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Inversion of the Iterated Operator $\mathcal{D}_m^k$ Related to the Klein–Gordon Kernel

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Abstract. In this paper, we introduce a class of convolution operators $\mathcal{D}_m^{(\alpha,\gamma,\nu)}$ acting on functions $f \in \mathcal{S}(\mathbb{R}^n)$, the Schwartz space, defined by $\mathcal{D}_m^{(\alpha,\gamma,\nu)}(f) = \mathcal{M}_{\alpha,\gamma,\nu} * f$, where $\mathcal{M}_{\alpha,\gamma,\nu}$ is a distribution kernel depending on complex parameters $\alpha, \gamma, \nu \in \mathbb{C}$, and $*$ denotes the convolution on $\mathbb{R}^n$. The primary goal is to construct and characterize the inverse operator $L^{(\alpha,\gamma,\nu)} = \left[\mathcal{D}_m^{(\alpha,\gamma,\nu)}\right]^{-1}$, such that, for every $\varphi \in \mathcal{S}(\mathbb{R}^n)$, if $\varphi = \mathcal{D}_m^{(\alpha,\gamma,\nu)}(f)$, then $L^{(\alpha,\gamma,\nu)}(\varphi) = f$. We also investigate the conditions under which the inverse exists in the sense of tempered distributions and provide explicit representations of the fundamental solutions associated with these operators.

Key Words and Phrases: Klein–Gordon kernel, fundamental solution, diamond Klein Gordon operator, distribution theory.
2010 Mathematics Subject Classifications: 46F10, 46F12

1. Introduction

The study of partial differential equations and their associated operators plays a crucial role in various fields of mathematics, physics, and engineering. Understanding the fundamental solutions of these operators is particularly important, as they provide powerful tools for solving complex differential equations through convolution. This introduction surveys the evolution of key operators, their fundamental solutions, and related concepts that form the background for the current work. We begin with the classical ultra-hyperbolic operator and its extensions, leading to the more recently developed diamond operators and their generalized forms.

Gelfand and Shilov [1] were the first to construct a fundamental solution to the equation

$$\Box^k u(x) = f(x), \tag{1}$$

where $u(x)$ and $f(x)$ are generalized functions on $\mathbb{R}^n$. The solution they proposed has a complicated form due to the structure of the operator $\Box^k$.

The operator $\Box^k$, called the k-times iterated ultra-hyperbolic operator, is defined by

$$\Box^k = \left(\sum_{j=1}^{p} \frac{\partial^2}{\partial x_j^2} - \sum_{j=p+1}^{p+q} \frac{\partial^2}{\partial x_j^2} \right)^k, \tag{2}$$

where $p+q=n$ is the dimension of $\mathbb{R}^n$ and k is a non-negative integer.

Trione [2] later showed that the generalized function $R_{2k}(x)$, defined by (5) with $\alpha = 2k$, is the unique fundamental solution to equation (1). Subsequently, Tellez [3] proved that this fundamental solution exists only when $n = p+q$ and p is odd.

Motivated by these results, Kananthai [4] introduced a new differential operator, denoted by $\Diamond^k$, called the iterated diamond operator, defined by

$$\Diamond^k = \left[\left(\sum_{i=1}^{p} \frac{\partial^2}{\partial x_i^2} \right)^2 - \left(\sum_{j=p+1}^{p+q} \frac{\partial^2}{\partial x_j^2} \right)^2 \right]^k,$$

where $n = p+q$ is the dimension of $\mathbb{R}^n$, and k is a non-negative integer.

This operator can be factorized as

$$\Diamond^k = \triangle^k \Box^k = \Box^k \triangle^k,$$

where $\Box^k$ is the iterated ultra-hyperbolic operator defined in (2), and

$$\triangle^k = \left(\sum_{j=1}^{n} \frac{\partial^2}{\partial x_j^2} \right)^k$$

is the k-times iterated Laplace operator on $\mathbb{R}^n$.

