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Spectral Analysis for the Almost Periodic Sturm–Liouville Operator with a Finite Number of Distribution Functions

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Abstract. This paper investigates the spectral properties of an almost periodic Sturm–Liouville operator on the real axis, where the differential expression includes a finite number of point interactions of δ and δ' type and an almost periodic potential. The presence of singular perturbations at prescribed points x_k leads to interface conditions that couple solution behavior across intervals.
Fundamental solutions of the associated differential equation are constructed and analyzed, including their analytic structure and asymptotic behavior. Using the method of variation of parameters, we explicitly derive the resolvent operator and show that the perturbation it induces is of finite rank. This leads to a complete description of the spectrum: the eigenvalues correspond to the squares of the zeros of the characteristic determinant $D(\lambda)$, and their total number does not exceed $m+n$. Furthermore, we prove that the continuous spectrum coincides with the positive real axis $[0,\infty)$, with possible spectral singularities only at points $\left(\frac{\Lambda_n}{2}\right)^2$. The results extend classical spectral theory to a class of almost periodic Sturm–Liouville operators with finitely many distributional interactions, providing a framework for analyzing similar operators that arise in quantum mechanics and related areas.
Key Words and Phrases: impulsive conditions, Sturm–Liouville operators, spectral singularities, almost periodic functions.

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1. Introduction

Sturm–Liouville operators with singular-point interactions have attracted considerable attention due to their fundamental role in mathematical physics, spectral theory, and quantum mechanics. Localized perturbations modeled by the Dirac delta (δ) and delta-prime (δ') distributions provide an effective framework for describing highly concentrated interactions, such as impurities in crystalline materials, quantum dots, nanowires, quantum waveguides, and quantum graphs. These operators also arise naturally in scattering theory, where localized defects modify the propagation of quantum particles.

We consider the differential operator $L_{m,n}$ in $L^2(\mathbb{R})$ formally generated by

$$\begin{aligned}(L_{m,n}f)(x) = &- f''(x) + q(x) f(x) \\ &+ \sum_{k=1}^{m} \alpha_k \delta(x - x_k) f(x) + \sum_{k=m+1}^{m+n} \beta_{k-m} \delta'(x - x_k) f(x), \quad x \in \mathbb{R},\end{aligned}$$

where, $\mathbb{R}$ is the real axis, $\alpha_k \in \mathbb{R}\,(k = 1, ..., m), \beta_j \in \mathbb{R}\,(j = 1, ..., n)$,

$$q(x) = \sum_{n=1}^{\infty} q_n e^{i\Lambda_n x}, \qquad \sum_{n=1}^{\infty} |q_n| < \infty, \tag{1}$$

and $x_0 = -\infty < x_1 < ... < x_{m+n+1} = +\infty$.

The set of exponents is a countable set of positive real numbers closed under addition

$$M = \{\Lambda_1,\, \Lambda_2,\, \Lambda_3,\, ...\Lambda_n, ...\}\,, \;\; \Lambda_n > 0, \;\; n \in \mathbb{N}.$$

The domain of the operator $L_{m,n}$ is defined by

$$\begin{aligned} D\big(L_{m,n}\big) = \Big\{ y \in L^2(\mathbb{R}) : y|_{(x_k, x_{k+1})} \in C^2(x_k, x_{k+1}),\ & k = 0, 1, \ldots, m+n; \\ & y(x_k \pm 0), y'(x_k \pm 0) \text{ exist and are finite;} \\ & y(x_k + 0) = y(x_k - 0),\ k = 1, \ldots, m; \\ & y'(x_k + 0) - y'(x_k - 0) = \alpha_k\, y(x_k),\ k = 1, \ldots, m; \\ & y'(x_k + 0) = y'(x_k - 0),\ k = m+1, \ldots, m+n; \\ & y(x_k + 0) - y(x_k - 0) = \beta_{k-m}\, y'(x_k), \\ & \ k = m+1, \ldots, m+n \Big\}. \end{aligned}$$

In this paper, we study the generalized Sturm–Liouville differential equation on the entire real axis

$$(L_{m,n}y)(x) = \lambda^2 y(x), \;\; x \in \mathbb{R} \setminus \{x_1, ..., x_{m+n}\}, \tag{2}$$

where λ is a complex number.

The operator $L_{m,n}$ arises in various areas such as quantum mechanics, one-dimensional quantum graphs, and nanostructures. This operator models particles moving in one dimension under localized "spikes" of potential at points x_k.

