Azerbaijan Journal of Mathematics
V. 16, No 2, 2026, July
ISSN 2218-6816
https://doi.org/10.59849/2218-6816.2026.2.195

On the Best Approximation of Bivariate Functions by Sums of Coordinate Functions on r-Convex Sets

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Abstract. Let $\mathcal{S} \subset \mathbb{R}^2$ be a compact set and $f : \mathcal{S} \to \mathbb{R}$ be a continuous function. We consider the problem of finding the best approximation error of $f(x, y)$ by functions of the form $\varphi(x) + \psi(y)$, where $\varphi, \psi \in C(\mathbb{R})$. In other words, the problem asks for the value of

$$E(f, \mathcal{S}) = \inf_{\varphi,\psi\in C(\mathbb{R})} \|f(x, y) - \varphi(x) - \psi(y)\|_{C(\mathcal{S})}.$$

In special cases, when $\mathcal{S}$ is an axis-aligned rectangle or a stairlike polygon, and for functions f with $\Delta_x\Delta_y f(u, v) \geq 0$ for any rectangle $[u; u + x] \times [v; v + y] \subset [0, 1]^2$, it is proven that the error can be calculated as a supremum of alternating functionals defined on the maximal polygons on $\mathcal{S}$. In this paper, we prove this result for a wider class of sets.

Key Words and Phrases: best approximation error, r-convex domains, maximal polygons.

2010 Mathematics Subject Classifications: 41A50, 41A30

1. Introduction

Let $\mathcal{S} \subset \mathbb{R}^2$ be a compact set and $f : \mathcal{S} \to \mathbb{R}$ be a continuous function. Consider the problem of approximating f by the sum of one-variable coordinate functions. It is a classic problem to evaluate how close the sum of two coordinate functions can get to the original function. In other words, the problem asks for the value

$$E(f, \mathcal{S}) = \inf_{\varphi,\psi\in C(\mathbb{R})} \|f(x, y) - \varphi(x) - \psi(y)\|_{C(\mathcal{S})}, \tag{1}$$

where the infimum runs over all pairs of functions $\varphi, \psi \in C(\mathbb{R})$.

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The problem (1) was first studied by S. P. Diliberto and E. G. Straus [9] in the case of $\mathcal{S} = [0,1]^2$ (the problem was proposed by RAND Corporation for the most efficient distribution of data in the small memory of the computers of the day). The same problem was investigated by Yu. P. Ofman [16] for bounded bivariant functions, and for more general sets by D. E. Marshall and A. G. O'Farrell in [15]. In [9], Diliberto and Straus gave a "leveling algorithm" to find the approximation error (1) in case $\mathcal{S} = [0,1]^2$. The addressed problem, together with the methods developed from its analysis, finds important applications in approximation by subalgebras and, in particular, in RBF approximation, ridge function approximation, and neural network models. Recent papers (see [1, 4, 5, 6, 7, 11]) improve the theory in connection with these directions. Moreover, based on the "levelling" algorithm, the ADI (alternating-direction-implicit) iteration method for numerical solution of differential equations is developed (see [18, 19, 20]).

The first precise formula to calculate the exact value (1) was given by T.J. Rivlin and R.J. Sibner [17]. They established under the condition $f_{xy} \geq 0$ that

$$E(f,[0,1]^2) = \frac{1}{4}\big(f(1,1) - f(1,0) - f(0,1) + f(0,0)\big). \tag{2}$$

Later, L. Flatto gave a different proof of this formula in his work [10], where a generalization to the approximation of an n variable function by $n-1$ variable functions was suggested under similar conditions. In [8], (2) is proven for functions f satisfying the condition

$$\Delta_x \Delta_y f(u,v) = f(u+x, v+y) - f(u+x, v) - f(u, y+v) + f(u,v) \geq 0,$$

for any rectangle $[u; u+x] \times [v; v+y] \subset [0,1]^2$. Recently, the paper [2] using results of [13] proved a formula to calculate the approximation error of the continuous three-variable functions satisfying similar conditions by sums of three coordinate functions in the set $[0,1]^3$.

In [15], D. E. Marshall and A. G. O'Farrell proved a duality formula(see Theorem 4) that expresses the error as a supremum of alternating functionals (see (8)) defined on lightning bolts (introduced by Arnold [3]) on $\mathcal{S}$. (2) is a straightforward corollary of this result. Later, V. Ismailov [12], using the results of [15], proved a more general theorem. We call a polygon P a stairlike polygon if $P = P_1 P_2 ... P_{2n}$ is a simple polygon with vertices

$$P_1 = (x_1, y_1), ..., P_{2i-1} = (x_i, y_i), P_{2i} = (x_{i+1}, y_i), ..., P_{2n} = (x_{n+1}, y_n), \tag{3}$$

for some real numbers, $x_{n+1} = x_1 < x_2 < ... < x_n$ and $y_1 > y_2 > ... > y_n$. The number of vertices of P is denoted by $|P|$.

