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Approximation of Analytic Functions on Parabolic Riemann Surfaces

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Abstract. This paper introduces a definition of polynomials on parabolic Riemann surfaces based on a special exhaustion function. We investigate approximation problems for analytic functions on parabolic Riemann surfaces that admit rich families of polynomials. For polynomially convex compact sets, an analogue of the classical Runge theorem is established.

Key Words and Phrases: parabolic Riemann surface, Evans-Selberg potential, polynomials, polynomially convex sets, regular parabolic Riemann surfaces.

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1. Introduction

The potential-theoretic and conformal classification of Riemann surfaces into elliptic, parabolic, and hyperbolic types is a classical topic in complex analysis [2, 7]. Parabolic Riemann surfaces form a particularly subtle class: they are non-compact, but they do not admit a Green function with a pole, and they possess no non-constant negative subharmonic functions [2, 8]. These properties impose strong restrictions on the behavior of harmonic and holomorphic functions, thereby distinguish parabolic surfaces sharply from hyperbolic ones.

The structure of parabolic manifolds has been extensively studied in various settings, including strictly parabolic manifolds [8], S-parabolic and parabolic Stein manifolds [9, 10, 11], and more recently in connection with splitting phenomena for parabolic manifolds [10]. In the case of analytic surfaces, the potential theory of parabolicity can be described through the behavior of Green-type functions and related extremal objects [5]. From this point of view, the Evans–Selberg potential plays a central role: for an open Riemann surface R_0, the existence of such a potential is equivalent to parabolicity and provides an effective substitute for the Green function [2, 5, 8, 11]. The Evans–Selberg potential is harmonic outside a single pole, has a logarithmic singularity at that point, and tends to $+\infty$ at the Aleksandrov boundary; hence it yields a natural exhaustion of the surface by compact sublevel sets.

This potential-theoretic viewpoint suggests a way to generalize the classical notion of polynomials from $\mathbb{C}$ to parabolic Riemann surfaces. Using a suitable Evans–Selberg type exhaustion function ρ on a parabolic surface R_0, one can define polynomial-type holomorphic functions as those whose growth is bounded by a fixed power of $1 + e^{\rho(z)}$. This approach was systematically developed in the context of parabolic and Stein manifolds in [1, 4, 11], where the resulting space of such functions is denoted by $\mathcal{P}_\rho(R_0)$. In favorable geometric situations—such as analytic graphs, algebraic manifolds, and their complements—these polynomial spaces turn out to be rich and are dense in the space $\mathcal{O}(R_0)$ of holomorphic functions [1, 3, 4]. On the other hand, there also exist parabolic Riemann surfaces on which every polynomial in this sense is constant [1], showing that the polynomial structure can be extremely poor in general.

These phenomena motivate the introduction of *regular* parabolic Riemann surfaces, namely those parabolic surfaces (R_0, ρ) for which the polynomial space $\mathcal{P}_\rho(R_0)$ is dense in $\mathcal{O}(R_0)$ [1, 3, 4]. On such surfaces it is natural to develop an approximation theory by elements of $\mathcal{P}_\rho(R_0)$, in close analogy with the classical theory of polynomial approximation in $\mathbb{C}^n$. In this context, the notions of polynomial hull and polynomial convexity with respect to $\mathcal{P}_\rho(R_0)$ become central objects. The notion of polynomial convexity is also closely related to extremal plurisubharmonic functions, capacities, and Bernstein–Walsh type estimates in several complex variables [13, 14, 15], which underlie many results in polynomial approximation and algebraicity criteria [12, 13].

The aim of the present paper is to develop a Runge-type approximation theory on regular parabolic Riemann surfaces. We first recall the construction of polynomial-type functions on a parabolic Riemann surface (R_0, ρ) and discuss examples where the space $\mathcal{P}_\rho(R_0)$ is trivial or dense. We then introduce polynomially convex compact sets with respect to $\mathcal{P}_\rho(R_0)$ and establish a separation theorem characterizing polynomial convexity in terms of polynomials that are uniformly small on a compact set and large at an exterior point. Finally, using a finite system of such separating polynomials, we construct a holomorphic mapping of R_0 into a unit polydisc in $\mathbb{C}^N$ whose image contains a closed analytic set. By applying holomorphic extension and approximation results from the theory of Stein manifolds and analytic sets [6], we obtain a Runge-type theorem: every function holomorphic in a neighborhood of a polynomially convex compact set $K \subset R_0$ can be uniformly approximated on K by polynomials from $\mathcal{P}_\rho(R_0)$. Thus, the classical Runge approximation theorem is extended to the setting of regular parabolic Riemann surfaces endowed with an Evans–Selberg exhaustion.