These models are used when the potential is extremely localized compared to the particle's wavelength, for example, quantum dots or wells in semiconductors, electron tunneling through atomic impurities, or modeling defects in nanowires or quantum waveguides. The terms involving $\delta(x-x_k)$ and $\delta'(x-x_k)$ represent singular perturbations of the underlying differential operator. Here, $\delta(x-x_k)$ denotes the Dirac delta distribution concentrated at the point x_k, and the term $\alpha_k\delta(x-x_k)f(x)$ describes a point (delta-type) interaction at $x=x_k$ with a coupling constant α_k.

Operators of the form $L_{m,n}$ are used as building blocks for quantum graphs-networks of one-dimensional wires joined at vertices. At each vertex δ or δ'-type conditions model vertex coupling, which determines how wave functions transmit or reflect between edges. It is also relevant in scattering theory that involves highly localized perturbations. Therefore, we focus on its spectral analysis.

The spectral theory of Sturm–Liouville operators has been extensively developed over the past decades, with particular emphasis on direct and inverse spectral problems, non-self-adjoint operators, and operators with almost periodic coefficients. Classical results of Gasymov [11] established the spectral analysis of a class of non-self-adjoint differential operators with complex periodic potentials, while Kadiev [13] investigated Schrödinger operators containing a finite number of distributional perturbations. The inverse problem for half-line Schrödinger operators was studied by Aktosun [1], providing constructive methods based on the Jost function that have become fundamental in inverse spectral theory. More recently, uniqueness results for inverse Dirac problems have also been obtained in [12, 14], demonstrating the continuing development of inverse spectral methods for differential operators.

In recent years, increasing attention has been devoted to Sturm–Liouville operators with impulsive conditions and almost periodic potentials. Spectral properties of impulsive Sturm–Liouville operators were investigated in [3], while the case of discontinuous operators with almost-periodic potentials was analysed in [5]. Further developments include operators with complex periodic coefficients [6], almost periodic operators with impulse [7], multiple impulse points [8], and quadratic operator pencils with impulses [2, 10]. Spectral and inverse problems for non-self-adjoint Sturm–Liouville operators have also been considered in [9].

Motivated by these advances, the present paper investigates the spectral structure of almost-periodic Sturm–Liouville operators with a finite number of δ- and δ'-type distributional interactions, extending several known results to this more general setting. In connection with important applications in quantum mechanics and solid-state physics, it is of significant interest to investigate the spectral characteristics of the operator $L_{m,n}$, which models a quantum particle moving in an almost periodic medium with finitely many localized singular interactions. Such operators naturally arise in the study of quasicrystals, nanostructures, quantum graphs, and scattering processes involving highly localized defects. As a rule, the problem under consideration is closely related to discontinuities and interface effects in the physical properties of a medium.

The spectral theory of the operator $L_{m,n}$ has been studied by many authors. Spectral characteristics for this operator, with a real-valued function $q(x)$ satisfying the condition

$$\int_{-\infty}^{\infty} (1+x^2)q(x)dx < \infty,$$

was considered in [1]. The inverse problem for this operator, satisfying the condition

$$\int_{-\infty}^{\infty} (1+|x|)|q(x)|dx < \infty,$$

where $q(x)$ is a real–valued function, whose value is zero for negative arguments, was considered in [2]. Also, spectral analysis of a non-self-adjoint operator pencil with periodic potentials was studied in [3], and with a Dirac delta function was studied in [4]. Parallel to this development, Sturm–Liouville operators with almost periodic potentials have attracted sustained interest due to their relevance in solid-state physics, quasicrystals, and aperiodic media. The cases $\alpha_k = 0$ and $\beta_{k-m} = 0$ were considered in [5], and in this paper, we study the operator with almost periodic potentials and the Dirac delta and Dirac delta prime functions.

In Section 2, the particular solutions are constructed, and in Section 3, the spectral analysis of the operator is studied.

2. Particular solutions to the equation $(L_{m,n})y(x) = \lambda^2 y(x)$

In this section, we construct the fundamental solutions of the differential equation $(L_{m,n}y)(x) = \lambda^2 y(x)$ and analyze their analytic and asymptotic properties in the presence of almost periodic potentials and finitely many distributional interactions.