Definition 1. *[12, Definition 2] An axis-aligned simple polygon $P \subset \mathcal{S}$ is called a* ***maximal polygon*** *on the set $\mathcal{S}$ if there are no axis-aligned polygon $Q \neq P$ with $P \subset Q \subset \mathcal{S}$ and $|P| \geq |Q|$.*

Theorem 1. *[12] For any stairlike polygon P and function f satisfying $\Delta_x \Delta_y f \geq 0$, the error $E(f, P)$ can be calculated as the maximum of alternating functionals defined on the maximal polygons on P.*

Since the only maximal polygon on an axis-aligned rectangle is the rectangle itself, this result is a natural generalization of the formula (2). However, this result does not cover all figures for which a similar theorem could be stated.

In this paper, we relax the conditions on $\mathcal{S}$ such that the formula obtained in [12] still applies.

2. Main Result

Definition 2. *A set $\mathcal{S} \subset \mathbb{R}^2$ is called* ***r-convex*** *(or convex with respect to coordinate axes) if for any two points $A, B \in \mathcal{S}$ such that the line AB is parallel to one of the coordinate axes, the line segment AB lies on $\mathcal{S}$.*

It is trivial that any convex set in $\mathbb{R}^2$ is an r-convex set. On the other hand, the stairlike polygons defined as (3) are r-convex too.

Definition 3. *Suppose $\mathcal{S}$ is a compact set. Let y_0 denote the smallest value of the projection of $\mathcal{S}$ onto the OY-axis. If $\mathcal{S}$ is an r-convex set and the equation*

$$\operatorname{Pr}_x \mathcal{S} = \{x | (x, y_0) \in \mathcal{S}\},$$

holds, then we call S a lower-extremal r-convex set, where Pr_x is the projection operator onto the Ox axes. We call the line segment $\{(x, y_0) | (x, y_0) \in \mathcal{S}\}$ the sharp edge of $\mathcal{S}$. Similarly, we can define upper-, right-, and left-extremal r-convex sets and their sharp edges. Together, we call them outer-extremal r-convex sets.

We denote by $M(\mathcal{S})$ the set of functions f such that

$$\Delta_x \Delta_y f(u, v) = f(u + x, v + y) - f(u + x, v) - f(u, v + y) + f(u, v) \geq 0,$$

for any rectangle $[u; u + x] \times [v; v + y] \subset S$.

The following theorem is a direct generalization of the results in [17] and [12].

Theorem 2. *For any outer-extremal r-convex finitely bolt-connected compact set $\mathcal{S} \subset \mathbb{R}^2$ and any function $f \in M(\mathcal{S})$ the approximation error can be calculated by the formula*

$$E(f, \mathcal{S}) = \sup_{b \in [\mathcal{S}]_p^{max}} |r(f, b)|. \tag{4}$$

The terms "finitely bolt-connected", $r(f,b)$, and $[S]_p^{max}$ encountered in Theorem 2 will be defined in Section 3.

3. Duality relations

The following well-known duality formula is a corollary of the Hahn-Banach Theorem and is widely used in problems of approximation by a linear subspace.

Theorem 3. *Let E be a Banach space and G be a linear subspace of E. Then for any $f \in E$*

$$\inf_{g \in G} \|f - g\| = \sup_{l \in G_\perp} \frac{|l(f)|}{\|l\|}, \tag{5}$$

where $G_\perp$ denotes the set of all linear functionals that are orthogonal to all elements of G.

In our case, E becomes the set $C(\mathcal{S})$ and $G = \{\varphi(x) + \psi(y), \varphi, \psi \in C(\mathbb{R})\}$. Various improvements on this formula were made by narrowing $G_\perp$ to its dense subsets. These relations rely on considering elements of $G_\perp$ as associated measures, and showing that the subset of measures with a finite support is dense on it. On the other hand, extremal elements among measures with a finite support are the ones supported on lightning bolts. Moreover, any measure supported on a lightning bolt is a convex combination of alternating measures on minimal lightning bolts. Below, we present the definitions and results in more detail.

