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Existence of Strong Solutions for Systems of Second Order Quasilinear Elliptic Equations

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Abstract. This paper is devoted to boundary value problems for second-order quasilinear elliptic equations and systems with a quasilinear elliptic principal operator in Sobolev spaces. An interpolation method is described for obtaining a priori estimates of strong solutions of quasilinear elliptic equations and systems with unbounded singularities on the right-hand side, provided that there is a first a priori estimate in the space of summable functions.The first part of the paper presents the main idea of the method through the example of a second-order quasilinear elliptic equation. In the second part, we establish an a priori estimate for second-order quasilinear elliptic systems. We also identify conditions under which the maximum principle holds for equations and systems of the above type and prove an existence theorem for solutions. The proposed approach extends the method for elliptic equations developed by S.I. Pohozhaev [1, 2].

Key Words and Phrases: interpolation method, upper and lower solution, quasilinear equation, Sobolev spaces.

1. Introduction. Problem statement and the main result

Let $R_+ = \left\{ t_0 \in R \,\middle|\, t_0 \geq 0 \right\}$. We adopt the following notation throughout the paper. Let $x = (x_1, \ldots, x_N)$ be a point in space R^N and let $\Omega \subset R^N$ be a bounded domain with boundary $\partial\Omega$ of class C^2.

Unless otherwise stated, all functions are assumed to be real-valued. We shall use the following function spaces (see [3, 9, 10, 11, 13, 15, 16]): the space of summable functions $L_p(\Omega)$, $p \geq 1$, with the norm

$$\|u\|_{p;\Omega} = \left(\int\limits_{\Omega} |u(x)|^p dx \right)^{1/p} ;$$

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anisotropic Sobolev space $W_p^2(\Omega)$ endowed with the norm

$$\|u\|_{W_p^2(\Omega)} = \|u\|_{p;\Omega} + \sum_{i=1}^{N} \Big\| \frac{\partial^2 u}{\partial x_i^2} \Big\|_{p;\Omega};$$

Sobolev space $W_p^{2-1/p}(\partial\Omega)$ endowed with the norm $\|u\|_{W_p^{2-1/p}(\partial\Omega)}$, which contains the traces of functions belonging to $W_p^2(\Omega)$.

By $D^i u$ we denote a vector of all partial derivatives $D^\alpha u$ of the function $u(x)$, $\alpha = (\alpha_1, \ldots, \alpha_N)$, $|\alpha| = i$, which has N components for $i = 1$ and N^2 for $i = 2$. For $\vec{u}(x) \equiv (u_1(x), \ldots, u_n(x))$, by $D^i \vec{u}$ we mean a vector $\left(D^i u_1, \ldots, D^i u_n\right)$. We will write $\vec{u}(x) \in \vec{L}_p(\Omega)$ if $u_j \in L_p(\Omega)$ for all $j = \overline{1,n}$ and $\|\vec{u}\|_{p;\Omega} \equiv \sum\limits_{j=1}^{n} \|u_j\|_{L_p(\Omega)}$. We will also adhere to similar agreements in the cases of spaces $\vec{C}(\Omega)$, $\vec{W}_p^{2-1/p}(\partial\Omega)$, and $\vec{W}_p^2(\Omega)$.

Let us consider the following boundary value problem for an equation

$$\begin{cases} \sum\limits_{i,j=1}^{N} a_{ij}(x,u,Du) \frac{\partial^2 u}{\partial x_i \partial x_j} = f(x,u,Du), \quad x \in \Omega, \\ u\Big|_{\partial\Omega} = \varphi(x), \quad x \in \partial\Omega, \end{cases} \tag{1}$$

where $u = u(x)$ and $Du = \left(\frac{\partial u}{\partial x_1}, \ldots, \frac{\partial u}{\partial x_N}\right)$.

We assume the following conditions:

A.1) Let the function $f(x,u,Du)$ be determined in $\Omega \times R \times R^N$ with the values in R and be a Caratheodorian function, i.e., measurable with respect to x for all $(u,Du) \in R \times R^N$ and continuous with respect to $(u,Du) \in R \times R^N$ almost for all $x \in \Omega$.

A.2) Let

$$|f(x,u,\xi)| \le b(x,u) + b_1(x,u) \cdot |\xi|^\mu,$$

almost for all $x \in \Omega$ and for all $u \in R$, $\xi \in R^N$ with such non-negative Caratheodorian functions $b(x,u)$, $b_1(x,u)$ that for any $r \ge 0$ the function

$$b_r(x) = \sup\left\{ b(x,u) \,\Big|\, |u| \le r \right\}$$

belong to $L_p(\Omega)$ with $p > N$, while the function

$$b_{1,r}(x) = \sup\left\{ b_1(x,u) \,\Big|\, |u| \le r \right\}$$

belong to $L_q(\Omega)$ with $q > p$ and $\mu > 1$.

A.3) Let $\mu = 2 - \frac{N}{q}$.

A.4) Let $L(u)$ be a second-order quasilinear elliptic operator of the form

$$Lu = \sum_{i,j=1}^{N} a_{ij}(x, u, Du) \frac{\partial^2 u}{\partial x_i \partial x_j},$$

the coefficients $a_{ij}(x, \xi_0, \xi_1)$, $i, j = \overline{1, N}$ are a real and continuous functions on $\overline{\Omega} \times R \times R^N$.

Consider the operator $L_v u$:

$$L_v u = \sum_{i,j=1}^{N} a_{ij}(x, v(x), Dv(x)) \frac{\partial^2 u}{\partial x_i \partial x_j},$$

for any function $v \in C^1\left(\overline{\Omega}\right)$ to be a linear elliptic operator with such real-valued coefficients that the linear problem (with respect to $u(x)$))

$$\begin{cases} L_v u = g(x), & x \in \Omega, \\ u\Big|_{\partial\Omega} = \varphi(x), & x \in \partial\Omega, \end{cases} \tag{2}$$

for any function $v \in C^1\left(\overline{\Omega}\right)$ to be coercive in the space $W_p^2(\Omega)$, i.e. the a priori estimation.

$$\|u\|_{W_p^2(\Omega)} \le C\left(\|g\|_{p;\Omega} + \|\varphi\|_{W_p^{2-1/p}(\partial\Omega)} + \|u\|_{p;\Omega}\right), \tag{3}$$

with a positive constant C independent of $g \in L_p(\Omega)$, of $\varphi \in W_p^{2-1/p}(\partial\Omega)$ and of the solution $u \in W_p^2(\Omega)$ of the linear problem (2) being fulfilled.

Herewith, the constant C can depend on the continuity module of the coefficients $a_{ij}(x)$, $i, j = \overline{1, N}$ and the continuity module of functions $v(x), Dv$.

Note that sufficient conditions on the operator coefficient L, ensuring the coercivity of the boundary value problem (2), which is linear with respect to $u(x)$, were established in the paper by S. Agmon, A. Douglis, and L. Nirenberg [3].

This problem is studied in the class of real-valued functions belonging to the Sobolev space $W_p^2(\Omega)$, where $p > N$. It is assumed that every strong solution $u \in W_p^2(\Omega)$ of the problem (1) satisfies the a priori estimate $\|u\|_\infty \le M$ with some $M > 0$, where $\|u\|_\infty = \operatorname*{ess\,sup}_{x\in\Omega} |u(x)|$.

