\Phi_k(x,y)f(y)\}])\,d\Omega_y = 0,$$

i.e., $Af \in C_{\perp}^{3}(\Omega)$. Let

$$Z_0(h) = h^{\alpha}|\ln h| + \int_0^h \frac{\omega(f,t)}{t^{1-\alpha}}dt + h^{\alpha}\int_h^{diam\Omega}\frac{\omega(f,t)}{t}dt.$$

Applying L'Hôpital's rule, we obtain:

$$\lim_{h\to 0} h^{\alpha}\int_h^{diam\Omega}\frac{\omega(f,t)}{t}dt = \lim_{h\to 0}\frac{\int_h^{diam\Omega}\frac{\omega(f,t)}{t}dt}{h^{-\alpha}} =$$

$$= \lim_{h\to 0}\frac{-\frac{\omega(f,h)}{h}}{-\alpha\, h^{-\alpha-1}} = -\frac{1}{\alpha}\lim_{h\to 0} h^{\alpha}\omega(f,h) = 0,$$

and therefore $\lim\limits_{h\to 0} Z_0(h) = 0$. Since the derivative of the function $Z_0(h)$ is non-negative, i.e., the function $Z_0(h)$ is non-decreasing, then from Theorem 1 we obtain that

$$\bar{\omega}(Af,h) = \max_{\substack{|x'-x''|\le h,\\ x',x''\in\Omega}}\left|(Af)(x') - (Af)(x'')\right| \le$$

$$\le M_f \max_{\substack{|x'-x''|\le h,\\ x',x''\in\Omega}} Z_0\left(\left|x'-x''\right|\right) = M_f Z_0(h),\, h\in(0,d/2].$$

Further, taking into account that the derivative of the function $Z_0(h)/h$ is non-positive, i.e., the function $Z_0(h)/h$ is non-increasing, we have:

$$\omega(Af,h) = h\sup_{\tau\ge h}\frac{\bar{\omega}(Af,\tau)}{\tau} \le h\sup_{\tau\ge h}\frac{M_f Z_0(\tau)}{\tau} =$$

$$= h\frac{M_f Z_0(h)}{h} = M_f Z_0(h),\quad h\in(0,d/2].$$

It is obvious that this estimate also holds for all $h\in(0,diam\Omega]$. The theorem is proved. ◄

Theorem 3. *Let $\Omega\subset R^3$ be a Lyapunov surface with exponent $0<\alpha\le 1$ and $f\in C_{\perp}^{3}(\Omega)$. Then $A^{*}f\in C_{\perp}^{3}(\Omega)$ and*

$$\omega(A^{*}f,h) \le M_f\left(h^{\alpha}|\ln h| + \int_0^h\frac{\omega(f,t)}{t^{1-\alpha}}dt + h^{\alpha}\int_h^{diam\Omega}\frac{\omega(f,t)}{t}dt\right),$$

where $h\in(0,diam\Omega]$ and M_f is a positive constant depending only on Ω, k and f.

Proof. Let us take any points $x',\ x'' \in \Omega$ such that the quantity $h = |x' - x''|$ would be less than $d/2$. It is known that for any vector function $g \in C^3(\Omega)$ the following relation holds:

$$\left[n\left(x'\right), g\left(x'\right)\right] - \left[n\ \left(x''\right), g\ \left(x''\right)\right] =$$
$$= \left[n\left(x'\right) - n\ \left(x''\right), g\left(x'\right)\right] + \left[n\ \left(x''\right), g\left(x'\right) - g\ \left(x''\right)\right].$$

Then, taking into account inequality (3), we obtain that

$$\omega\left(\left[n, f\right], h\right) \leq M\left(\|f\|_3 h^\alpha + \omega\left(f, h\right)\right).$$

Moreover, taking into account Theorem 1, we obtain:

$$\left|\left(A^* f\right)\left(x'\right) - \left(A^* f\right)\left(x''\right)\right| = \left|\left[n\left(x'\right), \left(A\left[n, f\right]\right)\left(x'\right)\right] - \left[n\left(x''\right), \left(A\left[n, f\right]\right)\left(x''\right)\right]\right| =$$
$$= \left|\left[n\left(x'\right) - n\left(x''\right), \left(A\left[n, f\right]\right)\left(x'\right)\right] + \left[n\left(x''\right), \left(A\left[n, f\right]\right)\left(x'\right) - \left(A\left[n, f\right]\right)\left(x''\right)\right]\right| \leq$$
$$\leq M\left(h^\alpha \left|\left(A\left[n, f\right]\right)\left(x'\right)\right| + \left|\left(A\left[n, f\right]\right)\left(x'\right) - \left(A\left[n, f\right]\right)\left(x''\right)\right|\right) \leq$$
$$\leq M\left(h^\alpha \|A\| \|f\|_3 + \|[n, f]\|_3 h^\alpha |\ln h| + \right.$$
$$\left. + \int_0^h \frac{\omega\left(\left[n, f\right], t\right)}{t^{1-\alpha}} dt + h^\alpha \int_h^{diam\Omega} \frac{\omega\left(\left[n, f\right], t\right)}{t} dt\right) \leq$$
$$\leq M\left(\|f\|_3 h^\alpha |\ln h| + \int_0^h \frac{\omega\left(f, t\right)}{t^{1-\alpha}} dt + h^\alpha \int_h^{diam\Omega} \frac{\omega\left(f, t\right)}{t} dt\right).$$

