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On the Existence of Solutions for a Fractional Differential Boundary Value Problem via Darbo-Type Fixed Point

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Abstract. In this work, we study the existence of solutions for a class of fractional boundary value problems involving the Caputo fractional derivative. Our approach is based on transforming the considered boundary value problems into an equivalent fixed-point equation by applying fixed-point results related to generalized Proinov-type contraction mappings defined through the measure of noncompactness. The obtained results substantially extend classical contraction principles and offer powerful tools for the analysis of nonlinear problems. An illustrative example is presented to demonstrate the significance and the applicability of the obtained results.

Key Words and Phrases: Caputo fractional derivative, generalized Proinov contraction, measure of noncompactness.

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1. Introduction

The aim of this article is to study the existence of solution to the boundary value problem (BVPs for short) for a nonlinear fractional differential equation:

$$ {}^{c}D_{0+}^{r}\chi(t) = \varphi(t, \chi(t)), \quad t \in I = [0, T], \tag{1} $$

$$ {}^{c}D_{0+}^{r-1}\chi(T) = \Phi(\chi), \tag{2} $$

$$ a\chi(0) - b\chi(T) = \Psi(\chi), \tag{3} $$

where $a, b \in R, (a < 0, b > 0)$, $1 < r < 2$, ${}^{c}D_{0+}^{r}$, is the Caputo derivative, $\varphi : I \times \mathbb{R} \to \mathbb{R}, \Phi : \mathcal{C}(I) \to \mathbb{R}$ and $\Psi : \mathcal{C}(I) \to \mathbb{R}$ are given continuous function.

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Fixed-point theory plays a central role in nonlinear analysis, with the Banach contraction principle and its numerous extensions providing powerful tools for establishing the existence and uniqueness of solutions. Recent developments have introduced several classes of generalized contractive mappings, including Proinov contractions, α-admissible mappings, and fixed-point principles formulated in terms of measures of noncompactness. At the same time, fractional differential equations involving the Caputo derivative have attracted considerable attention because of their broad range of applications, thereby motivating the study of the existence, uniqueness, and stability of solutions under various boundary conditions; see, for example, [1, 2, 4, 5, 8, 12, 13, 14, 15].

In [11], the boundary value problem $(1)-(3)$ was investigated by reformulating it as a fixed-point problem. Since the associated operator P, defined by $P\chi = \chi$, is difficult to show to be condensing, the authors decomposed it into the sum of two operators, $P\chi = F_1\chi + F_2\chi$. They proved that F_1 is completely continuous, whereas F_2 is α-Lipschitz, under relatively restrictive assumptions, including a Lipschitz condition. In the present paper, we establish the existence of solutions to the same problem under substantially weaker assumptions.

The remainder of this paper is organized as follows. In Section 2, we present several definitions and auxiliary lemmas that will be used to establish the existence of solutions to BVPs $(1) - (3)$. In Section 3, we prove the main existence theorem. Finally, in Section 4, we present a representative example illustrating the applicability of the obtained result.

2. Preliminaries

For the sake of clarity, we begin by recalling some fundamental concepts from fractional calculus which will be used in establishing the main results of this work (see [1] and [7]).

Let $\mathbb{E} = \mathbb{C}(I)$ denote the Banach space consisting of all continuous functions defined on I endowed with the sup-norm $\|\chi\|_\infty = \sup_{t\in I} |\chi(t)|$.

Definition 1 ([10])**.** *The Riemann–Liouville fractional integral of order $\mu > 0$ of a function $\varphi : (0, +\infty) \to \mathbb{R}$ is given by*

$$I_{0^+}^{\mu}\varphi(t) = \frac{1}{\Gamma(\mu)} \int_0^t (t-s)^{\mu-1} \varphi(s)\, ds,$$

where $\Gamma(\mu)$ denotes the Gamma function, provided that the integral on the right-hand side exists for every $t \in (0, +\infty)$.