A solution of the system

$$-y''(x) + q(x)\,y(x) = \lambda^2 y(x) \ \text{ for } x \in (x_k, x_{k+1}), \ k = \overline{0, m+n}, \tag{3}$$

$$y(x_k+0) = y(x_k-0) = y(x_k), \ y'(x_k+0) - y'(x_k-0) = \alpha_k y(x_k), \ k = \overline{1, m}, \tag{4}$$

$$\begin{gathered} y'(x_k+0) = y'(x_k-0) = y'(x_k), y(x_k+0) - y(x_k-0) = \beta_{k-m} y'(x_k), \\ k = \overline{m+1, m+n}, \end{gathered} \tag{5}$$

is called a solution of the equation

$$(L_{m,n}y)(x) = \lambda^2 y(x).$$

According to [5], for problem (3)-(5), the following theorem is satisfied:

Theorem 1. *The problem* (3)-(5) *with potential* $q(x)$ *in the form of* (1) *has fundamental solutions of the form*

$$f^{\pm}(x,\lambda) = e^{\pm i\lambda x}\left(1 + \sum_{n=1}^{\infty}\sum_{\alpha=n}^{\infty}\frac{V_{n\alpha}}{\Lambda_n \pm 2\lambda}e^{i\Lambda_\alpha x}\right).$$

The numbers $V_{n\alpha}$ *are determined by the following relations*

$$\Lambda_\alpha\left(\Lambda_\alpha - \Lambda_n\right)V_{n\alpha} + \sum_{\beta\oplus\gamma=n} V_{n\beta}q_\gamma = 0,$$

$$q_\alpha + \sum_{\beta\oplus\gamma=n} V_{n\beta}q_\gamma = 0,$$

and the series

$$\sum_{n=1}^{\infty}\Lambda_n^{-1}\sum_{\alpha=n}^{\infty}\Lambda_\alpha\left(\Lambda_\alpha - \Lambda_n\right)|V_{n\alpha}|,$$

converges.

Proof. It follows that the functions $f^{\pm}(x,\lambda)$ have simple poles at the points $\lambda = \mp\frac{\Lambda_n}{2}$, $n \in \mathbb{N}$. The functions $f^{+}(x,\lambda)$, $f^{-}(x,\lambda)$ are linearly independent, since their Wronskian is

$$W\left[f^{+}(x,\lambda),\, f^{-}(x,\lambda)\right] = -2i\lambda \text{ for } \lambda \neq 0, \pm\frac{\Lambda_n}{2}.$$

If $\lambda \neq -\frac{\Lambda_n}{2}$ and $\operatorname{Im}\lambda > 0$, then $f^{+}(x,\lambda) \in L^2(0,\infty)$. If $\lambda \neq \frac{\Lambda_n}{2}$ and $\operatorname{Im}\lambda > 0$, then $f^{-}(x,\lambda) \in L^2(-\infty,0)$. Taking into account that the potential $q(x)$ can be extended to the upper semiplane as an analytic function and

$$\lim_{\operatorname{Im} x\to\infty} f^{+}(x,\lambda)\,e^{-i\lambda x} = 1 \quad \text{for } \operatorname{Im}\lambda > 0,$$

$$\lim_{\operatorname{Im} x\to-\infty} f^{-}(x,\lambda)\,e^{i\lambda x} = 1 \quad \text{for } \operatorname{Im}\lambda > 0.$$

Since

$$W\left[f_n^{\pm}(x), f^{\mp}\left(x, \mp\frac{\Lambda_n}{2}\right)\right] = 0,$$

we obtain

$$f_n^{\pm}(x) = S_n^{\pm} f^{\mp}\left(x, \mp\frac{\Lambda_n}{2}\right). \tag{6}$$

Comparison of these relations yields

$$S_n^{\pm} = V_{nn}^{\pm}.$$

For the derivatives of the considered functions, the following relation can be derived from equation (6) by taking derivatives of both sides

$$f_n^{\pm\prime}(x) = S_n^{\pm} f^{\mp\prime}\left(x, \mp\frac{\Lambda_n}{2}\right).$$

The general solution of equation (3) $y(x, \lambda)$ can be expressed in the form

$$y(x, \lambda) = C_1 f^{+}(x, \lambda) + C_2 f^{-}(x, \lambda),$$

where C_1, C_2 are arbitrary constants. It follows that any solution of the problem (3) - (5) can be written as follows: for $-\infty < x < x_1$

$$y(x, \lambda) = C_0 f^{+}(x, \lambda) + C_1 f^{-}(x, \lambda),$$

for $x_k < x < x_{k+1}$

$$y(x, \lambda) = C_{2k} f^{+}(x, \lambda) + C_{2k+1} f^{-}(x, \lambda),$$

for $x_{m+n} < x < \infty$

$$y(x, \lambda) = C_{2(m+n)} f^{+}(x, \lambda) + C_{2(m+n)+1} f^{-}(x, \lambda),$$

where C_k $\left(k = \overline{0, 2(m+n)+1}\right)$ are constants chosen such that $y(x, \lambda)$ satisfies the given conditions. ◄