For the quasilinear elliptic problem (1) we consider the admissible growth of the subordinate nonlinear operator $f(x, u, Du)$ with respect to the derivatives of the solution. More precisely, we seek conditions under which the a priori estimates $\|u\|_\infty$ and $\|Du\|_\infty$ together with the given data of problem (1), imply an a priori estimate for $\|u\|_{W_p^2(\Omega)}$.

For second-order elliptic equations, this issue has been considered in papers [5, 8, 12, 13], among others.

It is known that, for the second-order equation

$$\Delta u = f\left(x, u, Du\right), \quad x \in \Omega,$$

subject to a boundary condition for which the maximum principle holds, and considered in the space $W_p^2\left(\Omega\right)$ with $p > N$, an a priori estimate for $\|u\|_\infty$ implies an a priori estimate for $\|u\|_{W_p^2(\Omega)}$ in terms of $\|u\|_\infty$ and the given data of the problem, provided that the continuous function $f\left(x, u, \xi\right)$ with $(x, u, \xi) \in \overline{\Omega} \times R \times R^N$ satisfies the S.N. Bernstein condition [8]:

$$\left|f\left(x, u, \xi\right)\right| \leq b\left(x, u\right)\left(1 + |\xi|^2\right),$$

where $b\left(x, u\right)$ is a continuous function.

However, as shown in [1], if the nonlinear function $f\left(x, u, Du\right)$ belongs to $L_p\left(\Omega\right)$, with $p > N$, for every $u\left(x\right) \in W_p^2\left(\Omega\right)$, then the Bernstein condition is no longer adequate. In this case, the Carathéodory function $f\left(x, u, \xi\right)$ should satisfy the condition

$$\left|f\left(x, u, \xi\right)\right| \leq b\left(x, u\right)\left(1 + |\xi|^\mu\right),$$

$\mu = 2 - \frac{N}{p}$. Here, $b\left(x, u\right)$ is a Carathéodory function such that, for every $r > 0$, the function

$$b_r\left(x\right) = \sup\left\{b\left(x, u\right) \,\middle|\, u \in R,\ |u| \leq r\right\},$$

belongs to $L_p\left(\Omega\right)$, where $p > N$.

In this paper, we establish analogous inequalities for the admissible power-growth exponent of the nonlinear function $f\left(x, u, \xi\right)$ in the case of the quasilinear problem (1).

A new sharp growth condition on the nonlinear function $f\left(x, u, \xi\right)$ with respect to $\xi \in R^N$ is established, under which an a priori estimate for $\|u\|_\infty$ for solutions of problem (1) yields an a priori estimate for $\|Du\|_\infty$.

The optimality of the corresponding growth exponent is demonstrated by a specific example.

The solvability theory for problems of the form (1) is developed under the assumption that upper and lower solutions exist. For this purpose, the maximum principle established by A.D. Alexandrov [4] is used together with the theorem providing an a priori estimate for $\|u\|_{W_p^2(\Omega)}$.

The embedding theorem for anisotropic Sobolev spaces $W_p^2\left(\Omega\right)$ (see [9, 16]) plays an essential role in the study of problem (1).

By the Gagliardo–Nirenberg interpolation inequality [6, 14], we have

$$\|Du\|_{S;\Omega} \leq c_1 \|u\|_{W_p^2(\Omega)}^{1/\mu} \cdot \|u\|_{\infty}^{1-1/\mu} + c_2 \cdot \|u\|_{\infty}, \tag{4}$$

where c_1 and c_2 are positive constants independent of the function $u\,(x) \in W_p^2\,(\Omega)$ and

$$\frac{1}{S} = \frac{1}{N} + \frac{1}{\mu}\left(\frac{1}{p} - \frac{2}{N}\right), \tag{5}$$

with $\frac{1}{2} \leq \frac{1}{\mu} < 1$ (or $1 < \mu \leq 2$).

Lemma 1. *Assume that conditions A.1)–A.3) are satisfied. Then the operator $F\,(u)\,(x) \equiv f\,(x, u\,(x)\,, Du\,(x))$ is completely continuous as a mapping from $W_p^2\,(\Omega)$ to $L_p\,(\Omega)$ with $p > N$.*

Proof. Let us estimate $\|F\,(u)\,\|_{p;\Omega}$ using conditions A.1) - A.3) and Hòlder's inequality. Condition A.2) implies that

$$\|F\,(u)\,\|_{p;\Omega} \leq \|b_r\|_{p;\Omega} + \|b_{1,r}\|_{q;\Omega} \cdot \|Du\|_{S;\Omega}^{\mu}, \tag{6}$$

with $r = \text{const} \cdot \|u\|_{\infty}$ and $S = \frac{\mu pq}{q-p}$.

To estimate $\|Du\|_{S;\Omega}$, we apply inequality (4), taking $S = \frac{\mu pq}{q-p}$ and $q \neq \infty$. Then

$$\|F\,(u)\,\|_{p;\Omega} \leq \|b_r\|_{p;\Omega} + \|b_{1,r}\|_{q;\Omega} \cdot \left[c_1 \cdot \|u\|_{W_p^2(\Omega)}^{1/\mu} \cdot \|u\|_{\infty}^{1-1/\mu} + c_2 \cdot \|u\|_{\infty}\right]^{\mu} \leq$$

$$\leq \|b_r\|_{p;\Omega} + \|b_{1,r}\|_{q;\Omega} \cdot 2^{\mu-1} \cdot \|u\|_{\infty}^{\mu-1} \cdot c_1^{\mu} \cdot \|u\|_{W_p^2(\Omega)} + \|b_{1,r}\|_{q;\Omega} \cdot 2^{\mu-1} \cdot c_2^{\mu} \cdot \|u\|_{\infty}^{\mu} \leq$$

$$\leq \|b_r\|_{p;\Omega} + \Phi_1\left(\|u\|_{\infty}\right) \cdot \|u\|_{W_p^2(\Omega)} + \Phi_2\left(\|u\|_{\infty}\right), \tag{7}$$

where Φ_1, $\Phi_2 \,:\, R_+ \to R_+$ are increasing functions determined by the given data.

Under the assumptions of this lemma, the embedding operator $F_1 \,:\, W_p^2\,(\Omega) \to L_S\,(\Omega)$ is completely continuous. By estimate (7), the operator $F \,:\, L_S\,(\Omega) \to L_p\,(\Omega)$ is bounded, and, by the standard properties of superposition operators, it is continuous. Therefore, the operator $F \,:\, W_p^2\,(\Omega) \to L_p\,(\Omega)$ is completely continuous as the composition of a completely continuous operator and a continuous operator. ◀

Let $u \in W_p^2\,(\Omega)$ be a solution of problem (1), and assume that conditions A.1)–A.4) are satisfied. Then problem (1) can be regarded as a linear problem with the fixed right-hand side $F\,(u) \equiv f\,(x, u\,(x)\,, Du\,(x)) \in L_p\,(\Omega)$ with $p > N$.