Definition 2 ([10]). *The fractional derivative of order $\mu > 0$ in the Riemann–Liouville sense of a function $\varphi : (0, +\infty) \to \mathbb{R}$ can be expressed as*

$$D_{0^+}^{\mu} \varphi(t) = \frac{1}{\Gamma(n-\mu)} \frac{d^n}{dt^n} \int_0^t \frac{\varphi(s)}{(t-s)^{\mu-n+1}} ds,$$

where $n = [\mu] + 1$, provided that the right-hand side is well defined pointwise on $(0, +\infty)$. The notation $[\mu]$ indicates the integer part associated with the real number μ.

For $\mu < 0$, we adopt the convention $D_{0^+}^{\mu} \varphi(t) = I_{0^+}^{-\mu} \varphi(t)$. If $0 \leq \varsigma \leq \mu$, we get $D_{0^+}^{\varsigma} I_{0^+}^{\mu} \varphi(t) = I_{0^+}^{\mu-\varsigma} \varphi(t)$.

The following lemma describes the action of the Riemann–Liouville fractional integral and derivative operators on power functions. In particular, these operators transform the power function t^{ϱ} into another power function

$$I_{0^+}^{\mu} t^{\varrho} = \frac{\Gamma(\varrho+1)}{\Gamma(\varrho+\mu+1)} t^{\varrho+\mu}.$$

Also

$$D_{0^+}^{\mu} t^{\varrho} = \frac{\Gamma(\varrho+1)}{\Gamma(\varrho-\mu+1)} t^{\varrho-\mu}.$$

It is important to note that $D_{0^+}^{\mu} t^{\varrho} = 0$, whenever $\varrho = \mu - i$, where $i = 1, 2, 3, \ldots, n$ and n denotes the smallest integer satisfying $n \geqslant \mu$. In this case

$$\frac{d^n t^{\varrho-\mu+n}}{dt^n} = \frac{d^n t^{n-i}}{dt^n} = 0.$$

We now introduce the following auxiliary lemmas.

Lemma 1 ([10]). *Let $\varphi \in L^1(0, +\infty)$ and suppose that α and β are positive real constants. Then $I_{0^+}^{\alpha} I_{0^+}^{\beta} \varphi(t) = I_{0^+}^{\alpha+\beta} \varphi(t)$, $D_{0^+}^{\alpha} I_{0^+}^{\alpha} \varphi(t) = \varphi(t)$ belongs to $D_{0^+}^{\alpha} \varphi(t) \in L^1(0, +\infty)$, then $I_{0^+}^{\alpha} D_{0^+}^{\alpha} \varphi(t) = \varphi(t) + \sum_{i=1}^{i=n} c_i t^{\alpha-i}$, where c_i are real constants in $\mathbb{R}$.*

Definition 3 ([3]). *We make use of the classical concept of a measure of noncompactness on a Banach space $\mathbb{E}$ For the precise definition and its main properties, the reader is referred to [3]. The Kuratowski measure of noncompactness Λ in [9] can be described as follows:*

$$\Lambda(B) = \inf \{\tau > 0, B = \cup B_i \quad diam(B_i) \leqslant \tau\}.$$

Definition 4 ([6]). *Consider the mapping $\eta : \mathbb{E} \times \mathbb{E} \to [0, +\infty)$. If the condition $\eta(\chi, y) \geqslant 1$ leads to $\eta(\mathcal{P}\chi, \mathcal{P}y) \geqslant 1$ for all $\chi, y \in \mathbb{E}$, then the operator $\mathcal{P} : \mathbb{E} \to \mathbb{E}$ is η-admissible.*

In what follows, we present two fixed point results related to measures of noncompactness Φ which will be applied in the sequel. where $\mathcal{P} : F \to F$ is continuous and $k \in [0, 1)$.

Theorem 1 ([8]). *If*

$$\Phi(\mathcal{P}\mathcal{B}) \leqslant k\Phi(\mathcal{B}),$$

then $\mathcal{P}$ possesses a fixed point.