Definition 4. *We call a finite or infinite ordered set $b = (b_1, b_2, b_3, \dots)$ of $\mathbb{R}^2$ with $b_i \neq b_j$ for $1 \le i < j \le n$ a* ***lightning bolt*** *if*

$$\mathrm{Pr}_x(b_{2k-1}) = \mathrm{Pr}_x(b_{2k}), \mathrm{Pr}_y(b_{2k}) = \mathrm{Pr}_y(b_{2k+1}) \quad \forall k = 1, 2, 3, ... \tag{6}$$

or

$$\mathrm{Pr}_y(b_{2k-1}) = \mathrm{Pr}_y(b_{2k}), \mathrm{Pr}_x(b_{2k}) = \mathrm{Pr}_x(b_{2k+1}) \quad \forall k = 1, 2, 3, ... \tag{7}$$

where Pr_x and Pr_y are projection operators onto the axes Ox and Oy, respectively. A finite lightning bolt $b = (b_1, b_2, b_3, \dots, b_n)$ is called ***closed*** *if $b' = (b_1, b_2, b_3, \dots, b_n, b_1)$ is a lightning bolt.*

Let us denote the set of all finite lightning bolts on the set $\mathcal{S}$ by $[\mathcal{S}]$ and the set of all closed lightning bolts on the set $\mathcal{S}$ by $[\mathcal{S}]^{cl.}$. When b is finite, the number of points in b is called the **length** of b and is denoted by $|b|$. It is clear that for any closed lightning bolt b, the length $|b|$ is even.

For any finite lightning bolt $b = (b_1, b_2, ..., b_n)$ we define the alternating functional

$$r(f, b) = \frac{\sum_{i=1}^{n} (-1)^i f(b_i)}{n}. \tag{8}$$

It is easy to prove that, for any lightning bolt b, $r(f, b) \in G_\perp$ and $\|r\| = 1$. Note that the same set of points $B \in \mathbb{R}^2$ could form multiple lightning bolts b. However, the value of $|r(f, b)|$ stays the same for all these lightning bolts. We denote this value by $r(f, B)$. Moreover, we define the functional

$$r^*(f, P) := |P| r(f, P),$$

for each polygon P, where $|P|$ denotes the number of sides of P.

The simplest examples of closed lightning bolts are simple polygons for which any edge is parallel to one of the coordinate axes. We call such a polygon an axis-aligned simple polygon. Throughout the paper, we will use simple polygons in two different meanings. Whenever they are treated as lightning bolts, we understand a lightning bolt made up from the vertices of the polygon while keeping the original order (with an arbitrary start). However, when they are treated as sets (especially when we are talking about inclusion, union, and intersection of polygons), we use them in the usual sense, that is, the set of plane points bounded by the closed piecewise smooth curve that the edges of the polygon constitute. Let $[\mathcal{S}]_p$ and $[\mathcal{S}]_p^{max}$ denote the set of all axis-aligned simple polygons and the set of all maximal polygons on $\mathcal{S}$, respectively.

The following theorem is due to Marshall and O'Farrell.

Theorem 4. *[15, Corollary 4] Let $\mathcal{S} \subset \mathbb{R}^2$ be a compact set. For any $f \in C(D)$ we have*

$$E(f, D) = \sup_{b \in [\mathcal{S}]} |r(f, b)|. \tag{9}$$

Definition 5. *A set $\mathcal{S} \subset \mathbb{R}^2$ is called* ***finitely bolt-connected*** *if there exists a positive integer N depending only on $\mathcal{S}$ such that, for any two points of $\mathcal{S}$, there exists a lightning bolt connecting them with a length not larger than N.*

The following corollary is a more ready-to-use form of (9) for finitely bolt-connected sets.

Corollary 1. *[14, Chapter 2, Corollary 4.27] Let $\mathcal{S} \subset \mathbb{R}^2$ be a finitely bolt-connected compact set. For any function $f \in C(\mathcal{S})$ we have*

$$E(f, \mathcal{S}) = \sup_{b \in [\mathcal{S}]^{cl}} |r(f, b)|. \tag{10}$$

A closed lightning bolt is called minimal if it has no proper subset that constitutes a closed lightning bolt. It is easy to see that a closed lightning bolt is minimal if and only if no three points of it lie on the same line parallel to a coordinate axis. Let us denote the set of all minimal lightning bolts on $\mathcal{S}$ by $[\mathcal{S}]^{min}$. The following results show that it is enough to consider minimal lightning bolts to find the approximation error.

Proposition 1. *[14, Chapter 2, Proposition 4.22] If a closed lightning bolt b is not minimal, then for any function $f \in C(\mathcal{S})$ there exists a minimal lightning bolt b^1 such that*

1. *All points of b^1 are among the points of b,*
2. *$r(f, b^1) \geq r(f, b)$.*

The following result is now a corollary of Corollary 1 and Proposition 1.