We define an operator $L_{m,n}$ in the Hilbert space $L^2(-\infty, \infty)$ generated by differential expression $(L_{m,n} f)$. The domain of the operator $L_{m,n}$ is the set of functions that belong to $L^2(-\infty, \infty)$ and satisfy the system (3)-(5). We denote the domain by $D(L_{m,n})$. We assume that the spectral characteristics of the operator $L_{m,n}$ coincide with those of equation (3).

3. Spectrum of the operator $L_{m,n}$

To analyze the spectrum of equation (3), it is necessary to construct its resolvent operator. Consider the non-homogeneous differential equation

$$\begin{cases} -y''(x) + q(x)\, y(x) = \lambda^2 y(x) - f(x), \quad x \neq x_i \ \left(i = \overline{1, m+n}\right), \\ y(x_i + 0) = y(x_i - 0) = y(x_i), \\ y'(x_k + 0) - y'(x_k - 0) = \alpha_k y(x_k) \quad \left(i = \overline{1, m}\right), \\ y'(x_i + 0) = y'(x_i - 0) = y'(x_i), \\ y(x_i + 0) - y(x_i - 0) = \beta_{i-m} y'(x_i) \quad \left(i = \overline{m+1, m+n}\right), \end{cases} \tag{7}$$

where $f(x)$ is an arbitrary function belonging to $L^2(-\infty, +\infty)$. We seek a solution of equation (7) in the form:

$$y(x,\lambda) = \begin{cases} C_0(x) f^+(x,\lambda) + C_1(x) f^-(x,\lambda) & \text{if } x \in (-\infty, x_1), \\ C_{2k}(x) f^+(x,\lambda) + C_{2k+1}(x) f^-(x,\lambda) & \text{if } x \in (x_k, x_{k+1}), \\ C_{2(m+n)}(x) f^+(x,\lambda) + C_{2(m+n)+1}(x) f^-(x,\lambda) & \text{if } x \in (x_{m+n}, +\infty), \end{cases}$$

where $f^+(x,\lambda)$ and $f^-(x,\lambda)$ are linearly independent solutions of equation (3), $k = \overline{1, m+n-1}$ and $C_j(x)$, $\left(j = \overline{0, 2(m+n)+1}\right)$ are unknown functions to be determined.

To determine the functions $C_j(x)$, $\left(j = \overline{0, 2(m+n)+1}\right)$ we employ the classical method of variation of parameters. Specifically, we solve the following system of linear equations to find ${C'}_{2k}(x)$, ${C'}_{2k+1}(x)$, $x \in (x_k, x_{k+1})$ $\left(k = \overline{1, m+n}\right)$.

$$\begin{cases} {C'}_{2k}(x)\, f^+(x,\lambda) + {C'}_{2k+1}(x)\, f^-(x,\lambda) = 0, \\ {C'}_{2k}(x)\, {f^+}'(x,\lambda) + {C'}_{2k+1}(x)\, {f^-}'(x,\lambda) = f(x), \end{cases} \tag{8}$$

for $k = \overline{0, m+n}$.

Since the Wronskian

$$W\left[f^+ f^-\right] = \left[f^+(x,\lambda), f^-(x,\lambda)\right] = -2i\lambda \neq 0,$$

linear system of equations (8) admits a unique solution. Applying Cramer's rule, the derivatives of the unknown functions can be expressed as

$${C'}_{2k}(x) = -\frac{f^-(x,\lambda)\, f(x)}{W\left[f^+ f^-\right]}, \quad {C'}_{2k+1}(x) = \frac{f^+(x,\lambda)\, f(x)}{W\left[f^+ f^-\right]}.$$

Integration then yields

$$C_{2k}(x) = -\frac{1}{W[f^+ f^-]}\left(\int\limits_{x_k}^{x} f^-(t,\lambda) f(t)\,dt + C_{2k}\right),$$

$$C_{2k+1}(x) = \frac{1}{W[f^+ f^-]}\left(\int\limits_{x_{k+1}}^{x} f^+(t,\lambda) f(t)\,dt + C_{2k+1}\right),$$

$x \in (x_k, x_{k+1})$, where C_{2k}, C_{2k+1} are arbitrary constants and $k = \overline{0, m+n}$.