Theorem 1. *Assume that conditions A.1)–A.4) are satisfied for some $p > N$, with $S = \frac{\mu pq}{q-p}$. Then there exists a function $\Phi : R_+^2 \to R_+$, bounded on each compact such that every solution $u \in W_p^2(\Omega)$ with $p > N$ of the problem* (1) *satisfying*

$$\|u\|_{W_p^2(\Omega)} \leq \Phi\left(M, \|\varphi\|_{W_p^{2-1/p}(\partial\Omega)}\right),$$

also satisfies $\|u\|_\infty \leq M$, where $R_+^2 = R_+ \times R_+$.

Proof. We have

$$Lu = f(x, u, Du) = \frac{f(x, u, Du)}{1 + |Du|^\mu} \cdot (1 + |Du|^\mu),$$

with $\mu = 2 - \frac{N}{q}$. Hence

$$Lu - c(x)u = \frac{f(x, u, Du)}{1 + |Du|^\mu} \cdot (1 + |Du|^\mu) - c(x)u(x),$$

where $c(x) = b_r(x) \geq 0$. Then for the function $u(x)$ we have the following

$$\begin{cases} Lu - c(x)u(x) = f_1(x) \cdot |Du|^\mu + f_0(x), & x \in \Omega, \\ u\Big|_{\partial\Omega} = \varphi(x), & x \in \partial\Omega. \end{cases} \tag{8}$$

Here $f_1(x) = \frac{f(x,u,Du)}{1+|Du|^\mu}$, $\quad f_0(x) = f_1(x) - c(x)u(x)$.

Our goal is to obtain an estimate for $\|u\|_{W_p^2(\Omega)}$ in terms of the quantities bounding $\|u\|_\infty$,, $\|f_0\|_{p;\Omega}$, $\|f_1\|_{q;\Omega}$ and $\|\varphi\|_{W_p^{2-1/p}(\partial\Omega)}$. For this purpose, in the space $W_p^2(\Omega)$, with $p > N$, we consider the following boundary value problem:

$$\begin{cases} Lv - c(x)v = f_1(x)|Dv|^\mu + \lambda \cdot f_0(x), & x \in \Omega, \\ v\Big|_{\partial\Omega} = \lambda \cdot \varphi(x), & x \in \partial\Omega, \end{cases} \tag{9}$$

with the parameter $\lambda \in [0, 1]$ and the functions $c(x)$, $f_0(x)$ and $f_1(x)$ determined above. The following lemma on the uniqueness of the solution is valid for the problem (9).

Lemma 2. *For any fixed λ, problem* (9) *has at most one solution from $W_p^2(\Omega)$ with $p > N$.*

Proof. Let us assume the opposite. Then for the difference $\mathcal{W} = v(x) - Z(x)$ of two possible solutions $v(x)$ and $Z(x)$ we have the following:

$$\begin{cases} L\mathcal{W} - c(x)\mathcal{W}(x) = f_1(x)\sum\limits_{i=1}^{N} h_i(x)D_i\mathcal{W}(x), & x \in \Omega, \\ \mathcal{W}(x)\Big|_{\partial\Omega} = 0, & x \in \partial\Omega, \end{cases}$$

Here $h_i(x) = \int\limits_0^1 H_i(x,\tau)\, d\tau$, where

$$H_i(x,\tau) = \mu \left[\sum_{k=1}^{N} (\tau \cdot D_k\mathcal{W} + D_k Z)^2 \right]^{\frac{\mu}{2}-1} \cdot (\tau \cdot D_i \mathcal{W} + D_i Z)(x),$$

for $\sum\limits_{k=1}^{N} (\tau \cdot D_k\mathcal{W} + D_k Z)^2 (x) \neq 0$ and $H_i(x,\tau) = 0$ for $\sum\limits_{k=1}^{N} (\tau \cdot D_k\mathcal{W} + D_k Z)^2 (x) = 0$.

Since $c(x) \geq 0$, $c(x) \in L_p(x)$, $f_1(x) \in L_q(\Omega)$, $h_i \in L_p(\Omega)$ $(i = 1, \ldots, N)$ and $p > N$. It follows from the results of A. D. Alexandrov's paper that $\mathcal{W}(x) \equiv 0$ (see [4]). ◄

Let v_1 and v_2 be solutions of problem (9) corresponding to the parameter values λ_1 and λ_2, respectively, where $\lambda_2 > \lambda_1$. Then, for the difference $\tilde{v} = v_2 - v_1$ we have the following

$$\begin{cases} L\tilde{v} - c(x)\,\tilde{v} = f_1(x) \sum\limits_{i=1}^{N} \tilde{h}_i(x)\, D_i\tilde{v} + (\lambda_2 - \lambda_1)\, f_0(x), \quad x \in \Omega, \\ \tilde{v}\Big|_{\partial\Omega} = (\lambda_2 - \lambda_1)\,\varphi(x), \quad x \in \partial\Omega. \end{cases}$$

Here $\tilde{h}_i(x) = \int\limits_0^1 \tilde{H}_i(x,\tau)\, d\tau$, where

$$\tilde{H}_i(x,\tau) = \mu \left[\sum_{k=1}^{N} (\tau \cdot D_k\tilde{v} + D_k v_1)^2 \right]^{\frac{\mu}{2}-1} \cdot (\tau \cdot D_i \tilde{v} + D_i v_1)(x),$$

for $\sum\limits_{k=1}^{N} (\tau \cdot D_k\tilde{v} + D_k v_1)^2 (x) \neq 0$ and $\tilde{H}_i(x,\tau) = 0$ for $\sum\limits_{k=1}^{N} (\tau \cdot D_k\tilde{v} + D_k v_1)^2 (x) = 0$.

Assume $K = (\lambda_2 - \lambda_1)(1 + \|u\|_\infty)$, $\lambda_2 > \lambda_1$.

Lemma 3. $\|v_2 - v_1\|_\infty \leq (\lambda_2 - \lambda_1)(1 + \|u\|_\infty)$.

Proof. For the function $(\tilde{v} - K)$ we have

$$\begin{cases} L(\tilde{v} - K) - c(x)(\tilde{v}(x) - K) = f_1(x) \sum\limits_{i=1}^{N} \tilde{h}_i(x)\, D_i(\tilde{v} - K) + \\ + (\lambda_2 - \lambda_1)\, f_0(x) + c(x)\, K, \quad x \in \Omega, \\ (\tilde{v} - K)\Big|_{\partial\Omega} = (\lambda_2 - \lambda_1)\,\varphi(x) - K, \quad x \in \partial\Omega. \end{cases}$$

In addition

$$c(x)K + (\lambda_2 - \lambda_1) f_0(x) = (\lambda_2 - \lambda_1)[c(x)(1 + \|u\|_\infty) + f_0(x)] =$$

$$= (\lambda_2 - \lambda_1) \cdot [c(x)(1 + \|u\|_\infty) + f_1(x) - c(x)u(x)] =$$

$$= (\lambda_2 - \lambda_1) \cdot [(\|u\|_\infty - u(x))c(x) + (c(x) + f_1(x))] \geq 0, \ \ x \in \Omega,$$

$$(\tilde{v} - K)\Big|_{\partial\Omega} = (\lambda_2 - \lambda_1)\varphi(x) - K = (\lambda_2 - \lambda_1)[\varphi(x) - (1 + \|u\|_\infty)] \leq$$

$$\leq (\lambda_2 - \lambda_1)[\varphi(x) - \|u\|_\infty] \leq 0, \ \ x \in \partial\Omega.$$

It then follows from the results of A.D. Alexandrov [4] that $\tilde{v}(x) \leq K$ in $\overline{\Omega}$.
The inequality $\tilde{v}(x) \geq -K, \ x \in \overline{\Omega}$ is proved in the same manner. ◀

We continue the proof of Theorem 1. For this purpose, we consider the parametric family of problems (9). Note that, by Lemma 2, the solution of problem (9) corresponding to $\lambda = 1$ coincides with the solution of problem (8), and hence with the solution of problem (1).