Corollary 2. *Under the conditions of Corollary 1, we have*

$$E(f, \mathcal{S}) = \sup_{b \in [\mathcal{S}]^{min}} |r(f, b)|. \tag{11}$$

Remark 1. *The corollary 2 is actually the best result that can be obtained in this direction. Indeed, for any closed minimal lightning bolt $b = (b_1, b_2, ..., b_{2n})$, there exist a finitely bolt-connected compact set $\mathcal{S}_b \in \mathbb{R}^2$ and a function $f \in C(\mathcal{S}_b)$ such that, b is the unique lightning bolt in $[\mathcal{S}_b]$ that the supremum* (1) *is attained. Indeed, let*

$$\mathcal{S}_b = \bigcup_{1 \leq i \leq 2n} b_i b_{i+1},$$

where $b_{2n+1} = b_1$ and $b_i b_{i+1}$ denote the edge connecting b_i and b_{i+1}. Obviously, S_b is compact and finitely bolt-connected. Let $f : S_b \to \mathbb{R}$ be defined such that $f(b_i) = (-1)^i$ for any $1 \leq i \leq 2n$ and f is linear on any edge $b_i b_{i+1}$, $1 \leq i \leq 2n$. Clearly, $r(f, b) = 1$ and moreover, for any other closed bolt $b^1 = (b_1^1, b_2^1, ..., b_{2m}^1) \in [\mathcal{S}_b]$ we have

$$|r(f, b^1)| = \left| \frac{\sum_{i=1}^{2m} (-1)^i f(b_i^1)}{2m} \right| < \frac{\sum_{i=1}^{2m} 1}{2m} = 1.$$

The last inequality follows from the fact that $|f(x,y)| \leq 1, (x,y) \in \mathcal{S}_b$ *with equality happening only at vertices of* b *and the fact that* b^1 *has at least one point that is not common with* b. *The motivation of this example comes from [14, Chapter 2, Theorem 2.6].*

4. Proof of The Theorem 2

The formulas in the previous chapter are valid for all continuous functions in a given compact set $\mathcal{S}$. In this section, we will use the geometric constraints to improve them and prove our theorem. To describe the set of lightning bolts for which the supremum can be attained in this special case, we give the following notation and lemmas.

Let $b = (b_1, b_2, ..., b_n) \in [\mathcal{S}]$, and $b_i = (x_i, y_i), b_{i+1} = (x_{i+1}, y_{i+1})$ be its two consecutive vertices. We call the edge $b_i b_{i+1}$ a positive edge if either i is odd and $x_i + y_i > x_{i+1} + y_{i+1}$ or i is even and $x_i + y_i < x_{i+1} + y_{i+1}$. We would call it a negative edge otherwise. In this definition, we consider $b_1 = b_{n+1}$ when necessary.

On the other hand, for any i, either $x_i = x_{i+1}$ or $y_i = y_{i+1}$ holds. Assume $x_i = x_{i+1}$ and denote

$$\begin{aligned}
&\xi_i = \min\big\{x, \{(x, y_i), (x, y_{i+1})\} \subset \mathcal{S}\big\},\\
&\gamma_i = \max\big\{x, \{(x, y_i), (x, y_{i+1})\} \subset \mathcal{S}\big\},\\
&b_i^l = (\xi_i, y_i), b_{i+1}^l = (\xi_i, y_{i+1}),\\
&b_i^r = (\gamma_i, y_i), b_{i+1}^r = (\gamma_i, y_{i+1}).
\end{aligned}$$

We define a left move of an edge $b_i b_{i+1} || Oy$ of b with respect to the edge as replacing the bolt with $b^l(i) = (b_1, ..., b_{i-1}, b_i^l, b_{i+1}^l, ..., b_n)$ and a right move as replacing the bolt with $b^r(i) = (b_1, ..., b_{i-1}, b_i^r, b_{i+1}^r, ..., b_n)$. Similarly, a left(upward) and a right(downward) move can be defined for the case $y_i = y_{i+1}$.

Proposition 2. *For any function* $f \in M(\mathcal{S})$ *and any bolt* $b \in [\mathcal{S}]$

1. *a left move of a positive edge of* b *not decreases* $r(f,b)$,
2. *a right move of a positive edge of* b *not increases* $r(f,b)$,
3. *a left move of a negative edge of* b *not increases* $r(f,b)$,
4. *a right move of a negative edge of* b *not decreases* $r(f,b)$.

Proof. Since the proofs of other parts are analogous to the proof of part (1) we will only prove (1). Thus, without loss of generality, assume that i is odd and $b_i b_{i+1}$ is a positive edge. Moreover, assume that the bolt b is defined via

(6). Then $x = x_{i+1}$ and therefore $y_i > y_{i+1}$. Hence, from $\xi_i < x_i$, $y_{i+1} < y_i$ and $f \in M(\mathcal{S})$ it follows that

$$r(f, b^l) - r(f, b) = f(\xi_i, y_{i+1}) - f(\xi_i, y_i) - f(x_i, y_{i+1}) + f(x_i, y_i) \geq 0.$$

◄

We say b is **vertically expandable** if there exists an index i, such that $b^l(i) \neq b$ and $b^r(i) \neq b$. Similarly, a **horizontally expandable** bolt can be defined via upward and downward moves. We call a lightning bolt **non-expandable** if it is not **vertically expandable** nor **horizontally expandable**. The following is a direct corollary of the previous proposition.