Note that $x_0 = -\infty$, $x_{m+n+1} = +\infty$. Consequently, the solution to equation (8) can be expressed in the form:

$$y(x,\lambda) = -\frac{1}{W[f^+ f^-]}\begin{cases} f^+(x,\lambda)\int\limits_{-\infty}^{x} f^-(t,\lambda) f(t)\,dt + f^-(x,\lambda)\int\limits_{x}^{x_1} f^+(t,\lambda) f(t)\,dt + \\ +C_0 f^+(x,\lambda) + C_1 f^-(x,\lambda), \ -\infty < x < x_1; \\ f^+(x,\lambda)\int\limits_{x_k}^{x} f^-(t,\lambda) f(t)\,dt + f^-(x,\lambda)\int\limits_{x}^{x_{k+1}} f^+(t,\lambda) f(t)\,dt + \\ +C_{2k} f^+(x,\lambda) + C_{2k+1} f^-(x,\lambda), \ x_k < x < x_{k+1}; \\ f^+(x,\lambda)\int\limits_{x_{m+n}}^{x} f^-(t,\lambda) f(t)\,dt + f^-(x,\lambda)\int\limits_{x}^{+\infty} f^+(t,\lambda) f(t)\,dt + \\ +C_{2(m+n)} f^+(x,\lambda) + C_{2(m+n)+1} f^-(x,\lambda), \ x_{m+n} < x < +\infty, \end{cases}$$

for $k = \overline{1, m+n+1}$.

Since $y(x,\lambda) \in L^2(-\infty,+\infty)$, $f^+(x,\lambda) \in L^2(0,\infty)$ and $f^-(x,\lambda) \in L^2(-\infty,0)$, $C_0 = C_{2(m+n)+1} = 0$. Then the general solution of equation (3) takes the form:

$$\begin{aligned} y(x,\lambda) = &-\frac{1}{W[f^+, f^-]}\int_{-\infty}^{\infty} G(x,t,\lambda) f(t)\,dt \\ &-\frac{1}{W[f^+, f^-]}\begin{cases} C_1 f^-(x,\lambda), & -\infty < x < x_1, \\ C_{2k} f^+(x,\lambda) + C_{2k+1} f^-(x,\lambda), & x_k < x < x_{k+1}, \\ C_{2(m+n)} f^+(x,\lambda), & x_{m+n} < x < +\infty, \end{cases} \end{aligned}$$

where $G(x,t,\lambda) = \begin{cases} f^+(x,\lambda) f^-(t,\lambda), \ t \le x; \\ f^+(t,\lambda) f^-(x,\lambda), \ x \le t, \end{cases}$ and $C_j \left(j = \overline{1, 2(m+n)}\right)$ are arbitrary constants and $k = 1, \ldots, m+n-1$.

To determine the constants $C_j \left(j = \overline{1, 2(m+n)}\right)$, we apply conditions (4) and (5). This yields the following system of equations:

$$\left\{\begin{array}{l}
-C_1 f^-(x_1,\lambda)+C_2 f^+(x_1,\lambda)+C_3 f^-(x_1,\lambda)=0;\\
-C_1\left(f^-(x_1,\lambda)+\alpha_1 f^{-\prime}(x_1,\lambda)\right)+C_2 f^{+\prime}(x_1,\lambda)+C_3 f^{-\prime}(x_1,\lambda)\\
\quad =\alpha_1\int\limits_{-\infty}^{\infty}G(x_1,t,\lambda)f(t)\,dt;\\
\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\\
-C_{2m-2}f^+(x_m,\lambda)-C_{2m-1}f^-(x_m,\lambda)+C_{2m}f^+(x_m,\lambda)+\\
+C_{2m+1}f^-(x_m,\lambda)=0;\\
-C_{2m-2}\left(f^{+\prime}(x_m,\lambda)+\alpha_m f^+(x_m,\lambda)\right)-\\
-C_{2m-1}\left(f^{-\prime}(x_m,\lambda)+\alpha_m f^-(x_m,\lambda)\right)+\\
\quad +C_{2m}f^{+\prime}(x_m,\lambda)+C_{2m+1}f^{-\prime}(x_m,\lambda)=\alpha_m\int\limits_{-\infty}^{\infty}G(x_m,t,\lambda)f(t)\,dt;\\
-C_{2m}f^{+\prime}(x_{m+1},\lambda)-C_{2m+1}f^{-\prime}(x_{m+1},\lambda)\\
\quad +C_{2m+2}f^{+\prime}(x_{m+1},\lambda)+C_{2m+3}f^{-\prime}(x_{m+1},\lambda)=0;\\
-C_{2m}\left(f^+(x_{m+1},\lambda)+\beta_1 f^{+\prime}(x_{m+1},\lambda)\right)-\\
-C_{2m+1}\left(f^-(x_{m+1},\lambda)+\beta_1 f^{-\prime}(x_{m+1},\lambda)\right)+\\
\quad +C_{2m+2}f^+(x_{m+1},\lambda)+C_{2m+3}f^-(x_{m+1},\lambda)=\\
=\beta_1\int\limits_{-\infty}^{\infty}G'(x_{m+1},t,\lambda)f(t)\,dt;\\
\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\\
-C_{2(m+n)-2}f^{+\prime}(x_{m+n},\lambda)-C_{2(m+n)-1}f^{-\prime}(x_{m+n},\lambda)+\\
+C_{2(m+n)}f^{+\prime}(x_{m+n},\lambda)=0;\\
-C_{2(m+n)-2}\left(f^+(x_{m+n},\lambda)+\beta_n f^{+\prime}(x_{m+n},\lambda)\right)-C_{2(m+n)-1}\left(f^-(x_{m+n},\lambda)\right.\\
\left.+\beta_n f^{-\prime}(x_{m+n},\lambda)\right)+C_{2(m+n)}f^+(x_{m+n},\lambda)=\beta_n\int\limits_{-\infty}^{\infty}G'(x_{m+n},t,\lambda)f(t)\,dt.
\end{array}\right. \tag{9}$$

Furthermore, by employing the representations,

$$Q_k(f) = \begin{cases} \int\limits_{-\infty}^{\infty} G(x_k, t, \lambda) f(t)\, dt & \left(k = \overline{1, m}\right); \\ \int\limits_{-\infty}^{\infty} G'(x_k, t, \lambda) f(t)\, dt & \left(k = \overline{m+1, m+n}\right), \end{cases} \tag{10}$$

$$Z_k = \begin{cases} \alpha_k & \left(k = \overline{1, m}\right); \\ \beta_{k-m} & \left(k = \overline{m+1, m+n}\right), \end{cases}$$

and denoting by $M_{m+n}(\lambda)$ the $2(m+n)$ by $2(m+n)$ matrix composed of the coefficients of the unknown constants $C_1, C_2, ..., C_{2(m+n)}$ in system (10), the latter can be succinctly rewritten in matrix form as

$$M_{m+n}(\lambda)\, C = ZQ,$$

where $C = \left(C_1, C_2, ..., C_{2(m+n)}\right)^T$, $ZQ = (0, Z_1Q_1, 0, Z_2Q_2, ..., 0, Z_{m+n}Q_{m+n})^T$.

Let us denote $\det(M_{m+n}(\lambda))$ by $D(\lambda)$ and define the set $\Gamma = \{\lambda : \operatorname{Im} \lambda \neq 0, D(\lambda) = 0\}$. For $\lambda \notin \Gamma$, the constants C_j are given by:

$$C_j = \frac{1}{D(\lambda)} \sum_{k=1}^{m+n} Z_k M_{m+n}^{2k,j} Q_k,$$

where $M_{m+n}^{i,j}$ is the algebraic cofactor m_{ij} of the matrix

$$M_{m+n}(\lambda) = (m_{ij})_{2(m+n)\times 2(m+n)}.$$

Consider the following representation:

$$Y_k(x, \lambda) = \begin{cases} Z_1 M_{m+n}^{2k,1}(\lambda) f^-(x, \lambda), & x \in (-\infty, x_1); \\ Z_j \left[M_{m+n}^{2k,2j-2}(\lambda) f^+(x, \lambda) + M_{m+n}^{2k,2j-1}(\lambda) f^-(x, \lambda)\right], & x \in (x_j, x_{j+1}); \\ Z_{m+n} M_{m+n}^{2k,2(m+n)}(\lambda) f^+(x, \lambda), & x \in (x_{m+n}, +\infty), \end{cases}$$

for $k = \overline{1, m+n}$ and $j = \overline{1, m+n-1}$. Using the above expressions, the solution to equation (3) can finally be written as:

$$\begin{aligned} \left(R_\lambda^{\langle m,n\rangle} f\right)(x) &\equiv y(x, \lambda) \\ &= -\frac{1}{W[f^+ f^-]} \left[\int\limits_{-\infty}^{\infty} G(x, t, \lambda) f(t)\, dt + \frac{1}{D(\lambda)} \sum_{k=1}^{m+n} Y_k(x, \lambda) Q_k(f)\right], \end{aligned} \tag{11}$$

where $Y_k(x,\lambda) \in L^2(-\infty,\infty)$ $\left(k=\overline{1,m+n}\right)$, $\lambda \neq 0$, $\lambda \notin \Gamma$ and $Q_k(f)$ are defined by formula (10).