Dominguez and Trione [5] introduced the distributional functions $H_\alpha(P \pm i0, n)$, which are causal (anti-causal) analogues of the elliptic kernel of Marcel Riesz [6]. Later, Cerutti and Trione [7] defined the causal (anti-causal) generalized Marcel Riesz potentials of order α, $\alpha \in \mathbb{C}$, by

$$R^\alpha \varphi = H_\alpha(P \pm i0, n) * \varphi,$$

where $\varphi \in \mathcal{S}$, $\mathcal{S}$ denotes the Schwartz space of functions [8], and $H_\alpha(P \pm i0, n)$ is given by

$$H_\alpha(P \pm i0, n) = \frac{e^{\mp \alpha \pi i/2} e^{\pm q \pi i/2} \Gamma((n-\alpha)/2)}{\alpha \pi^{n/2} \Gamma(\alpha/2)} (P \pm i0)^{(\alpha - n)/2}.$$

Here, P is defined by

$$P = P(x) = x_1^2 + x_2^2 + \cdots + x_p^2 - x_{p+1}^2 - x_{p+2}^2 - \cdots - x_{p+q}^2,$$

where q is the number of negative terms in the quadratic form P. The distributions $(P \pm i0)^\lambda$ are defined by

$$(P \pm i0)^\lambda = \lim_{\epsilon \to 0} (P \pm i\epsilon |x|)^\lambda,$$

where $\epsilon > 0$, $\lambda \in \mathbb{C}$, and $|x|^2 = x_1^2 + x_2^2 + \cdots + x_n^2$, see [1]. They also studied the inverse operator of R^α, denoted by $(R^\alpha)^{-1}$, such that if $f = R^\alpha \varphi$, then $(R^\alpha)^{-1} f = \varphi$.

Aguirre [9] later defined the ultra-hyperbolic Marcel Riesz operator M^α acting on a function $f \in \mathcal{S}$ by

$$M^\alpha(f) = R_\alpha * f,$$

where R_α is the ultra-hyperbolic kernel defined in (5). He also introduced the inverse operator $N^\alpha = (M^\alpha)^{-1}$ satisfying $N^\alpha(\varphi) = f$ whenever $M^\alpha(f) = \varphi$.

In a related development, Kananthai [10] introduced the diamond kernel of Marcel Riesz, denoted by $K_{\alpha,\beta}(x)$, defined via the convolution

$$K_{\alpha,\beta}(x) = S_\alpha(x) * R_\beta(x), \tag{3}$$

where $S_\alpha(x)$ is the elliptic kernel (see (6)), and $R_\beta(x)$ is the ultra-hyperbolic kernel (see (5)).

Tellez and Kananthai [11] proved that $K_{\alpha,\beta}(x)$ exists as a tempered distribution and belongs to the space of rapidly decreasing distributions. Moreover, they demonstrated that this family of convolution kernels $K_{\alpha,\beta}(x)$ is intrinsically connected with the diamond operator, thereby providing a foundation for further analysis in distribution theory.

Maneetus and Nonlaopon [12] later introduced the *diamond Marcel Riesz operator* of order $(\alpha, \beta) \in \mathbb{C}$, defined on functions $f \in \mathcal{S}$ by

$$M^{(\alpha,\beta)}(f) = K_{\alpha,\beta} * f,$$

where $K_{\alpha,\beta}$ is the diamond kernel given in (3). They also established the existence of its inverse operator $N^{(\alpha,\beta)} = [M^{(\alpha,\beta)}]^{-1}$ which satisfies

$$N^{(\alpha,\beta)}(\varphi) = f \quad \text{whenever} \quad M^{(\alpha,\beta)}(f) = \varphi.$$

Moreover, Maneetus and Nonlaopon defined the *Bessel ultra-hyperbolic Marcel Riesz operator* of order $\alpha \in \mathbb{C}$ acting on a function $f \in \mathcal{S}$ by

$$U^\alpha(f) = R_\alpha^B * f,$$

where R_α^B is the Bessel ultra-hyperbolic kernel of Marcel Riesz; see [13] for further details.

Subsequently, Tariboon and Kananthai [14] introduced the *diamond Klein-Gordon operator* iterated k times, denoted by $(\diamondsuit + m^2)^k$, defined as

$$(\diamondsuit + m^2)^k = \left[\left(\sum_{i=1}^{p} \frac{\partial^2}{\partial x_i^2}\right)^2 - \left(\sum_{j=p+1}^{p+q} \frac{\partial^2}{\partial x_j^2}\right)^2 + m^2\right]^k,$$

where $p+q=n$ is the dimension of $\mathbb{R}^n$, $m \geq 0$, k is a non-negative integer, and $x=(x_1,x_2,\ldots,x_n) \in \mathbb{R}^n$.