Corollary 3. *The following assertions hold for any function $f \in M(\mathcal{S})$*

1. *For any $b = (b_1, b_2, \dots, b_n) \in [\mathcal{S}]$ and any $1 \leq i \leq n$, either*

$$|r(f, b^r(i))| \geq |r(f, b)|$$

or

$$|r(f, b)| \leq |r(f, b^l(i))|.$$

2. *For any expandable bolt b, there exists a non-expandable bolt b^*, such that b^* is minimal, $|b^*| \leq |b|$ and,*

$$|r(f, b)| \leq |r(f, b^*)|.$$

Moreover, if b is closed, then b^* can be chosen to be closed.

It is clear that at least one of any two consecutive vertices of any non-expandable lightning bolt from $[\mathcal{S}]$ belongs to the boundary of $\mathcal{S}$. The above corollary implies a further improvement of the formulas (9), (10), (11). However, these results are still weaker than the results in [17] and [12]. Indeed, for some sets, non-extensibility does not mean that they are maximal in the polygonal sense.

Lemma 1. *Let P be an axis-aligned polygon in $\mathbb{R}^2$ and $f \in M(P)$ be any function. Let $R_1, R_2, ..., R_s$ be some axis-aligned rectangles such that any two of them have no common interior points and $P = \bigcup_{i=1}^{s} R_i$. Then*

$$r^*(f, P) = \sum_{i=1}^{n} r^*(f, R_i).$$

Proof. It is easy to see that if R is a rectangle and T is an axis-aligned polygon such that R and T have at least an infinite number of common points but have no common interior points (in other words, if their intersection can be shown as a union of a finite number of line segments of their boundaries), then $T \cup R$ is an axis-aligned polygon and

$$r^*(f, T \cup R) = r^*(f, T) + r^*(f, R).$$

The proof ends straightforwardly from here. ◄

Lemma 2. *Let P and Q be two axis-aligned polygons such that $Q \subset P$ and $|Q| \geq |P|$. We have*

$$r(f, P) \geq r(f, Q)$$

for any $f \in M(P)$.

Proof. By drawing a line parallel to each coordinate axis passing through each vertex of P and each vertex of Q, we obtain a division of Q into rectangles. Moreover, this division separates P and $Q \backslash P$. That is $P = R_1 \cup R_2 \cup \cdots \cup R_m$ and $Q = P \cup R_{m+1} \cup \cdots \cup R_s$ for some m, s and some non-overlapping rectangles $R_i, 1 \leq i \leq s$. Thus, by Lemma 1, we have:

$$r^*(f, P) - r^*(f, Q) = \sum_{i=1}^{s} r^*(f, R_i) - \sum_{i=1}^{m} r^*(f, R_i) = \sum_{i=m+1}^{s} r^*(f, R_i) \geq 0,$$

where the last equality follows from the fact that $f \in M(P)$. Now, since $|P| \geq |Q|$ we have

$$|r(f, Q)| = \frac{|r^*(f, Q)|}{|Q|} \geq \frac{|r^*(f, P)|}{|P|} = |r(f, P)|.$$

◄

Lemma 3. *Let $\mathcal{S}$ be an r-convex set and P be axis-aligned simple polygon in $\mathcal{S}$, then there exists a maximal polygon Q in $\mathcal{S}$ (that is $Q \in [\mathcal{S}]_p^{max}$), such that $P \subset Q$ and $|P| \geq |Q|$.*

Proof. Call an edge of the polygon P free if it is expandable in both directions. Note that if an edge of P is free, then either a left or a right move of that edge will still result in a simple polygon but with fewer free edges (edges that can be moved in both directions). Thus after applying a finite number moves we get a simple polygon that each edge is not free any more. If the result Q' is not the desired polygon Q, then, since Q' has no more edges than P, we have that Q' is not maximal, and there should exist another polygon Q'' such that $Q' \subset Q'' \subset \mathcal{S}$

and $|Q''| \leq |Q'|$. But since each edge of Q' is not free, at least one of any two consecutive vertices of Q' belongs to the boundary of $\mathcal{S}$. If any Q_i of the vertices of Q' on boundary of $\mathcal{S}$ is not a vertex of Q'', then it should belong to the interior of Q''. In this case, if at least one of the neighbours of Q_i belong to the interior of Q'' or an edge of it, then the corresponding edge becomes free. Thus, both Q_{i-1} and Q_{i+1} belongs to the set of vertices of Q''.