From expression (11), it follows that the associated operator $R_\lambda^{\langle m,n\rangle} - R_\lambda$ is finite-dimensional, with rank at most $m+n$.

Theorem 2. *The eigenvalues of the operator $L_{m,n}$ are the squares of the zeros of the function $D(\lambda)$, and their number does not exceed $m+n$.*

Proof. Let $R_\lambda^{\langle m,n\rangle}$ denote the resolvent of the operator $L_{m,n}$ and let R_λ denote the resolvent of the corresponding unperturbed almost periodic Sturm–Liouville operator. According to the resolvent representation (11), we have

$$R_\lambda^{\langle m,n\rangle} f = R_\lambda f - \frac{1}{W[f^+,f^-]\,D(\lambda)} \sum_{k=1}^{m+n} Y_k(x,\lambda) Q_k(f). \tag{12}$$

Hence

$$R_\lambda^{\langle m,n\rangle} - R_\lambda = -\frac{1}{W[f^+,f^-]\,D(\lambda)} \sum_{k=1}^{m+n} Y_k(x,\lambda) Q_k(\cdot),$$

which is a finite-rank operator whose range is contained in span $\{Y_1(\cdot,\lambda),\ldots, Y_{m+n}(\cdot,\lambda)\}$. Therefore, rank $\left(R_\lambda^{\langle m,n\rangle} - R_\lambda\right) \leq m+n$.

Suppose now that $D(\lambda_0)=0$. Then the matrix $M_{m+n}(\lambda_0)$ is singular, and consequently the homogeneous algebraic system

$$M_{m+n}(\lambda_0)B = 0,$$

admits a non-trivial solution

$$B = (B_1, B_2, \ldots, B_{2(m+n)})^T \neq 0.$$

Define

$$y(x,\lambda_0) = \begin{cases} B_1 f^-(x,\lambda_0), & x \in (-\infty, x_1), \\ B_{2k} f^+(x,\lambda_0) + B_{2k+1} f^-(x,\lambda_0), & x \in (x_k, x_{k+1}), \\ B_{2(m+n)} f^+(x,\lambda_0), & x \in (x_{m+n}, +\infty). \end{cases}$$

Since the coefficients satisfy $M_{m+n}(\lambda_0)B=0$, the above function satisfies all interface conditions at the points $x_1,\ldots,x_{m+n}$. Moreover, for $\operatorname{Im}\lambda_0 > 0$,

$$f^+(\cdot,\lambda_0) \in L^2(0,\infty), \qquad f^-(\cdot,\lambda_0) \in L^2(-\infty,0),$$

and therefore $y(\cdot, \lambda_0) \in L^2(\mathbb{R})$. Thus,

$$L_{m,n} y = \lambda_0^2 y,$$

showing that λ_0^2 is an eigenvalue.

Conversely, let λ_0^2 be an eigenvalue of $L_{m,n}$ and let $y \neq 0$ be a corresponding eigenfunction. On every interval (x_k, x_{k+1}), the differential equation admits the representation

$$y = C_1 f^+ + C_2 f^-.$$

Since $y \in L^2(\mathbb{R})$, the coefficients of the non-square integrable solutions in the exterior intervals must vanish. Consequently, y has exactly the piecewise form given above. Substituting this representation into the interface conditions yields

$$M_{m+n}(\lambda_0) B = 0.$$

Because $B \neq 0$, we conclude that $\det M_{m+n}(\lambda_0) = D(\lambda_0) = 0$. Hence,

$$\lambda_0^2 \text{ is an eigenvalue } \iff D(\lambda_0) = 0.$$

Finally, the eigenspace is generated by vectors contained in the range of the finite-rank perturbation. Therefore its dimension does not exceed $m+n$. Consequently, the total number of eigenvalues (counted without multiplicity) cannot exceed $m+n$. This completes the proof. ◀