Let v_1 and v_2 be solutions of problem (9) corresponding to the parameter values λ_1 and λ_2, respectively, where $\lambda_2 > \lambda_1$. Then, by Lemma 3, we have

$$\|v_2 - v_1\|_\infty \leq (\lambda_2 - \lambda_1)(1 + \|u\|_\infty). \tag{10}$$

On the other hand, this function is the solution to the problem

$$\begin{cases} L\tilde{v} - c(x)\tilde{v} = f_1(x)(|Dv_2|^\mu - |Dv_1|^\mu) + (\lambda_2 - \lambda_1) f_0(x), \ \ x \in \Omega, \\ \tilde{v}\Big|_{\partial\Omega} = (\lambda_2 - \lambda_1)\varphi(x), \ \ x \in \partial\Omega. \end{cases}$$

Obviously

$$\|L\tilde{v} - c(x)\tilde{v}\|_{p;\Omega} \leq 2^{\mu-1} \cdot \|f_1\|_{q;\Omega} \cdot \|D\tilde{v}\|^\mu_{S;\Omega} +$$

$$+2^\mu \cdot \|f_1\|_{q;\Omega} \cdot \|Dv_1\|^\mu_{S;\Omega} + (\lambda_2 - \lambda_1)\|f_0\|_{p;\Omega}, \tag{11}$$

$$\|\tilde{v}\|_{W_p^{2-1/p}(\partial\Omega)} \leq (\lambda_2 - \lambda_1)\|\varphi\|_{W_p^{2-1/p}(\partial\Omega)}, \tag{12}$$

for $0 \leq \lambda_1 < \lambda_2 \leq 1$.

The inequality (4) for the function $\tilde{v}(x)$ of $W_p^2(\Omega)$ with $p > N$ produces

$$\|D\tilde{v}\|_{S;\Omega} \leq c_1 \cdot \|\tilde{v}\|^{1/\mu}_{W_p^2(\Omega)} \cdot \|\tilde{v}\|^{1-1/\mu}_\infty + c_2\|\tilde{v}\|_\infty, \tag{13}$$

with positive constants c_1 and c_2 independent of the function $\tilde{v}(x)$ from $W_p^2(\Omega)$.

On the other hand, by the classical linear theory of elliptic boundary value problems, the following estimate holds:

$$\|\tilde{v}\|_{W_p^2(\Omega)} \le c_3 \left(\|L\tilde{v} - c\,(x)\,\tilde{v}\|_{p;\Omega} + \|\tilde{v}\|_{W_p^{2-1/p}(\partial\Omega)} \right).$$

Here $c_3 = c_3\,(\Omega, N, p, q, \|c\|_{p;\Omega})$. Then, using inequalities (10)-(13), we get

$$\|\tilde{v}\|_{W_p^2(\Omega)} \le c_3 \Big[2^{\mu-1} \cdot \|f_1\|_{q;\Omega} \cdot \|D\tilde{v}\|_{S;\Omega}^{\mu} + 2^{\mu} \cdot \|f_1\|_{q;\Omega} \cdot \|Dv_1\|_{S;\Omega}^{\mu} +$$

$$+ (\lambda_2 - \lambda_1) \cdot \|f_0\|_{p;\Omega} + (\lambda_2 - \lambda_1) \cdot \|\varphi\|_{W_p^{2-1/p}(\partial\Omega)} \Big] \le$$

$$\le c_3 2^{\mu-1} \cdot \|f_1\|_{q;\Omega} \cdot \left[c_1 \cdot \|\tilde{v}\|_{W_p^2(\Omega)}^{1/\mu} \cdot \|\tilde{v}\|_{\infty}^{1-1/\mu} + c_2 \|\tilde{v}\|_{\infty} \right]^{\mu} +$$

$$+ c_3 \cdot 2^{\mu} \cdot \|f_1\|_{q;\Omega} \cdot \|Dv_1\|_{S;\Omega}^{\mu} + c_3 \cdot (\lambda_2 - \lambda_1) \cdot \|f_0\|_{p;\Omega} + c_3\,(\lambda_2 - \lambda_1) \cdot \|\varphi\|_{W_p^{2-1/p}(\partial\Omega)} \le$$

$$\le c_3 \cdot 2^{2\mu-2} \cdot c_1^{\mu} \cdot \|f_1\|_{q;\Omega} \cdot \|\tilde{v}\|_{\infty}^{\mu-1} \cdot \|\tilde{v}\|_{W_p^2(\Omega)} +$$

$$+ c_3 \cdot 2^{2\mu-2} \cdot c_2^{\mu} \cdot \|f_1\|_{q;\Omega} \cdot \|\tilde{v}\|_{\infty}^{\mu} + c_3 \cdot 2^{\mu} \cdot \|f_1\|_{q;\Omega} \cdot \|Dv_1\|_{S;\Omega}^{\mu} +$$

$$+ c_3 \cdot (\lambda_2 - \lambda_1) \left(\|f_0\|_{p;\Omega} + \|\varphi\|_{W_p^{2-1/p}(\partial\Omega)} \right) \le$$

$$\le c_3 \cdot 2^{2\mu-2} \cdot c_1^{\mu} \cdot \|f_1\|_{q;\Omega} \cdot (\lambda_2 - \lambda_1)^{\mu-1} \cdot (1 + \|u\|_{\infty})^{\mu-1} \cdot \|\tilde{v}\|_{W_p^2(\Omega)} +$$

$$+ c_3 \cdot 2^{2\mu-2} \cdot c_2^{\mu} \cdot \|f_1\|_{q;\Omega} \cdot (\lambda_2 - \lambda_1)^{\mu} \cdot (1 + \|u\|_{\infty})^{\mu} + c_3 \cdot 2^{\mu} \cdot \|f_1\|_{q;\Omega} \cdot \|Dv_1\|_{S;\Omega}^{\mu} +$$

$$+ c_3 \cdot (\lambda_2 - \lambda_1) \left(\|f_0\|_{p;\Omega} + \|\varphi\|_{W_p^{2-1/p}(\partial\Omega)} \right).$$

We require that the following inequalities hold:

$$c_3 \cdot 2^{2\mu-2} \cdot c_1^{\mu} \cdot \|f_1\|_{q;\Omega} \cdot (\lambda_2 - \lambda_1)^{\mu-1} \cdot (1 + \|u\|_{\infty})^{\mu-1} \le \frac{1}{2}.$$

Then we have

$$\|\tilde{v}\|_{W_p^2(\Omega)} \le \Phi_1 + \Phi_2 \cdot \|Dv_1\|_{S;\Omega}^{\mu}, \tag{14}$$

for

$$0 < \lambda_2 - \lambda_1 \le h, \tag{15}$$

where

$$h = c_3^{-1!(\mu-1)} \cdot 2^{-(2\mu-1)/(\mu-1)} \cdot c_1^{-\mu/(\mu-1)} \cdot \|f_1\|_{q;\Omega}^{-\mu/(\mu-1)} \cdot (1 + M)^{-1},$$

and Φ_1, $\Phi_2 : R_+ \to R_+$ are increasing functions determined by the known data. Here λ_1 and λ_2 are any numbers in the interval $[0, 1]$ satisfying the inequality (15), and v_1, v_2 are the solutions corresponding to them from $W_p^2(\Omega)$ of the problem (9).