Thus, Q'' shares at least one of any two consecutive vertices of Q' with it and since $|Q'| \geq |Q''|$ the remaining vertices can have a finite number of possible places, and we can find the maximal construction among them uniquely. ◀

Proof. [Proof of the Theorem 2] Without loss of generality, assume that $\mathcal{S}$ is lower-extremal. Let $b^0 = (b_1, b_2, ..., b_n) \in [\mathcal{S}]^{cl}$. By Corollary 1 our theorem is equivalent to prove that

$$|r(f, b^0)| \leq \sup_{b \in [\mathcal{S}]_p^{max}} |r(f, b)|.$$

Assume the contrary, and moreover, suppose b^0 is a bolt with smallest length not satisfying the inequality. The last assumption also implies minimality of b^0. Let $\gamma = \min\{\mathrm{Pr}_y(b_i)\}$ and $\xi = \max\{\mathrm{Pr}_x(b_i) | \mathrm{Pr}_y(b_i) = \gamma\}$. Without loss of generality, let $b_1 = (\xi, \gamma)$ and also the bolt b^0 satisfies the equations (6). Thus, b_2 is placed on above of b_1 and b_n is placed on left of it. It is clear that, $b_1 b_2$ and $b_n b_1$ are both negative edges and $b_{n-1} b_n$ is a positive edges. We claim that $r(f, b^0) > 0$ and there is no other negative edges parallel to the x-axis. Indeed, if $r(f, b^0) > 0$ and there were two distinct negative edges parallel to x-axis we can apply a right move to both of them and move them to the sharp edge of $\mathcal{S}$. Since $r(f, b^0) > 0$, these moves will not decrease $|r(f, b)|$ by Proposition 2. On the other hand, the resulting bolt becomes non-minimal and by Corollary 2 there exists a minimal bolt b^* satisfying $|b^0| > |b^*|$ and

$$|r(f, b^*)| \geq |r(f, b^0)| > \sup_{b \in [\mathcal{S}]_p^{max}} |r(f, b)|.$$

The last inequalities contradict our assumption on b^0 having smallest possible length. Moreover, if $r(f, b^0) < 0$, we consider two cases:

1. If $\mathrm{Pr}_y(b_2) > \mathrm{Pr}_y(b_{n-1})$ let $b^* = (b_1^* b_2, b_3, ..., b_{n-2})$, where
$$b_1^* = (\mathrm{Pr}_x(b_1), \mathrm{Pr}_y(b_{n-1})),$$
2. If $\mathrm{Pr}_y(b_2) < \mathrm{Pr}_y(b_{n-1})$ let $b^* = (b_3, ..., b_{n-2}, b_{n-1}, b_n^*)$, where
$$b_n^* = (\mathrm{Pr}_x(b_n), \mathrm{Pr}_y(b_2)).$$

In both cases

$$|r(f, b^*)| > |r(f, b^0)|$$

and $|b^*| < |b^0|$. Hence, we get a contradiction in the same way.

Now, since all edges $b_{2k}b_{2k+1}, 1 \leq k \leq \lfloor n-2 \rfloor$ are horizontal by equation (6) and all of them are positive edges we have $\mathrm{Pr}_x(b_{2k}) > \mathrm{Pr}_x(b_{2k+1}) = \mathrm{Pr}_x(b_{2k+2})$.

The last relation implies that the bolt $b^0 \in [\mathcal{S}]_p$. Indeed, the edge $b_1 b_n$ cannot intersect any other edge as it has the smallest coordinate y among all the edges in b^0. On the other hand, for any two edges $b_i b_{i+1}$ and $b_j b_{j+1}$ for some $1 \leq i < i+1 < j < n$ we have

$$\mathrm{Pr}_x(b_i) \geq \mathrm{Pr}_x(b_{i+1}) \geq \mathrm{Pr}_x(b_j) \geq \mathrm{Pr}_x(b_{j+1}),$$

and thus, they cannot intersect.

Hence, by Lemma 3 and Lemma 2 there exists a polygon $|P| \in [\mathcal{S}]_P^{max}$ such that

$$r(f, P) = |r(f, b^P)| \geq |r(f, b^*)|.$$

Here, b^P denotes a closed lightning bolt with the same vertices as P. Therefore,

$$\sup_{b \in [\mathcal{S}]_p^{max}} |r(f, b)| \geq |r(f, b^P)| \geq r(f, b^0).$$

We get a contradiction to our assumption, and the theorem is proven. ◀

We present three examples below, that show why our theorem does not generalize to all figures and explains the motivation for r-convex and outer-extremal sets.