2. Parabolic Riemann surface and the Evans–Selberg potential

Definition 1. *A Riemann surface R is called* parabolic *if there exist no non-constant negative subharmonic functions on R. All open Riemann surfaces that are not parabolic are referred to as* hyperbolic*.*

There are several alternative characterizations of parabolic Riemann surfaces, some of which appear in the following statements.

Theorem 1. *(see:[2]) A Riemann surface is parabolic if any one of the following conditions holds:*

1. *The maximum principle is valid on the surface.*
2. *The Green function does not exist.*

One of the important aspects in the characterization of parabolic Riemann surfaces is the existence of a special potential function.

Let R_0 be an open Riemann surface. A function $h(z)$ satisfying the following three properties is called an Evans–Selberg potential on R_0 with a pole at a point $\xi \in R_0$:

(a) The function h is harmonic on R_0 except at the point ξ.

(b) The function
$$h(z) + \log|z - \xi|$$
extends harmonically to a punctured neighborhood around ξ.

(c) When the point z approaches the Aleksandrov boundary point a_∞ of the open Riemann surface R_0, the function $h(z)$ tends to $+\infty$.

Theorem 2. *(see:[2]) For an open Riemann surface R_0 to be parabolic, it is necessary and sufficient that an Evans–Selberg potential exists on it.*

Thus, every parabolic Riemann surface R_0 admits a potential function $h(z)$ that is harmonic everywhere except at one point θ, and subharmonic on the entire surface. As $z \to \theta$, the function satisfies the asymptotic relation
$$h(z) \sim \ln|z - \theta|.$$

When the point z approaches the boundary of the parabolic Riemann surface R_0, we have $h(z) \to \infty$. This implies that for any number $c \in \mathbb{R}$, the set

$$\{z \in R_0 : h(z) < c\},$$

is compact.

The existence of such a special potential function $h(z)$ plays a fundamental role in the structure theory of parabolic Riemann surfaces.

Polynomial-type functions (polynomials) make it possible to describe simple functions on parabolic Riemann surfaces. Let R_0 be a parabolic Riemann surface and let $\rho(z)$ be the corresponding special potential function. We denote by $\mathcal{O}(R_0)$ the space of all holomorphic functions on R_0.

Definition 2. *For a function $f \in \mathcal{O}(R_0)$, suppose there exist positive constants c_f and d such that for all $z \in R_0$,*

$$|f(z)| \leq c_f\big(1 + e^{\rho(z)}\big)^d. \tag{1}$$

If inequality (1) holds, then f is called a polynomial *on R_0. Among all integers d satisfying (1), the smallest one is called the* degree *of the polynomial.*

We denote by $\mathcal{P}_\rho(R_0)$ the space of all polynomials on R_0 defined via the potential function ρ.

Every parabolic manifold admits only a limited family of such polynomials. On the other hand, some of the most important examples of parabolic manifolds—analytic graphs, algebraic surfaces, and their complements—have the property that their polynomial spaces are dense in the class of analytic functions (see [3, 4, 5]).

A. Sadullaev and A. Aytuna constructed an example of a parabolic Riemann surface whose polynomial space consists only of constant functions.

Theorem 3. *(see:[1]) Let $K \subset \mathbb{C}$ be a polar compact set, and let $u(z)$ be a subharmonic function on the complex plane such that: u is harmonic on $\mathbb{C}\setminus K$,and $u|_K = -\infty$, and moreover,*

$$\lim_{z \to K} \frac{u(z)}{\ln \operatorname{dist}(z, K)} = 0.$$

Then the stated relation holds.