Theorem 2 ([6]). *Let $F \in \Omega(\mathbb{E})$ and $\eta : \mathbb{E} \times \mathbb{E} \to [0, +\infty)$. Suppose that $\mathcal{P} : F \to F$ is continuous and η-admissible and satisfying the condition*

$$\begin{aligned} &\eta(\chi, \mathcal{P}\chi)\omega\left(diam\left(\mathcal{P}\mathcal{B}_1\right)\right) \\ &\leqslant \Theta\left(\max\left\{diam\left(\mathcal{B}_1\right), diam\left(\mathcal{P}\mathcal{B}_1\right), diam\left(\mathcal{P}\mathcal{B}_2\right), \tfrac{1}{2}diam\left(\mathcal{P}\mathcal{B}_1 \cup \mathcal{P}\mathcal{B}_2\right)\right\}\right), \end{aligned} \tag{4}$$

for each $\mathcal{B}_1, \mathcal{B}_2 \subset F$. Then $\mathcal{P}$ then possesses a distinct fixed point.

The detailed assumptions and proofs can be found in [6, 8].

3. Existence result

Let $E = \mathcal{C}(I; \mathbb{R})$ denote the space of continuous functions on I, Define

$$F = \{\chi \in E : \parallel \chi \parallel < \rho\}.$$

To investigate the solvability of the boundary value problem (1)–(3), we use the following result:

Theorem 3. *Suppose that the following assumptions are satisfied:*

- *(H_1) there exists a function $d : [0; +\infty[\to [0; +\infty[$ which is semicontinuious and strictly non decreasing such that $d(0) = 0; d(t) < t, \forall t > 0$, and there are exists $C_1, C_2, C_3 \geqslant 0$ with $4TC_1 + 2C_2 + T^{1-r}C_3 \leqslant \Gamma(r)T^{1-r}$ such that for all $\chi, y \in R$ and $t \in [0; T]$*

$$\begin{aligned} |\varphi(t, \chi) - \varphi(t, y)| &\leqslant \frac{4C_1T^r}{\Gamma(r)}d(|\chi - y|), \\ |\Phi(t, \chi) - \Phi(t, y)| &\leqslant \frac{2C_2T^{r-1}}{\Gamma(r)}d(|\chi - y|), \\ |\Psi(t, \chi) - \Psi(t, y)| &\leqslant \frac{C_3T^{1-r}}{\Gamma(r)}d(|\chi - y|). \end{aligned}$$

- (H_2)

$$\begin{aligned} M_1 &:= \sup_{t\in I} |\varphi(t,\chi(t))| < +\infty, \\ M_2 &:= \sup_{t\in I} |\Phi(t,\chi(t))| < +\infty, \\ M_3 &:= \sup_{t\in I} |\Psi(t,\chi(t))| < +\infty. \end{aligned}$$

- (H_3) *Assume that There exists a constant* $\rho > 0$ *satisfying*

$$\frac{T^{r-1}}{\Gamma(r)} \left(4TM_1 + 2M_2 + T^{1-r}M_3\right) \leqslant \rho.$$

- (H_4) *For any* $\chi, y \in F$ *and* $t \in I$, *suppose that:*

$$\eta(\chi(t), y(t)) \geqslant 1 \longrightarrow \eta(\mathcal{P}\chi(t), \mathcal{P}y(t)) \geqslant 1,$$

where $\mathcal{P} : F \longrightarrow F$ *is defined by*

$$\mathcal{P}\chi(t) = I^r\varphi(t,\chi(t)) - \alpha I^r\varphi(T,\chi(T)) - \beta I^1\varphi(T,\chi(T)) + \beta\Phi(\chi) - \frac{\alpha}{b}\Psi(\chi).$$

Here:

$$\alpha = \frac{b}{b-a} < 1, \beta = T^{r-2}\Gamma(3-r)(t - \frac{bT}{b-a}),$$

and

$$I^r\varphi(t,\chi(t)) = \frac{1}{\Gamma(r)} \int_0^t (t-s)^{r-1} \varphi(s,\chi(s))ds,$$

$\Phi(\chi), \Psi(\chi)$ *are defined in* (2),(3).