Setting $\lambda^{(k-1)} = \lambda_1$, $\lambda^{(k)} = \lambda_2$ and $v^{(k-1)} = v_1$, $v^{(k)} = v_2$, we obtain from inequality (14) that

$$\|v^{(k)}\|_{W_p^2(\Omega)} \leq \Phi_1 + \|v^{(k-1)}\|_{W_p^2(\Omega)} + \Phi_2 \cdot \|Dv^{(k-1)}\|_{S;\Omega}^{\mu},$$

for $0 < \lambda^{(k)} - \lambda^{(k-1)} \leq h,\ \lambda^{(k-1)}, \lambda^{(k)} \in [0,1]$. Since h is independent of the particular values of $\lambda^{(k-1)}$ and $\lambda^{(k)}$, and since the embedding $W_p^2(\Omega) \to C^1(\Omega)$ with $p > N$ implies

$$\|Dv^{(k-1)}\|_{S;Q_T} \leq \text{const} \cdot \|v^{(k-1)}\|_{W_p^2(\Omega)},$$

the assertion of the theorem follows after a finite number of iterations. The first iteration, corresponding to $(k = 1)$, starts with $\lambda^{(0)} = 0$ and $v^{(0)} = 0$. ◀

As follows from Theorem 1, the key point of this method is the condition under which the embedding theorem holds. Since we seek the maximal admissible value of μ for which the conditions of the embedding theorem are satisfied, we obtain the following implicit equation for determining μ:

$$\frac{1}{S} = \frac{1}{N} + \frac{1}{\mu}\left(\frac{1}{p} - \frac{2}{N}\right), \quad S = \frac{\mu pq}{q-p},$$

$$\frac{q-p}{\mu pq} = \frac{1}{N} + \frac{1}{\mu p} - \frac{2}{\mu N}.$$

The solution of this equation is calculated explicitly in the form:

$$\mu = 2 - \frac{N}{q}. \tag{16}$$

To obtain a genuinely nonlinear problem, we assume that $\mu > 1$.

2. Optimality of the Exponent $\mu = 2 - \frac{N}{q}$

In this section, we present an example of a boundary value problem of the form (1) for which all the assumptions of Theorem 1, except condition A.3), are satisfied. For this counterexample, we establish the corresponding inequality and show that the conclusion of Theorem 1 does not hold. Thus, without additional assumptions, the exponent $\mu = 2 - \frac{N}{p}$ in condition A.3) cannot be improved. To demonstrate this, we consider the following counterexample (see [1]).

Let $N=1,\ \Omega=[0,1]\subset R$, and

$$\begin{cases} u'(x)\cdot u''(x)=b(x)\,|u'|^{\mu}, & x\in\Omega,\\ u(0)=0, u(1)=(1+\varepsilon)^{\nu}-\varepsilon^{\nu}. \end{cases} \tag{17}$$

Here $0<\nu<1, 0<\varepsilon<1, \mu>2-\frac{1}{p}$ and $b(x)=\nu^{2-\mu}\cdot(\nu-1)\cdot(x+\varepsilon)^{2\nu-3-\mu(\nu-1)},\ \ p>1$.

For the given parameters, this problem has a unique solution.

$$u(x)=(x+\varepsilon)^{\nu}-\varepsilon^{\nu}.$$

For the norm $\|b\|_{q;\Omega}$, we have

$$\|b\|_{q;\Omega}=\nu^{2-\mu}\cdot(1-\nu)\cdot\left(\int\limits_0^1 (x+\varepsilon)^{q[2\nu-3-\mu(\nu-1)]}\,dx\right)^{1/q}=$$

$$=\nu^{2-\mu}\cdot(1-\nu)\cdot\left|\frac{(x+\varepsilon)^k}{k}\Bigg|_0^1\right|^{1/q}=\nu^{2-\mu}\cdot(1-\nu)\cdot\left|\frac{(1+\varepsilon)^k-\varepsilon^k}{k}\right|^{1/q},$$

where $k=[\mu(1-\nu)+2\nu-3]\cdot q+1$.

Hence, for $k>0:$

$$\mu>\frac{3-\frac{1}{q}-2\nu}{1-\nu}=\frac{2-\frac{1}{q}}{1-\nu}+\frac{1-2\nu}{1-\nu} \tag{18}$$

and for the indicated values of the parameters, the inequality

$$\|b\|_{q;\Omega}\le\nu^{2-\mu}\cdot 2^{k/q}\cdot k^{-1/q},$$

is fulfilled for all ε from the interval $(0,1)$.

For the function $\tilde{\varphi}(x)=[(1+\varepsilon)^{\nu}-\varepsilon^{\nu}]\cdot x$ (such that $\tilde{\varphi}\Big|_{\partial\Omega}=u\Big|_{\partial\Omega}$) we have

$$\|\tilde{\varphi}\|_{W_p^2(\Omega)}=\|\tilde{\varphi}\|_{p;\Omega}+\|\tilde{\varphi}'\|_{p;\Omega}+\|\tilde{\varphi}''\|_{p;\Omega}=$$

$$=\left(\int\limits_0^1[(1+\varepsilon)^{\nu}-\varepsilon^{\nu}]^p\cdot x^p dx\right)^{1/p}+[(1+\varepsilon)^{\nu}-\varepsilon^{\nu}]\cdot\left(\int\limits_0^1 dx\right)^{1/p}=$$

$$=[(1+\varepsilon)^{\nu}-\varepsilon^{\nu}]\cdot\left[\frac{x^{p+1}}{p+1}\Bigg|_0^1\right]^{1/p}+[(1+\varepsilon)^{\nu}-\varepsilon^{\nu}]=$$

$$= [(1+\varepsilon)^{\nu} - \varepsilon^{\nu}] \cdot (p+1)^{1/p} + [(1+\varepsilon)^{\nu} - \varepsilon^{\nu}] =$$

$$= [(1+\varepsilon)^{\nu} - \varepsilon^{\nu}] \cdot \left(\frac{1}{(p+1)^{1/p}} + 1 \right) \leq 2^{1+\nu} < 4,$$

for the given values of parameters.