Consider the manifold $W = \overline{\mathbb{C}} \setminus K$, where K is a compact set satisfying the condition of the previous theorem. Let us take the special function $\phi(z) = -u(z)$.

Then $\phi(z)$ is harmonic on $W \setminus \{\infty\}$, satisfies $\phi(\infty) = -\infty$, and as $z \to K$ we have $\phi(z) \to \infty$. Thus the surface $(W, \phi) = R_0$ is parabolic.

Any holomorphic function f on W must satisfy an estimate of the form

$$\ln |f(z)| \leq C + d\,\phi(z), \qquad d \in \mathbb{N},$$

which implies that f belongs to a polynomial class. Using Cauchy's integral formula and strong estimates, one proves that such functions must be trivial, i.e.,

$$f = \text{const}.$$

Therefore, W contains no nontrivial polynomials, and hence is a non-regular parabolic Riemann surface.

3. Polynomially convex sets on regular parabolic Riemann surfaces

Definition 3. *If $\overline{\mathcal{P}_\rho(R_0)} = \mathcal{O}(R_0)$, then the parabolic Riemann surface R_0 is called* regular.

Let (R_0, ρ) be a regular parabolic Riemann surface. For a compact set $K \subset R_0$, we define its polynomial hull $\widehat{K}$ by

$$\widehat{K} := \{z \in P_\rho : |P(z)| \leq \|P\|_K, \quad \forall P \in \mathcal{P}_\rho(R_0)\}.$$

Definition 4. *If a compact set $K \subset R_0$ satisfies $\widehat{K} = K$, then K is called a* polynomially convex set *with respect to $\mathcal{P}_\rho$.*

Polynomial convexity is an important structural property in the theory of polynomial approximation on regular parabolic Riemann surfaces.

Theorem 4. *Let (R_0, ρ) be a regular parabolic Riemann surface, and let $K \subset R_0$ be a compact set. The set K is polynomially convex with respect to $\mathcal{P}_\rho$ if and only if for every point $z^0 \in R_0 \setminus K$, there exists a polynomial $P(z) \in \mathcal{P}_\rho(R_0)$ such that*

$$\|P(z)\|_K \leq \frac{1}{2} \qquad \text{and} \qquad |P(z^0)| > 2.$$

Proof. **Necessity.** Assume that $K \subset R_0$ is polynomially convex. Then for each $z^0 \in R_0 \setminus K$, one can find a polynomial $Q(z)$ such that

$$\|Q(z)\|_K < |Q(z^0)|.$$

Define the function

$$T(z) = \frac{1}{\|Q(z)\|_K} Q(z).$$

Then

$$\|T(z)\|_K = 1 < |T(z^0)|.$$

Choose $m \in \mathbb{N}$ such that

$$|T(z^0)|^m > 4.$$

If we set

$$P(z) = \frac{1}{2} T(z)^m,$$

then this polynomial satisfies

$$\|P(z)\|_K \leq \frac{1}{2}, \qquad |P(z^0)| > 2,$$

which proves the necessity.

Sufficiency. This direction is immediate. If

$$\|P(z)\|_K \leq \frac{1}{2} \qquad \text{and} \qquad |P(z^0)| > 2,$$

then z^0 cannot lie in the polynomial hull $\widehat{K}$. Since this holds for all $z^0 \in R_0 \setminus K$, we obtain $\widehat{K} = K$. ◄

4. Main Result

The main result of this work is formulated below.

Theorem 5. *Let (R_0, ρ) be a regular parabolic Riemann surface, and let $K \subset R_0$ be a polynomially convex compact set. If a function f is holomorphic in a neighborhood of a polynomially convex compact set $K \subset R_0$, then for every $\varepsilon > 0$ there exists a polynomial*

$$P(z) \in \mathcal{P}_\rho(R_0),$$

such that

$$\|f(z) - P(z)\|_K < \varepsilon.$$

Proof. Let $K \subset R_0$ be a polynomially convex compact set, and let $U \supset K$ be a neighborhood in which the function $f \in \mathcal{O}(U)$ is holomorphic.