For the proof of how the problem can be transformed into a fixed-point problem, the reader is referred to reference [11].

If there is an $\chi_0 \in F$ such that $\mathcal{P}$ satisfies $\gamma(\chi_0(t), \mathcal{P}\chi_0(t)) \geqslant 1$, then a solution to problem(1)-(3) is in F.

Proof. First, we verify that the operator $\mathcal{P}$ is well-defined.

$$|I^r\varphi(t,\chi(t)) - \alpha I^r\varphi(T,\chi(T))| \leqslant \frac{1}{\Gamma(r)} \int_0^t (t-s)^{r-1} |\varphi(s,\chi(s))| ds +$$

$$+\frac{1}{\Gamma(r)}\int_0^T (T-s)^{r-1}\left|\varphi(s,\chi(s))\right| ds \leqslant \frac{2T^r}{r\Gamma(r)}M_1, \tag{5}$$

and

$$\left|\beta I^1\varphi(t,\chi(t))\right| \leqslant T^{r-1}\int_0^T \left|\varphi(s,\chi(s))\right| ds \leqslant 2T^r M_1, \tag{6}$$

and

$$\left|\beta\Phi(\chi) - \frac{\alpha}{b}\Psi(\chi)\right| \leqslant 2T^{r-1}M_2 + M_3. \tag{7}$$

By using (5), (6), (7) we get

$$\begin{aligned} |\mathcal{P}\chi(t)| &\leqslant \frac{2T^r}{r\Gamma(r)}M_1 + 2T^r M_1 + 2T^{r-1}M_2 + M_3 \\ &\leqslant \frac{2T^r}{\Gamma(r)}M_1 + \frac{2T^{r-1}}{\Gamma(r)}M_2 + \frac{M_3}{\Gamma(r)} \\ &= \frac{T^{r-1}}{\Gamma(r)}\left(4TM_1 + 2M_2 + T^{1-r}M_3\right) \leqslant \rho. \end{aligned} \tag{8}$$

Consequently, the operator $\mathcal{P}$ is well defined.

Next we prove that the operator $\mathcal{P}$ is continuous. To this hand, suppose that $\chi, y \in F$ and use $\varepsilon > 0$ such that $|\chi - y| < \varepsilon$. We get

$$\begin{aligned} |I^r\varphi(t,\chi(t)) - \alpha I^r\varphi(t,y(t))| &= \frac{1}{\Gamma(r)}\int_0^t (t-s)^{r-1}\left|\varphi(s,\chi(s)) - \varphi(s,y(s))\right| ds \\ &\leqslant \frac{T^r}{r\Gamma(r)}\left|\left|\varphi(t,\chi(t)) - \varphi(t,y(t))\right|\right| \\ &= \frac{T^r C_1}{\Gamma(r)} d(|\chi - y|), \end{aligned} \tag{9}$$

and

$$\begin{aligned} |I^r\varphi(T,\chi(T)) - \alpha I^r\varphi(T,y(T))| &= \frac{1}{\Gamma(r)}\int_0^T (T-s)^{r-1}\left|\varphi(s,\chi(s)) - \varphi(s,y(s))\right| ds \\ &\leqslant \frac{T^r}{r\Gamma(r)}\left|\left|\varphi(t,\chi(t)) - \varphi(t,y(t))\right|\right| \end{aligned}$$

$$= \frac{T^r C_1}{\Gamma(r)} d(|\chi - y|), \tag{10}$$

$$\begin{aligned}
\left|\beta(I^1\varphi(T,\chi(T)) - I^1\varphi(T,y(T))\right| &= \beta \int_0^T |\varphi(s,\chi(s)) - \varphi(s,y(s))|\, ds \\
&\leqslant 2T^r \,||\varphi(t,\chi(t)) - \varphi(t,y(t))|| \\
&\leqslant \frac{2T^r C_1}{\Gamma(r)} d(|\chi - y|), \qquad (11)
\end{aligned}$$