By virtue of the inequality (18) for any $p > 1$ and $\mu > 2 - \frac{1}{q}$ there exists a ν_*, $0 < \nu_* < 1$ such that the inequality (18) is fulfilled.

Then, for the parameters of given values $p > 1$, $\mu > 2 - \frac{1}{q}$, and the corresponding inequalities ν_*:

$$\begin{aligned} &\|u\|_{\infty} \leq 2,\ \|b\|_{q;\Omega} \leq \nu^{2-\mu} \cdot 2^{k/q} \cdot k^{-1/q}, \\ &\|\varphi\|_{W_p^{2-1/p}(\partial\Omega)} \leq c_0 \|\tilde{\varphi}\|_{W_p^2(\Omega)} \leq 4c_0 \end{aligned} \tag{19}$$

are fulfilled uniformly with respect to ε from $(0,1)$. Here, the constant c_0 is independent of the function $\tilde{\varphi}$.

On the other hand, for the norm $\|Du\|_{\infty}$, we have

$$\|Du\|_{\infty} = \sup_{\Omega} |Du| = \sup_{\Omega} |\nu\,(x+\varepsilon)^{\nu-1}| = \nu_* \varepsilon^{-(1-\nu_*)} \to +\infty \text{ as } \ \varepsilon \to 0\ (\varepsilon > 0)\,,$$

and consequently, $\|u\|_{W_p^2(\Omega)} \to +\infty$ as $\varepsilon \to 0$ $(\varepsilon > 0)$.

In the same way, the statement of Theorem 1 for any $\mu > 2 - \frac{1}{q}$ is not valid.

3. Solvability of the problem (1)

Let us consider the boundary value problem (1) under conditions A.1)–A.4) and the following Lipschitz condition.

A.5) Assume that

$$|f\,(x,u,\xi) - f\,(x,u,\eta)\,| \leq b_2\,(x,u,\xi,\eta) \cdot |\xi - \eta|,$$

for almost every $x \in \Omega$ and for all $(u,\xi,\eta) \in R \times R^N \times R^N$, where $b_2\,(x,u,\xi,\eta)$ is measurable with respect to x for all $(u,\xi,\eta) \in R \times R^N \times R^N$, is continuous with respect to (u,ξ,η) almost for all $x \in \Omega$ and for any fixed $r > 0$

$$b_{2,r}\,(\cdot) \equiv \sup \left\{ b_2\,(x,u,\xi,\eta) \,\middle|\, |u| \leq r,\ |\xi| \leq r,\ |\eta| \leq r \right\},$$

and $b_{2,r} \in L_q\,(\Omega)$ with $q > N$.

We recall the following standard definition.

Definition 1. *The function $u^{+}(x)$ from $W_p^2(\Omega)$ with $p > N$ is called an upper solution of the problem* (1) *if*

$$\begin{cases} -Lu^{+} + f\left(x, u^{+}, Du^{+}\right) \geq 0, \;\; x \in \Omega, \\ u^{+}\Big|_{\partial\Omega} \geq \varphi(x), \;\; x \in \partial\Omega. \end{cases}$$

Definition 2. *The function $u^{-}(x)$ from $W_p^2(\Omega)$ with $p > N$ is called the down solution of the problem* (1) *if*

$$\begin{cases} -Lu^{-} + f\left(x, u^{-}, Du^{-}\right) \leq 0, \;\; x \in \Omega, \\ u^{-}\Big|_{\partial\Omega} \leq \varphi(x), \;\; x \in \partial\Omega. \end{cases}$$

Theorem 2. *Assume that conditions A.1)–A.5) are satisfied for some $p > N$, and that the boundary function $\varphi(x)$ belongs to $W_p^{2-1/p}(\Omega)$. Suppose that problem* (1) *admits an upper solution $u^{+}(x)$ and a lower solution $u^{-}(x)$ such that $u^{+}(x) \geq u^{-}(x)$ in $\overline{\Omega}$. Then problem* (1) *has a solution $u(x)$ from $W_p^2(\Omega)$ satisfying*

$$u^{-}(x) \leq u(x) \leq u^{+}(x), \;\; x \in \overline{\Omega}.$$

The proof of this theorem is based on the following lemmas.

Lemma 4. *Let the real-values function $F_0(x, u, \xi)$ defined on $\Omega \times R \times R^N$, satisfy the Carathéodory condition A.1), with $f = F_0$, and*

$$\sup_{(u,\xi)\in R\times R^N} \left|F_0(x, u, \xi)\right| \in L_p(\Omega), \;\; p > N. \tag{20}$$

Then the boundary value problem

$$\begin{cases} Lu = F_0(x, u, Du), \;\; x \in \Omega, \\ u\Big|_{\partial\Omega} = \varphi(x), \;\; x \in \partial\Omega, \end{cases}$$

where $\varphi(x) \in W_p^{2-1/p}(\partial\Omega)$ has the solution $u \in W_p^2(\Omega)$.

Proof. Let us consider the following boundary value problem

$$\begin{cases} L_v u = F_0(x, v, Dv), \;\; x \in \Omega, \\ u\Big|_{\partial\Omega} = \varphi(x), \;\; x \in \partial\Omega, \end{cases}$$

for the arbitrary function $v(x) \in C^1\left(\overline{\Omega}\right)$. Then, by the classical L_p-theory of linear elliptic boundary value problems, for every function $v(x)$ of $C^1\left(\overline{\Omega}\right)$ this problem admits a solution $u(x)$ of $W_p^2(\Omega)$, $u = T(v)$ which, by (20), satisfies the following inequality:

$$\|u\|_{W_p^2(\Omega)} \le C, \tag{21}$$

where the constant C is independent of $v(x)$. The operator $T : C^1\left(\overline{\Omega}\right) \to W_p^2(\Omega)$ is continuous and as a result of the compact embedding $W_p^2(\Omega) \to C^1\left(\overline{\Omega}\right)$ $(p > N)$ is a completely continuous operator from $C^1\left(\overline{\Omega}\right)$ to $C^1\left(\overline{\Omega}\right)$. Due to estimate (21) there exists a sphere in the space $C^1\left(\overline{\Omega}\right)$ that the operator takes to itself. Then, by the Schauder‘s known theorem, the operator T in the $C^1\left(\overline{\Omega}\right)$ space has a fixed point $u(x)$ that by the definition of the operator T belongs to the space $W_p^2(\Omega)$. ◀

We now determine for the function $u(x)$ from $W_p^2(\Omega)$ with $p > N$ the cut off operator σ by the relation

$$\sigma u(x) = \begin{cases} u^+(x) & \text{for } u(x) > u^+(x), \\ u(x) & \text{for } u^-(x) \le u(x) \le u^+(x), \\ u^-(x) & \text{for } u(x) < u^-(x), \end{cases}$$

and consider the following boundary value problem

$$\begin{cases} Lu = f(x, \sigma u, Du), \quad x \in \Omega, \\ u\Big|_{\partial\Omega} = \varphi(x), \quad x \in \partial\Omega. \end{cases} \tag{22}$$