Theorem 3. *The continuous spectrum of the operator $L_{m,n}$ consists of the positive half – axis $[0, \infty)$ and spectral singularities may occur at points $\left(\frac{\Lambda_n}{2}\right)^2$ on this axis.*

Proof. From formula (12), $R_\lambda^{\langle m,n \rangle} - R_\lambda$ is a finite-rank operator. Since every finite-rank operator is compact, the resolvent difference is compact. Therefore, by Weyl's theorem on the invariance of the essential spectrum under compact perturbations,

$$\sigma_{\text{ess}}(L_{m,n}) = \sigma_{\text{ess}}(L_0),$$

where

$$L_0 = -\frac{d^2}{dx^2} + q(x),$$

denotes the corresponding unperturbed almost periodic Sturm–Liouville operator. Since $\sigma_{\text{ess}}(L_0) = [0, \infty)$, it follows that $\sigma_{\text{ess}}(L_{m,n}) = [0, \infty)$.

It remains to be determined what type of spectrum it is.

Let $\lambda \in \mathbb{R}$, $\lambda \neq \frac{\Lambda_n}{2}$. Then every solution of equation (3) has the form

$$y(x, \lambda) = C_1 f^+(x, \lambda) + C_2 f^-(x, \lambda).$$

The principal parts of the fundamental solutions are $e^{i\lambda x}, e^{-i\lambda x}$, which are oscillatory and do not decay as $|x| \to \infty$. The remaining almost periodic factors are bounded. Hence no non-trivial solution belongs to $L^2(\mathbb{R})$. Therefore,

$$\sigma_p(L_{m,n}) \cap [0, \infty) = \varnothing.$$

Next, we show that the residual spectrum is empty. Indeed, for every densely defined closed operator

$$\sigma_r(L) = \left\{\lambda : \overline{\lambda} \in \sigma_p(L^*)\right\}.$$

The adjoint operator satisfies the same differential equation together with the adjoint interface conditions. Repeating the previous argument shows that the adjoint operator has no real eigenvalues. Consequently,

$$\sigma_r(L_{m,n}) = \varnothing.$$

Hence, every point of $[0, \infty)$ belongs to the continuous spectrum.

Finally, from the representation

$$f^{\pm}(x, \lambda) = e^{\pm i\lambda x} \left(1 + \sum_{n=1}^{\infty} \sum_{\alpha=n}^{\infty} \frac{V_{n\alpha}^{\pm}}{\Lambda_n \pm 2\lambda} e^{i\Lambda_\alpha x}\right),$$

the only possible singularities occur when $\Lambda_n \pm 2\lambda = 0$, that is, $\lambda = \pm\frac{\Lambda_n}{2}$. Consequently, in terms of the spectral parameter $\mu = \lambda^2$, possible singularities occur only at

$$\mu = \left(\frac{\Lambda_n}{2}\right)^2.$$

Since no square-integrable eigenfunctions exist for real values of λ, these points cannot be eigenvalues and may only appear as spectral singularities embedded in the continuous spectrum. Therefore, $\sigma_c(L_{m,n}) = [0, \infty)$, and possible spectral singularities occur only at $\left(\frac{\Lambda_n}{2}\right)^2$, $n \in \mathbb{N}$. The proof is complete. ◄

4. Conclusion

In this paper, we have investigated the spectral properties of the almost periodic Sturm–Liouville operator $L_{m,n}$ on the real axis in the presence of finitely many singular perturbations of δ- and δ'-type. By constructing fundamental solutions and analyzing their analytic and asymptotic behavior, we established a framework for studying such operators with distributional interactions. Using the method of variation of parameters, we derived an explicit representation of the resolvent operator and demonstrated that the perturbation induced by the point interactions is of finite rank. This result plays a crucial role in characterizing the spectral structure of the operator.

In particular, we proved that the eigenvalues are determined by the zeros of the characteristic determinant $D(\lambda)$, and their total number does not exceed $m+n$. Furthermore, we showed that the continuous spectrum coincides with the positive half-axis $[0,\infty)$, while possible spectral singularities may occur only at specific points related to the almost periodic structure of the potential. These findings extend classical spectral theory to a broader class of Sturm–Liouville operators that incorporate both almost periodic potentials and finitely many localized singularities. The results obtained in this work provide a rigorous mathematical foundation for the analysis of models arising in quantum mechanics, nanostructures, and related fields, in which localized defects and almost-periodic media play a significant role. Future research may focus on inverse spectral problems, stability analysis, and extensions to more general classes of distributional perturbations.

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