and

$$\begin{aligned}
|\beta(\Phi(t,\chi(t)) - \Phi(t,y(t)))| &\leqslant 2T^{r-1} \,||\Phi(t,\chi(t)) - \Phi(t,y(t))|| \qquad (12) \\
&\leqslant \frac{2T^{r-1}C_2}{\Gamma(r)} d(|\chi - y|),
\end{aligned}$$

and

$$\begin{aligned}
\left|\frac{\alpha}{b}(\Psi(t,\chi(t)) - \Psi(t,y(t)))\right| &\leqslant ||\Psi(t,\chi(t)) - \Psi(t,y(t))|| \\
&\leqslant \frac{2T^{1-r}C_3}{\Gamma(r)} d(|\chi - y|). \qquad (13)
\end{aligned}$$

$$\begin{aligned}
|\mathcal{P}\chi(t) - \mathcal{P}y(t)| &\leqslant \frac{4T^r C_1}{\Gamma(r)} d(|\chi - y|) \\
&\leqslant +\frac{2T^{r-1}C_2}{\Gamma(r)} d(|\chi - y|) + \frac{2T^{1-r}C_3}{\Gamma(r)} d(|\chi - y|) \\
&\leqslant \frac{T^{r-1}}{\Gamma(r)} \left(4TC_1 + 2C_2 + C_3\right) d(|\chi - y|) \qquad (14) \\
&\leqslant d(|\chi - y|) \leqslant |\chi - y| \leqslant \varepsilon.
\end{aligned}$$

Now, show that $\mathcal{P}$ fulfils (5), suppose that $B_1, B_2 \in F$, and take $\chi, y \in B_1$. We get:

$$|\mathcal{P}\chi(t) - \mathcal{P}y(t)| \leqslant d(|\chi - y|).$$

Consequently

$$d^{-1} |\mathcal{P}\chi - \mathcal{P}y| \leq |\chi - y| \leqslant diam(B_1) \leq$$
$$\max\left\{diam(B_1), diam(\mathcal{P}B_1), diam(\mathcal{P}B_2), \frac{1}{2} diam(\mathcal{P}B_1 \cup \mathcal{P}B_2)\right\}.$$

Then

$$\omega\left(diam(\mathcal{P}B_1)\right) = d^{-1}(diam(\mathcal{P}B_1)) \leq$$
$$\Theta\left(\max\left\{diam(B_1), diam(\mathcal{P}B_1), diam(\mathcal{P}B_2), \frac{1}{2}diam(\mathcal{P}B_1 \cup \mathcal{P}B_2)\right\}\right).$$

Here we take $\omega(t) = d^{-1}(t)$ and $\Theta(t) = t$.

Furthermore, define the function $\eta : F \times F \longrightarrow [0, +\infty[$ by

$$\eta(\chi, y) = \begin{cases} 1 & \gamma(\chi(t), y(t)) \geqslant 1, \\ 0 & else. \end{cases}$$

Therefore

$$\eta(\chi, \mathcal{P}\chi)\omega\left(diam(\mathcal{P}B_1)\right) \leq$$
$$\Theta\left(\max\left\{diam(B_1), diam(\mathcal{P}B_1), diam(\mathcal{P}B_2), \frac{1}{2}diam(\mathcal{P}B_1 \cup \mathcal{P}B_2)\right\}\right).$$

Hence, the operator $\mathcal{P}$ satisfies condition (5).

Assume that $\eta(\chi, y) \geqslant 1$ for all $\chi, y \in F$. Then it follows that $\gamma(\chi, y) \geqslant 1$. Using assumption (H_4), we obtain $\gamma(\mathcal{P}\chi, \mathcal{P}y) \geqslant 1$, which leads to $\eta(\mathcal{P}\chi, \mathcal{P}y) \geqslant 1$. Hence, the mapping $\mathcal{P} : F \longrightarrow F$ is η-admissible for all $\chi, y \in F$.