Lemma 5. *Let the function $f(x, \sigma u, Du)$ satisfy conditions A.1) -A.5) with $p > N$ and $\varphi \in W_p^{2-1/p}(\Omega)$. Let $u(x)$ from $W_p^2(\Omega)$ be the solution of the problem* (22). *Then*

$$u^-(x) \le u(x) \le u^+(x), \quad x \in \overline{\Omega}. \tag{23}$$

Proof. For the function $\mathcal{W}(x) = u(x) - u^+(x)$ we have

$$L\mathcal{W} \ge f(x, \sigma u, Du) - f\left(x, u^+, Du^+\right), \quad x \in \Omega, \tag{24}$$

$$\mathcal{W}\Big|_{\partial\Omega} \le 0, \quad x \in \Omega. \tag{25}$$

We now assume the inverse statement of the lemma. Then set $G = \Big\{x \in \Omega \,\Big|\, \omega(x) > 0\}$ is not empty, and

$$L\mathcal{W} \ge f\left(x, u^+, Du\right) - f\left(x, u^+, Du^+\right) \ge -\tilde{b}_{2,r}(x)\,|D\mathcal{W}|, \quad x \in \Omega,$$

where $\tilde{b}_{2,r}(x) = \sup\{b_2(x, u^+, Du, Du^+)\}$ and $\tilde{b}_{2,r} \in L_q(\Omega)$ with $q > p > N$. Then it follows from A.D. Alexandrov‘s paper [4] that $\mathcal{W}(x)$ achieves a strong positive maximum on the boundary $\partial\Omega$. This contradicts the boundary condition (25). Thus, it is proved that $u(x) \leq u^+(x)$ in the domain Ω.

Inequality $u^-(x) \leq u(x)$ is proved in the domain Ω in the same way. ◄

Proof of Theorem 2. Now, let $M = \max\limits_{\Omega}\left\{\max\limits_{\Omega} u^+(x), -\min\limits_{\Omega} u^-(x)\right\}$. Then $\|u\|_\infty \leq M$ and by virtue of Theorem 1

$$\|u\|_{W_p^2(\Omega)} \leq \Phi\left(M, \|\varphi\|_{W_p^{2-1/p}(\partial\Omega)}\right).$$

By virtue of the Sobolev embedding theorem [16], we have

$$\|u\|_{C^1(\overline{\Omega})} \leq c_2 \cdot \|u\|_{W_p^2(\Omega)} \quad (p > N), \tag{26}$$

where the positive constant c_2 is independent of the function $u(x) \in W_p^2(\Omega)$. Then we obtain the following.

$$\max_{\Omega}|Du(x)| \leq M_1, \;\; M_1 = c_2 \cdot \Phi\left(M, \|\varphi\|_{W_p^{2-1/p}(\partial\Omega)}\right).$$

We now define the function

$$F_1(x,u,\xi) = \begin{cases} f(x,u,\xi) & \text{for } |\xi| \leq M_2, \\ f\left(x,u,M_2\frac{\xi}{|\xi|}\right) & \text{for } |\xi| > M_2, \end{cases}$$

where

$$M_2 = \max_{\Omega}\left\{M_1, \max|Du^+(x)|, \max|Du^-(x)|\right\}.$$

This function satisfies conditions A.1)–A.3), together with the following corresponding inequality:

$$|F_1(x,u,\xi)| \leq \begin{cases} b(x,u) + b_1(x,u)|\xi|^\mu & \text{for } |\xi| \leq M_2, \\ b(x,u) + b_1(x,u)M_2^\mu & \text{for } |\xi| > M_2. \end{cases}$$

The function $F_1(x,u,\xi)$ also satisfies the condition A.5) with the appropriate function $b_{2,r}(x)$.

Let us consider the following boundary value problem

$$\begin{cases} Lu = F_1(x,\sigma u, Du), \;\; x \in \Omega, \\ u\Big|_{\partial\Omega} = \varphi(x), \;\; x \in \partial\Omega. \end{cases} \tag{27}$$

The function

$$\overline{F}_1(x,u,\xi) = \begin{cases} F_1(x,u^+(x),\xi) & \text{for} \quad u > u^+(x), \\ F_1(x,u(x),\xi) & \text{for} \;\; u^-(x) \leq u(x) \leq u^+(x), \\ F(x,u^-(x),\xi) & \text{for} \quad u < u^-(x), \end{cases}$$

satisfies the conditions of Lemma 4 and

$$\overline{F}_1 (x, u(x), Du(x)) = F_1 (x, \sigma u(x), Du(x)).$$

Consequently, to the problem (27) we can apply Lemma 4 by virtue of which there exists a solution $u(x)$ of this problem from the space $W_p^2(\Omega)$ with $p > N$.

The function $u^+(x)$ is the upper solution and $u^-(x)$ is the lower solution of the problem.

$$\begin{cases} Lu = F_1 (x, u, Du), \quad x \in \Omega, \\ u\Big|_{\partial\Omega} = \varphi(x), \; x \in \partial\Omega. \end{cases} \tag{28}$$

By the Lemma 4, we have

$$u^-(x) \leq u(x) \leq u^+(x), \quad x \in \overline{\Omega},$$

and, consequently, $\sigma u(x) = u(x)$ so that the obtained solution $u(x)$ of problem (27) is also the solution of problem (28).

We now apply the theorem to the solution $u(x)$ of problem (28) and obtain the following.

$$\|u\|_{W_p^2(\Omega)} \leq \Phi\left(M, \|\varphi\|_{W_p^{2-1/p}(\partial\Omega)}\right).$$

Then from the inequality (26) it follows that

$$\max_{\Omega} |Du(x)| \leq M_1, \quad M_1 = c_2 \cdot \Phi\left(M, \|\varphi\|_{W_p^{2-1/p}(\partial\Omega)}\right).$$

Since for $|\xi| \leq M_1 \leq M_2$ the function $F_1 = f$, we finally find that the solution $u(x)$ of problem (28) is also the solution of problem (1). ◄

4. The maximum principle and condition A.5)

Let $g(x, \xi)$ be a function satisfying the Carathéodory condition; i.e. $g(x, \xi)$ is measurable with respect to x for every $\xi \in R^N$ and continuous with respect to ξ for almost every $x \in \Omega$. Assume, moreover, that for every fixed $r > 0$, the function $\sup\limits_{|\xi| \leq r} |g(x, \xi)|$ belongs to the space $L_p(\Omega)$ with $p > N$.

For an arbitrary fixed function $v(x) \in W_p^2(\Omega)$ we consider the following inequality for the function $\mathcal{W}(x)$ from $W_p^2(\Omega)$:

$$\begin{cases} L\mathcal{W} \geq g(x, Dv + D\mathcal{W}) - g(x, Dv), \quad x \in \Omega, \\ \mathcal{W}(x)\Big|_{\partial\Omega} \leq 0, \; x \in \partial\Omega. \end{cases} \tag{29}$$

If the function $g(x, Dv + D\mathcal{W})$ satisfies only the above conditions, then (29) does not yield the inequality

$$\mathcal{W}(x) \leq 0, \ \ x \in \Omega, \tag{30}$$

in the general case.