As a result, proceeding in exactly the same way as in the proof of Theorem 2, we establish the validity of the theorem. ◄

3. Some properties of the operators generated by the vector potentials (1) and (2)

Let us introduce the following class of functions defined on $(0, diam\Omega]$:

$$J\left(\Omega\right) = \left\{\varphi:\ \ \varphi\uparrow,\ \lim_{h\to 0}\varphi\left(h\right) = 0,\ \varphi\left(h\right)/h\downarrow\right\}$$

and consider the function

$$Z\left(h, \varphi\right) = h^\alpha |\ln h| + \int_0^h \frac{\varphi\left(t\right)}{t^{1-\alpha}} dt + h^\alpha \int_h^{diam\Omega} \frac{\varphi\left(t\right)}{t} dt,$$

where the symbol $\uparrow$ denotes non-decreasing functions, and the symbol $\downarrow$ denotes non-increasing functions. Where no confusion may arise, we will sometimes write

$Z(h)$ and $Z(\varphi)$ instead of the full notation $Z(h,\varphi)$. In the proof of Theorem 2, it was shown that $Z \in J(\Omega)$.

Let $\varphi \in J(\Omega)$. Denote by $H^3(\varphi)$ the linear space of all vector functions $f \in C^3(\Omega)$, satisfying the condition

$$|f(x) - f(y)| \le M_f \varphi(|x-y|), x, y \in \Omega,$$

where M_f is a positive constant depending on f, but not on the points x and y. It is known (see [5, p. 60]) that the space $H^3(\varphi)$ is a Banach space with the norm

$$\|f\|_{H(\varphi)} = \|f\|_3 + \sup_{\substack{x,y\in\Omega,\\ x\neq y}} \frac{|f(x) - f(y)|}{\varphi(|x-y|)}. \tag{6}$$

It is evident that the space

$$H^3_\perp(\varphi) = \left\{ f \in H^3(\varphi) \mid (f(x), n(x)) = 0, \;\; x \in \Omega \right\},$$

is also a Banach space with the norm (6).

The following theorem follows from Theorems 2 and 3:

Theorem 4. *Let $\varphi \in J(\Omega)$. Then the operators A and A^* act boundedly from the space $H^3_\perp(\varphi)$ to the space $H^3_\perp(Z(\varphi))$.*

Let $\varphi_1, \varphi_2 \in J(\Omega)$. It is known (see [5, p. 70]) that if $\varphi_1(h) = o(\varphi_2(h))$, $h \to 0$, then the space $H^3(\varphi_1)$ is compactly embedded in the space $H^3(\varphi_2)$. Then the following holds:

Theorem 5. *Let $\varphi \in J(\Omega)$ and $Z(h,\varphi) = o(\varphi(h))$, $h \to 0$. Then the operators $A : H^3_\perp(\varphi) \to H^3_\perp(\varphi)$ and $A^* : H^3_\perp(\varphi) \to H^3_\perp(\varphi)$ are compact.*

Let us denote by $H^3_\beta(\Omega)$ the Hölder space with exponent $0 < \beta \le 1$, i.e., the space $H^3_\beta(\Omega)$ is the space $H^3(\varphi)$ for $\varphi(h) = h^\beta$. Moreover, let us assume that

$$H^3_{\perp,\beta}(\Omega) = \left\{ f \in H^3_\beta(\Omega) \mid (f(x), n(x)) = 0, \;\; x \in \Omega \right\}.$$

It is obvious that if $f \in H^3_{\perp,\beta}(\Omega)$, then $Z_0(h) = h^\alpha |\ln h|$.

From Theorems 4 and 5, we obtain:

Corollary 1. *The operators A and A^* act boundedly from the space $H^3_{\perp,\beta}(\Omega)$ to the space $H^3_{\perp,\alpha}(\Omega)$. Moreover, if $0 < \beta < \alpha$, then the operators A and A^* are compact in the space $H^3_{\perp,\beta}(\Omega)$.*

It should be noted that, in particular, if $\alpha = 1$, then from Corollary 1 we obtain the result from [2, p. 74] concerning the compactness of the operators A and A^* in the space $H^3_{\perp,\beta}(\Omega)$.

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