Moreover, since there exists $\chi_0 \in F$ such that $\alpha(\chi_0, \mathcal{P}\chi_0) \geqslant 1$, it follows from (2), $\mathcal{P}$ that P admits a fixed point in F. ◄

4. Example

In order to illustrate the obtained result, we consider the following fractional boundary value problem:

$$\begin{aligned} {}^cD^{\frac{3}{2}}\chi(t) &= \tfrac{t+1}{80}\ln(1+|\chi(t)|)., t \in \left[0; \tfrac{\pi}{4}\right] \\ -\tfrac{1}{2}\chi(0) - \tfrac{1}{2}\chi(\tfrac{\pi}{4}) &= \tfrac{\ln(1+|\cos\chi(0)|)}{50}; \\ {}^cD^{\frac{1}{2}}\chi(\tfrac{\pi}{4}) &= \tfrac{\ln(1+\left|\sin\chi(\frac{\pi}{4})\right|)}{30}, \end{aligned} \tag{15}$$

where $a = -\frac{1}{2}, b = \frac{1}{2}, \alpha = \frac{b}{b-a} = \frac{1}{2} < 1, \beta = t - \frac{\pi}{8}$ and $\chi \in F = \left\{\chi \in C([0, \frac{\pi}{4}]; \mathbb{R}) : \parallel \chi \parallel \leqslant \frac{\pi}{4}\right\}$.

- (H_1) Consider the function $d : [0; +\infty[\to [0; +\infty[$ is semicontinuious and strictly non decreasing and defined by $d(\chi) = \ln(1 + |\chi|)$ with $d(0) = 0; d(t) < t, \forall t > 0$.

  Let $\varphi : I \times \mathbb{R} \to \mathbb{R}$, and $\Phi, \Psi : C(I) \to \mathbb{R}$ be continuous operators defined as follows:

$$\varphi(t, \chi(t)) = \frac{t+1}{80}\ln(1 + |\chi(t)|);$$

$$\begin{aligned}
\Psi(t,\chi(t)) &= \frac{\ln(1+|\cos\chi(0)|)}{50};\\
\Phi(t,\chi(t)) &= \frac{\ln(1+\left|\sin\chi(\frac{\pi}{4})\right|)}{30}.
\end{aligned}$$

For every $\chi, y \in F$ and $t \in [0; \frac{\pi}{4}]$, we have

$$\begin{aligned}
|\varphi(t,\chi(t)) - \varphi(t,y(t))| &= \frac{t+1}{80}\left[\ln(1+|\chi(t)|) - \ln(1+|y(t)|)\right]\\
&\leqslant \frac{\pi+4}{320}\ln\left(\frac{1+|\chi(t)|}{1+|y(t)|}\right) \leqslant \frac{4C_1T^r}{\Gamma(r)}\ln(1+|\chi-y|)\\
&= \frac{4C_1(\frac{\pi}{4})^{\frac{3}{2}}}{\frac{\sqrt{\pi}}{2}}\ln(1+|\chi-y|).
\end{aligned}$$

$$\begin{aligned}
|\Phi(t,\chi(t)) - \Phi(t,y(t))| &= \frac{1}{30}\left[\ln(1+\left|\sin\chi(\frac{\pi}{4})\right|) - \ln(1+\left|\sin y(\frac{\pi}{4})\right|)\right]\\
&\leqslant \frac{1}{30}\ln\left(\frac{1+|\chi(t)|}{1+|y(t)|}\right) \leqslant \frac{2C_2T^{r-1}}{\Gamma(r)}\ln(1+|\chi-y|)\\
&= \frac{2C_2(\frac{\pi}{4})^{\frac{1}{2}}}{\frac{\sqrt{\pi}}{2}}\ln(1+|\chi-y|).
\end{aligned}$$