Furthermore, suppose that the Lipschitz condition A.5) is replaced by the following Hölder condition:

A. 6) $|f(x,u,\xi) - f(x,u,\eta)| \leq b_2(x,u,\xi,\eta) \cdot |\eta - \xi|^{\lambda}, \ \ 0 < \lambda < 1$, where b_2 is the same function as in condition A.5). Then, for any exponent λ, $0 < \lambda < 1$, inequality (29) does not, in general, imply inequality (30) for the function (g).

This fact is demonstrated by the following counterexample (see [1]).

For any λ, $0 < \lambda < 1$ and any $p > N$ there exists a function $\mathcal{W}(x) = 1 - |x|^{\alpha+1}$ with $\alpha > \left(1 - \frac{N}{p}\right) \Big/ (1 - \lambda)$ from the space $W_p^2(\Omega)$ with $p > N$ satisfying the following conditions:

$$\Delta\mathcal{W} = g_0(x)\,|D\mathcal{W}|^{\lambda}, \ \ x \in \Omega = \left\{ x \in R^N \,\middle|\, |x| < 1 \right\},$$

$$\mathcal{W}\Big|_{\partial\Omega} = 0, \ \ x \in \partial\Omega = \left\{ x \in N^N \,\middle|\, |x| = 1 \right\},$$

with $g_0(x) = -(1+\alpha)^{1-\lambda}(\alpha + N - 1) \cdot |x|^{\alpha(1-\lambda)-1}$, $g_0 \in L_q(\Omega)$ for $\alpha > (1 - N/p)\,/\,(1-\lambda)$.

Here conditions (29) are met for $v(x) \equiv 0$ and $g(x,\xi) = g_0(x) \cdot |\xi|^{\lambda}$, but inequality (30) is not valid $\Omega = \left\{ x \in R^N \,\middle|\, |x| < 1 \right\}$.

5. Some applications

Choosing the function from various sets of trial functions as upper and lower solutions, we can obtain various theorems on the existence of solutions to boundary value problems of the form (1).

1) We take the function as a "trial" set,

$$u(x) = \lambda,$$

where λ is a real number.

Theorem 3. *Assume that conditions A.1)–A.5) are satisfied for some $p > N$, and that the boundary function $\varphi(x)$ belongs to the space $W_p^{2-1/p}(\partial\Omega)$. Suppose that the functions $f(x,u,Du)$ and $\varphi(x)$ are such that there exist two numbers λ^+ and λ^-, with $\lambda^+ \geq \lambda^-$, such that*

$$\begin{cases} f(x,\lambda^-,0) \leq 0 \leq f(x,\lambda^+,0), \ \ x \in \Omega, \\ \lambda^- \leq \varphi(x) \leq \lambda^+, \ x \in \partial\Omega. \end{cases}$$

Then there exists a solution $\overline{u}(x)$ of the boundary value problem (1) *from the space $W_p^2(\Omega)$ and*

$$\lambda^- \leq \overline{u}(x) \leq \lambda^+,\ x \in \Omega.$$

To prove this theorem, it suffices to apply Theorem 2 $u^+ = \lambda^+$ and $u^- = \lambda^-$.

2) We have the function as a "trial" set,

$$u(x) = \lambda \frac{|x|^2}{2},$$

where λ is a real number.

Theorem 4. *Assume that conditions A.1)–A.5) are satisfied for some $p > N$, and that the boundary function $\varphi(x)$ belong to the space $W_p^{2-1/p}(\partial\Omega)$. Suppose that the functions $f(x, u, Du)$ and $\varphi(x)$ are such that there exist two numbers λ^+ and λ^-, with $\lambda^+ \geq \lambda^-$, such that*

$$\begin{cases} f\left(x, \lambda^+\frac{|x|^2}{2}, \lambda^+ x\right) \geq \sum\limits_{i,j=1}^{N} a_{ij}\left(x, \lambda^+\frac{|x|^2}{2}, \lambda^+ x\right)\lambda^+,\ x \in \Omega, \\ f\left(x, \lambda^-\frac{|x|^2}{2}, \lambda^- x\right) \leq \sum\limits_{i,j=1}^{N} a_{ij}\left(x, \lambda^-\frac{|x|^2}{2}, \lambda^- x\right)\lambda^-,\ x \in \Omega, \\ \lambda^-\frac{|x|^2}{2} \leq \varphi(x) \leq \lambda^+\frac{|x|^2}{2},\ x \in \partial\Omega. \end{cases}$$

Then there exists a solution $\overline{u}(x)$ to the boundary value problem (1) *from the space $W_p^2(\Omega)$, and*

$$\lambda^-\frac{|x|^2}{2} \leq \overline{u}(x) \leq \lambda^+\frac{|x|^2}{2},\ x \in \Omega.$$

To prove this theorem, it suffices to apply Theorem 2 with $u^+ = \lambda^+\frac{|x|^2}{2}$ and $u^- = \lambda^-\frac{|x|^2}{2}$.

An analogous solvability theorem for the boundary value problem (1) can be obtained by choosing trial functions of the form

$$u(x) = \frac{\lambda}{2}|x - x_0|^2,$$

where $\lambda^-, \lambda^+ \in R$ are chosen appropriately and $x_0, x \in R^N$.

As a "trial" functions set we take the function of the form

$$u(x) = u_0(x) + \lambda\psi_1(x),$$

where $u_0(x)$ is the solution of the problem

$$\begin{cases} Lu_0 = 0,\ x \in \Omega, \\ u_o\Big|_{\partial\Omega} = \varphi(x),\ x \in \partial\Omega. \end{cases}$$

λ is a real number and $\psi_1(x)$ is a first eigenfunction of the following boundary value problem:

$$\begin{cases} L\psi_1 + \lambda_1\psi_1 = 0, & x \in \Omega, \\ \psi_1\Big|_{\partial\Omega} = 0, & x \in \partial\Omega, \end{cases}$$

with $\psi_1 > 0$ in Ω.

Theorem 5. *Assume that conditions A.1)–A.5) are satisfied for some $p > N$, and that the boundary function $\varphi(x)$ belong to the space $W_p^{2-1/p}(\partial\Omega)$. Suppose that the functions $f(x, u, Du)$ and $\varphi(x)$ are such that there exist two real numbers λ^+ and λ^-, with $\lambda^+ \geq \lambda^-$, such that*

$$f\left(x, u_0(x) + \lambda^+\psi_1(x), Du_0(x) + \lambda^+ D\psi_1(x)\right) + \lambda_1\lambda^+\psi(x) \geq 0, \ x \in \Omega,$$

$$f\left(x, u_0(x) + \lambda^-\psi_1(x), Du_0(x) + \lambda^- D\psi_1(x)\right) + \lambda_1\lambda^-\psi(x) \leq 0, \ x \in \Omega.$$

Then there exists a solution $\overline{u}(x)$ to the boundary value problem (1) *from the space $W_p^2(\Omega)$, and*

$$u_0(x) + \lambda^-\psi_1(x) \leq \overline{u} \leq u_0(x) + \lambda^+\psi_1(x), \ x \in \Omega.$$

To prove this theorem, it suffices to apply Theorem 2 with $u^+ = u_0 + \lambda^+\psi_1$ and $u^- = u_0 + \lambda^-\psi_1$.

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