Later, Nonlaopon et al. [15] studied the fundamental solution of the operator $(\diamondsuit + m^2)^k$, which they called the *diamond Klein-Gordon kernel*. They also investigated its Fourier transform and convolution properties; see [16].

In addition, Sattaso and Nonlaopon [17] defined the diamond Klein-Gordon operator of order α acting on $f \in \mathcal{S}$ by

$$D^\alpha(f) = T_\alpha * f,$$

where T_α is defined in (10) and $\alpha \in \mathbb{C}$. Their objective was to analyze the convolution structure of T_α and to construct the inverse operator $L^\alpha = [D^\alpha]^{-1}$ such that

$$D^\alpha(f) = \varphi \quad \Rightarrow \quad L^\alpha(\varphi) = f.$$

Bupasiri [18] studied the fundamental solution of the operator $\mathcal{D}_m^k$, defined by:

$$\begin{aligned}
\mathcal{D}_m^k &= \left[\left(\left(\sum_{i=1}^{p} \frac{\partial^2}{\partial x_i^2}\right)^2 + \frac{m^2}{2}\right)^2 - \left(\left(\sum_{j=p+1}^{p+q} \frac{\partial^2}{\partial x_j^2}\right)^2 - \frac{m^2}{2}\right)^2\right]^k \\
&= \left[\left(\sum_{i=1}^{p} \frac{\partial^2}{\partial x_i^2}\right)^2 - \left(\sum_{j=p+1}^{p+q} \frac{\partial^2}{\partial x_j^2}\right)^2 + m^2\right]^k \left[\left(\sum_{i=1}^{p} \frac{\partial^2}{\partial x_i^2}\right)^2 + \left(\sum_{j=p+1}^{p+q} \frac{\partial^2}{\partial x_j^2}\right)^2\right]^k \\
&= (\diamondsuit + m^2)^k L_1^k L_2^k = (\diamondsuit + m^2)^k L^k.
\end{aligned} \tag{4}$$

He showed that the fundamental solution of the equation

$$\mathcal{D}_m^k \left(T_{2k}(x,m) * M_{2k}(w) * N_{2k}(z)\right) = \delta$$

is given by the convolution of the kernels T_{2k}, M_{2k}, and N_{2k}.

In this paper, we define the kernel $\mathcal{M}_{\alpha,\gamma,\nu}$ and the operator $\mathcal{D}_m^{(\alpha,\gamma,\nu)}$ of order (α,γ,ν) acting on a function $f \in \mathcal{S}$ (the Schwartz space) by $\mathcal{D}_m^{(\alpha,\gamma,\nu)}(f) = \mathcal{M}_{\alpha,\gamma,\nu} * f$, where $\alpha,\gamma,\nu \in \mathbb{C}$ and $*$ denotes the convolution. The main objective is to determine the inverse operator $L^{(\alpha,\gamma,\nu)} = \left[\mathcal{D}_m^{(\alpha,\gamma,\nu)}\right]^{-1}$, such that, for any $\varphi = \mathcal{D}_m^{(\alpha,\gamma,\nu)}(f)$, we have $L^{(\alpha,\gamma,\nu)}(\varphi) = f$.

2. Preliminaries

We begin by introducing the essential definitions and notation needed for our main results. Central to our discussion are the *Riesz kernels*, which underlie the theory of fundamental solutions, particularly for ultra-hyperbolic and elliptic operators.

We first recall the *ultra-hyperbolic kernel*, associated with the forward light cone and adapted to indefinite quadratic forms. Next, we present the *elliptic kernel*, corresponding to positive definite forms.

To handle operators with complex powers and mixed signatures, we extend these classical kernels to complex-valued versions, defined over suitable domains.

The following definitions and lemmas form the basis of our subsequent analysis.

Definition 1 (Ultra-hyperbolic kernel of Marcel Riesz). *Let $x = (x_1, x_2, \dots, x_n) \in \mathbb{R}^n$ and define*

$$u = x_1^2 + x_2^2 + \cdots + x_p^2 - x_{p+1}^2 - x_{p+2}^2 - \cdots - x_{p+q}^2, \quad \textit{where } p+q=n.$$

Denote by $\Gamma_+ = \{x \in \mathbb{R}^n : x_1 > 0 \text{ and } u > 0\}$ the interior of the forward cone, and $\overline{\Gamma_+}$ its closure.