Example 1. *Let $\mathcal{S} = \bigcup_{i=1}^3 R_i$, where*

$$R_1 = [1, 2] \times [1, 4], R_2 = [1, 4] \times [1, 2], R_3 = [3, 4] \times [1, 4].$$

Also, let $f : \mathcal{S} \to \mathbb{R}$ be defined as

$$f(x, y) = \int_0^x \int_0^y g(x, y) dx dy,$$

where g is defined as

$$g(x, y) = \begin{cases} 1 & \textit{if} \quad 1 \leq x \leq 4 \textit{ and } 1 \leq y \leq 3, \\ -1 & \textit{if} \quad 2 \leq x \leq 3 \textit{ and } 3 \leq y \leq 4, \\ 0 & \textit{otherwise.} \end{cases}$$

We have $|r(f,b)| = \frac{3}{2}$ *for the closed bolt* b *with vertices* $b_1 = (1,3), b_2 = (4,3), b_3 = (4,1), b_4 = (1,1)$. *On the other hand, the only maximal polygons in* $\mathcal{S}$ *are of the form*

$$\bigcup_{i=N_1}^{N_2} R_i$$

for some $1 \le N_1 \le N_2 \le 3$. *The respective value of* r *for these polygons is as follows*

$$r(f,R_1) = \frac{1}{2}, r(f,R_2) = \frac{3}{4}, r(f,R_3) = \frac{1}{2},$$
$$r(f,R_1 \cup R_2) = r(f,R_2 \cup R_3) = \frac{2}{3},$$
$$r(f,R_1 \cup R_2 \cup R_3) = \frac{5}{8}.$$

Thus, indeed, the relation (4) *does not hold for this case. In addition, we can show that* $E(f,\mathcal{S}) = \frac{3}{2}$. *Now, suppose we change the definition of a maximal polygon by requiring only the vertices of a polygon belonging to* $\mathcal{S}$ *instead of the area it covers. Consequently, the only maximal polygon becomes* $[1,4] \times [1,4]$. *Since* $r\big(f,[1,4] \times [1,4]\big) = \frac{5}{4}$, *the counter-example still works.*

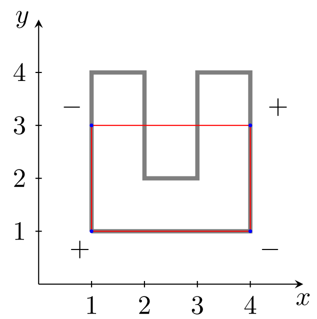

Figure 1: Example 1

A similar counterexample can be found for any set which is not r-convex. The next example provides motivation for considering outer-extremal sets.

Example 2. *Let* $\mathcal{S} = R_1 \cup R_2 \cup R_3$, *where* $R_1 = [1,3] \times [2,4]$, $R_2 = [1,4] \times [2,3]$ *and* $R_3 = [2,3] \times [1,4]$ *and* $f : \mathcal{S} \to \mathbb{R}$ *be defined as in the previous example, with*

$$g(x,y) = \begin{cases} 1, & \text{if} \quad (x,y) \in [1,3] \times [2,4], \\ 2, & \text{if} \quad (x,y) \in \mathcal{S} \backslash [1,3] \times [2,4], \\ 0, & \text{otherwise.} \end{cases}$$

Let b be the bolt with vertices $b_1 = (4, 2), b_2 = (4, 3), b_3 = (2, 3), b_4 = (2, 1), b_5 = (3, 1), b_6 = (3, 4), b_7 = (1, 4), b_8 = (1, 2)$. $|r(f, b)| = \frac{9}{8}$ and again the maximum is obtained on the self-intersecting bolt b but not in any maximal polygons of $\mathcal{S}$. Indeed, we have

$$\begin{aligned} r(f, R_1) &= r(f, R_2) = r(f, R_3) \\ = r(f, R_1 \cup R_2) &= r(f, R_2 \cup R_3) = 1, \\ r(f, R_1 \cup R_2 \cup R_3) &= \frac{8}{10}. \end{aligned}$$

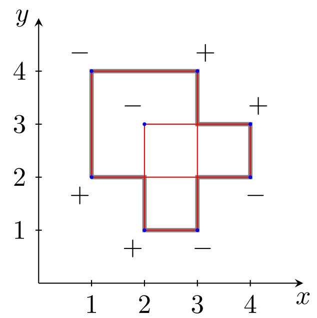

Figure 2: Example 2

In the previous example we have $f \in M(\mathbb{R}^2)$. Hence, the main reason the set does not satisfy the desired result is its geometry, which does not allow us to perform some necessary expansions, and thus, the supremum is obtained on self-intersecting bolts.