$$\begin{aligned}
|\Psi(t,\chi(t)) - \Psi(t,y(t))| &= \frac{1}{50}\left[\ln(1+|\cos\chi(0)|) - \ln(1+|\cos y(0)|)\right]\\
&\leqslant \frac{1}{50}\ln\left(\frac{1+|\chi(t)|}{1+|y(t)|}\right) \leqslant \frac{C_3T^{1-r}}{\Gamma(r)}\ln(1+|\chi-y|)\\
&= \frac{C_3(\frac{\pi}{4})^{\frac{-1}{2}}}{\frac{\sqrt{\pi}}{2}}\ln(1+|\chi-y|),
\end{aligned}$$

with

$$C_1 = \frac{1}{120\sqrt{\pi}}, \quad C_2 = \frac{\sqrt{\pi}}{80}, \quad C_3 = \frac{\pi}{60}$$

and

$$4TC_1 + 2C_2 + T^{1-r}C_3 = \frac{\sqrt{\pi}}{15} \leqslant \Gamma(r)T^{1-r} = 1.$$

- (H_2) We evaluate the following bounds:

$$\begin{aligned}
M_1 &:= \sup_{0\leqslant t\leqslant \frac{\pi}{4}} |\varphi(t,\chi(t))| < \frac{1}{40} \parallel \chi \parallel < \frac{1}{40} < +\infty, \\
M_2 &:= \sup_{0\leqslant t\leqslant \frac{\pi}{4}} |\Phi(t,\chi(t))| < \frac{1}{30} \parallel \chi \parallel < \frac{1}{30} < +\infty, \\
M_3 &:= \sup_{0\leqslant t\leqslant \frac{\pi}{4}} |\Psi(t,\chi(t))| < \frac{1}{50} \parallel \chi \parallel < \frac{1}{50} < +\infty.
\end{aligned}$$

- (H_3) There exists $\rho > 0$ satisfying:

$$\begin{aligned}
\frac{T^{r-1}}{\Gamma(r)}\left(4TM_1 + 2M_2 + T^{1-r}M_3\right) &\leqslant \pi\frac{\parallel \chi \parallel}{40} + 2\frac{\parallel \chi \parallel}{30} + \frac{2}{\sqrt{\pi}}\frac{\parallel \chi \parallel}{50} \\
&\leqslant \frac{15\pi\sqrt{\pi} + 40\sqrt{\pi} + 24}{600\sqrt{\pi}} \parallel \chi \parallel \\
&\leqslant \frac{15\pi^2 + 40\pi + 24\sqrt{\pi}}{2400} = \rho,
\end{aligned}$$

and there are exists $C_1, C_2, C_3 \geqslant 0$ with $4TC_1 + 2C_2 + T^{1-r}C_3 \leqslant \Gamma' r)T^{1-r}$ such that for all $\chi, y \in R$ and $t \in [0;T]$.

- (H_4) For all $\chi, y \in F$; $t \in I$:

$$\gamma(\chi(t), y(t)) \geqslant 1 \longrightarrow \gamma(\mathcal{P}\chi(t), \mathcal{P}y(t)) \geqslant 1,$$

where the operator $\mathcal{P} : F \longrightarrow F$ is introduced as:

$$\left| I^{\frac{3}{2}}\varphi(t,\chi(t)) - \alpha I^{\frac{3}{2}}\varphi(T,\chi(T) \right| \leqslant$$

$$\left[\frac{1}{80\Gamma(\frac{3}{2})} + \frac{1}{160\Gamma(\frac{3}{2})}\right] \int_0^{\frac{\pi}{4}} (\frac{\pi}{4} - s)^{\frac{1}{2}} (s+1) \left|\ln(1 + |\chi(t)|)\right| ds$$

$$\leqslant \frac{3\pi^2 + 12\pi}{2560} \parallel \chi \parallel_\infty . \tag{16}$$

$$\left| I^1\varphi(T,\chi(T)) \right| \leqslant$$

$$\frac{1}{80}\left|\frac{\pi}{8} - t\right| \int_0^{\frac{\pi}{4}} (\frac{\pi}{4} - s)^{\frac{1}{2}} (s+1) \left|\ln(1 + |\chi(t)|)\right| ds$$