Then we can find a system of polynomials

$$P_1, P_2, P_3, \ldots, P_N,$$

such that

$$\|P_j\|_K < 1, \qquad j = 1, 2, \ldots, N,$$

and for every boundary point $\xi \in \partial U$ we have

$$\max_{1 \le j \le N} |P_j(\xi)| > 1.$$

This follows from applying Theorem 4. Indeed, if we choose finitely many boundary points

$$\xi_1, \xi_2, \ldots, \xi_n$$

that lie densely on ∂U, then Theorem 4 provides, for each such point ξ_s, a polynomial

$$P_{\xi_s}(z) \in \mathcal{P}_\rho(R_0), \qquad s = 1, 2, \ldots,$$

such that

$$\|P_{\xi_s}(z)\|_K \le \frac{1}{2} \qquad \text{and} \qquad |P_{\xi_s}(\xi_s)| > 2.$$

Using the sets

$$U_j = \big\{ z \in U : |P_{\xi_j}(z)| > 2 \big\},$$

we obtain an open covering of ∂U. Since ∂U is compact, a finite subcover

$$\{U_j\}, \qquad j = 1, 2, \ldots, n,$$

may be chosen.

Define

$$P_k(z) = P_{\xi_k}(z), \qquad k = 1, 2, \ldots, n.$$

Then the required inequalities above hold for this finite collection.

Now consider the mapping

$$\pi = (P_1, P_2, \ldots, P_N),$$

which is of the form

$$P : R_0 \longrightarrow \mathbb{C}^N_w.$$

(Here the argument continues with the analysis of this mapping.) ◄

We define

$$\Pi = \{ z \in R_0 : |P_1(z)| < 1, \ |P_2(z)| < 1, \ \ldots, \ |P_N(z)| < 1 \},$$

and assume

$$K \subset\subset U \subset R_0.$$

Let

$$W = \left\{ w = (w_1, w_2, \dots, w_N) \in \mathbb{C}^N_w : |w_1| < 1,\ |w_2| < 1,\ \dots,\ |w_N| < 1 \right\},$$

denote the unit polydisc in $\mathbb{C}^N$.

Consider the mapping

$$\pi : R_0 \longrightarrow \mathbb{C}^N_w, \qquad \pi(z) = (P_1(z), P_2(z), \dots, P_N(z)).$$

Then $\pi(\Pi) \subset W$, and its image contains a closed analytic surface

$$A \subset W.$$

If necessary, the number of polynomials may be increased so that π becomes a proper mapping.

Since $f \in \mathcal{O}_\rho(R_0)$, the function f may be regarded as the restriction to A of a holomorphic function defined on W. By the Oka–Cartan theorem on holomorphic continuation, there exists a holomorphic function

$$g(w) = g(w_1, w_2, \dots, w_N) \in \mathcal{O}(W),$$

such that

$$f(z) = g(\pi(z)).$$

We expand the function g into its absolutely convergent power series:

$$g(w) = \sum_{|k|=0}^{\infty} c_k\, w_1^{k_1} w_2^{k_2} \dots w_N^{k_N},$$

where

$$k = (k_1, k_2, \dots, k_N), \qquad k_j \in \{0, 1, 2, \dots\},$$

is a multi-index, and

$$|k| = k_1 + k_2 + \dots + k_N.$$

This series converges absolutely and uniformly on each compact subset of W. Since $K \subset\subset \Pi$, we have

$$\pi(K) \subset\subset W.$$

Thus

$$f(z) = g(\pi(z)) = \sum_{|k|=0}^{\infty} c_k\, P_1(z)^{k_1}\, P_2(z)^{k_2} \dots P_N(z)^{k_N}.$$

The series converges absolutely and uniformly on K. Hence, for every $\varepsilon > 0$ there exists an integer m such that

$$\left\| f(z) - \sum_{|k|=0}^{m} c_k \, P_1(z)^{k_1} P_2(z)^{k_2} \dots P_N(z)^{k_N} \right\|_K < \varepsilon.$$

If we define

$$P(z) = \sum_{|k|=0}^{m} c_k \, P_1(z)^{k_1} P_2(z)^{k_2} \dots P_N(z)^{k_N},$$

then $P(z)$ is a polynomial in the sense of $\mathcal{P}_\rho(R_0)$. Thus the theorem is fully proved.

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