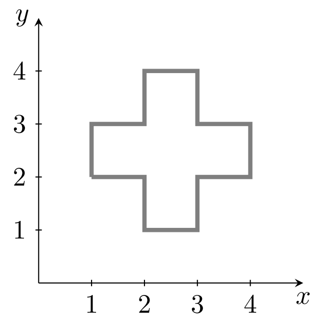

Figure 3: Example 3

Example 3. *Let $\mathcal{S} = R_1 \cup R_2$, where $R_1 = [2, 3] \times [1, 4]$ and $R_2 = [1, 4] \times [2, 3]$. Then, for any $b \in [\mathcal{S}]$ and any $f \in M(\mathcal{S})$ we have*

$$|r(f, b)| \leq \max(r(f, R_1), r(f, R_2), r(f, \mathcal{S})).$$

Since only maximal polygons in $\mathcal{S}$ are $\mathcal{S}, R_1$ and R_2, the relation (4) *holds in this case too. However, the figure is not outer-extremal. Therefore, we conclude that being outer-maximal is not necessary, unlike being r-convex.*

References

[1] Akhmedov, A., Ismailov, V.E. (2025) *A Note on the Problem of Straight-Line Interpolation by Ridge Functions*, Real Anal. Exchange, Advance Publication, 1–11.

[2] Aliev, R.A., Guliyev, V.A., Jabiyev, A.F. (2025) *Best approximation of a three-variable function by sum of one-variable coordinate functions*, arXiv preprint, arXiv:2507.04747.

[3] Arnold, V.I. (1957) *On functions of three variables*, Dokl. Akad. Nauk SSSR, **114**, 679–681. English transl.: Amer. Math. Soc. Transl., **28** (1963), 51–54.

[4] Asgarova, A.Kh. (2025) *Approximation with RBF neural networks using unit smoothing factors and translations*, Adv. Studies: Euro-Tbilisi Math. J., **18(3)**, 79–90.

[5] Asgarova, A.Kh., Huseynli, A.A., Ismailov, V.E. (2023) *A Chebyshev-type alternation theorem for best approximation by a sum of two algebras*, Proc. Edinb. Math. Soc., **66(4)**, 971–978.

[6] Asgarova, A.K., Ismailov, V.E. (2025) *A Note on the Approximation by a Sum of Two Algebras*, Math. Notes, **118(3)**, 442–449.

[7] Asgarova, A.Kh., Maharov, I.K. (2024) *On the error of approximation by RBF neural networks with two hidden nodes*, Proc. Inst. Math. Mech. Natl. Acad. Sci. Azerb., **50(1)**, 152–161.

[8] Babaev, M.-B.A. (1970) *Approximation of polynomials in two variables by functions of the form $\varphi(x) + \psi(y)$*, Dokl. Akad. Nauk SSSR, **193**, 967–969. English transl.: Soviet Math. Dokl., **11**, 1034–1036.

[9] Diliberto, S.P., Straus, E.G. (1951) *On the approximation of a function of several variables by the sum of functions of fewer variables*, Pacific J. Math., **1(2)**, 195–210.

[10] Flatto, L. (1966) *The approximation of certain functions of several variables by sums of functions of fewer variables*, Amer. Math. Monthly, **73(4)**, 131–132.

[11] Ismailova, A., Ismayilov, M. (2024) *On the universal approximation property of radial basis function neural networks*, Ann. Math. Artif. Intell., **92**, 691–701.

[12] Ismailov, V.E. (2006) *On methods for computing the exact value of the best approximation by sums of functions of one variable*, Siberian Math. J., **47(5)**, 883–888.

[13] Ismailov, V.E. (2009) *On the theorem of M. Golomb*, Proc. Indian Acad. Sci. Math. Sci., **119(1)**, 45–52.

[14] Khavinson, S.Ya. (1997) *Best Approximation by Linear Superpositions (Approximate Nomography)*, Translations of Mathematical Monographs, Vol. 159, American Mathematical Society, Providence, RI. Translated from the Russian manuscript by D. Khavinson.

[15] Marshall, D.E., O'Farrell, A.G. (1983) *Approximation by a sum of two algebras. The lightning bolt principle*, J. Funct. Anal., **52(3)**, 353–368.

[16] Ofman, Ju.P. (1961) *On the best approximation of functions of two variables by functions of the form* $\phi(x) + \psi(x)$, Izv. Akad. Nauk SSSR Ser. Mat., **25**, 239–252. English transl.: Amer. Math. Soc. Transl. (2), **44** (1963), 55–59.

[17] Rivlin, T.J., Sibner, R.J. (1965) *The degree of approximation of certain functions of two variables by a sum of functions of one variable*, Amer. Math. Monthly, **72**, 1101–1103.

[18] Tang, L., Tang, Y. (2025) *Finite Element Method for solving the screened Poisson equation with a delta function*, Mathematics, **13(8)**, Article 1360.

[19] Wachspress, E.L. (1994) *Three-variable alternating-direction-implicit iteration*, Comput. Math. Appl., **27(3)**, 1–7.

[20] Wachspress, E.L. (2013) *The ADI Model Problem*, Springer New York, NY.

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Received 06 January 2026
Accepted 22 May 2026