$$\leqslant \frac{\pi^3+4\pi^2}{10240} \parallel \chi \parallel_\infty . \tag{17}$$

$$\left|\frac{\alpha}{b}\Psi(\chi)\right| \leqslant \frac{1}{30}\left|\ln(1+\left|\sin\chi(\frac{\pi}{4})\right|)\right| ds \leqslant \frac{1}{30} \parallel \chi \parallel_\infty . \tag{18}$$

$$|\beta\Phi(\chi)| \leqslant \frac{1}{50}\frac{\pi}{8} \leqslant \frac{\pi}{400}. \tag{19}$$

Combining inequalities (16), (17), (18) and (19), we deduce that:

$$\begin{aligned} |\mathcal{P}\chi(t)| &\leqslant \left(\frac{3\pi^2+12\pi}{2560}+\frac{\pi^3+4\pi^2}{10240}+\frac{1}{30}\right) \parallel \chi \parallel_\infty + \frac{\pi}{400} \\ &\leqslant \left(\frac{3\pi^2+12\pi}{2560}+\frac{\pi^3+4\pi^2}{10240}+\frac{1}{30}\right)\frac{\pi}{4}+\frac{\pi}{400} \\ &\leqslant \frac{\pi^4+16\pi^3+48\pi^2}{40960}+\frac{520\pi}{48000} = 0.05 < \rho = 0.132. \end{aligned}$$

Hence, $\parallel \mathcal{P}\chi \parallel \leqslant \frac{\pi^4+16\pi^3+48\pi^2}{40960}+\frac{520\pi}{48000}$. Similarly, we can show that

$$\parallel \mathcal{P}y \parallel \leqslant \frac{\pi^4+16\pi^3+48\pi^2}{40960}+\frac{520\pi}{48000}.$$

Consequently

$$\parallel \mathcal{P}\chi \parallel + \parallel \mathcal{P}y \parallel \leqslant \frac{2\pi^4+32\pi^3+96\pi^2}{40960}+\frac{1040\pi}{48000} = 0.1 < 1,$$

which guarantees that $\gamma(\mathcal{P}\chi(t), \mathcal{P}y(t)) \geqslant 1$, for all $\chi, y \in F$ and $t \in I$.
Next, define the function $\chi_0[0, \frac{\pi}{4}] \to \mathbb{R}$ by

$$\chi_0(t) = \frac{\pi^4+16\pi^3+48\pi^2}{40960}+\frac{520\pi}{48000}.$$

Then

$$\parallel \chi_0 \parallel + \parallel \mathcal{P}\chi_0 \parallel = \frac{2\pi^4+32\pi^3+96\pi^2}{40960}+\frac{1040\pi}{48000} = 0.1 < 1.$$

Therefore, there exists $\chi_0 \in F$ satisfying this condition $\eta(\chi_0, \mathcal{P}\chi_0 \geqslant 1$. By Theorem 2, we conclude that the considered equation admits at least one solution in the space F.

5. Conclusion

In this paper, we investigated the existence of solutions for a class of nonlinear fractional boundary value problems involving the Caputo fractional derivative. By transforming the considered boundary value problem into an equivalent fixed–point formulation, we applied fixed–point results associated with generalized Proinov-type contraction mappings expressed through the measure of noncompactness.

The obtained results extend classical fixed-point techniques and allow the treatment of the problem under weaker assumptions compared with those used in earlier works. In particular, our approach avoids the decomposition of the associated operator into separate components and does not require strong Lipschitz-type conditions. Consequently, the developed framework provides a more flexible tool for establishing the existence of solutions for nonlinear fractional differential equations with nonlocal boundary conditions.

Furthermore, an illustrative example was presented to demonstrate the applicability of the theoretical results and to confirm the effectiveness of the proposed method. The techniques developed in this work may be useful for studying other classes of fractional differential equations and boundary value problems arising in applied sciences. Future research may focus on extending these results to uniqueness, stability analysis, or to systems of fractional differential equations under more general boundary conditions.

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