For any complex number α, define

$$R_\alpha(u) = \begin{cases} \dfrac{u^{(\alpha-n)/2}}{K_n(\alpha)}, & x \in \Gamma_+, \\ 0, & \textit{otherwise}, \end{cases} \tag{5}$$

where

$$K_n(\alpha) = \frac{\pi^{\frac{n-1}{2}} \Gamma\left(\frac{2+\alpha-n}{2}\right) \Gamma\left(\frac{1-\alpha}{2}\right) \Gamma(\alpha)}{\Gamma\left(\frac{2+\alpha-p}{2}\right) \Gamma\left(\frac{p-\alpha}{2}\right)}.$$

The function R_α was first introduced by Nozaki [19, page 72]. It is an ordinary function when $\operatorname{Re}(\alpha) \geq n$, *and a distribution otherwise. Its support satisfies*

$$\operatorname{supp} R_\alpha(u) \subseteq \overline{\Gamma_+}.$$

Definition 2 (Elliptic kernel of Marcel Riesz). *Let $x = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n$ and define*

$$v = x_1^2 + x_2^2 + \cdots + x_n^2.$$

For any complex number β, define

$$S_\beta(v) = \frac{v^{(\beta-n)/2}}{W_n(\beta)}, \tag{6}$$

where

$$W_n(\beta) = \pi^{n/2} 2^\beta \frac{\Gamma(\beta)}{\Gamma\left(\frac{n-\beta}{2}\right)}.$$

The function S_β is called the elliptic kernel of Marcel Riesz. It is an ordinary function if $\operatorname{Re}(\beta) \geq n$, *and a distribution otherwise.*

Definition 3. *Let $x = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n$. Define*

$$w = x_1^2+x_2^2+\cdots+x_p^2-i\left(x_{p+1}^2+\cdots+x_{p+q}^2\right), \quad z = x_1^2+x_2^2+\cdots+x_p^2+i\left(x_{p+1}^2+\cdots+x_{p+q}^2\right),$$

where $p + q = n$ and $i = \sqrt{-1}$.

For any complex numbers γ, ν, define

$$M_\gamma(w) = \frac{w^{(\gamma-n)/2}}{W_n(\gamma)}, \tag{7}$$

$$N_\nu(z) = \frac{z^{(\nu-n)/2}}{W_n(\nu)}, \tag{8}$$

with

$$W_n(\gamma) = \frac{\pi^{n/2} 2^\gamma \Gamma\left(\frac{\gamma}{2}\right)}{\Gamma\left(\frac{n-\gamma}{2}\right)}, \quad W_n(\nu) = \frac{\pi^{n/2} 2^\nu \Gamma\left(\frac{\nu}{2}\right)}{\Gamma\left(\frac{n-\nu}{2}\right)}.$$

Lemma 1 ([20]). *Let M_γ and N_ν be the distributions defined by Equations* (7) *and* (8)*, respectively. Then the following convolution identities hold:*

$$M_\gamma * M_{\gamma'} = i^{q/2} M_{\gamma+\gamma'},$$
$$N_\nu * N_{\nu'} = (-i)^{q/2} N_{\nu+\nu'}.$$

In particular, for $\gamma = 0$ and $\nu = 0$, we have

$$M_0(w) = i^{q/2}\delta, \quad N_0(z) = (-i)^{q/2}\delta.$$

Detailed proofs of Lemmas 2, 3, and 4 are provided in [17].

Lemma 2 (Convolution of $T_\alpha(x,m)$ and $T_\beta(x,m)$)**.** *Let $\beta = -\alpha$. Then the convolution $T_\alpha(x,m) * T_\beta(x,m)$ satisfies the following cases:*

1. *If p is odd and q is even, then*

$$T_\alpha * T_\beta = T_{\alpha+\beta} + A_{\alpha,\beta}(x,m),$$

where $A_{\alpha,\beta}(x,m)$ is defined below.

2. *If both p and q are odd, then*

$$T_\alpha * T_\beta = T_{\alpha+\beta} + A_{\alpha,\beta}(x,m).$$

3. *If p is even and q is odd, then*

$$T_\alpha * T_\beta = \frac{\cos(\alpha\pi/2)\cos(\beta\pi/2)}{\cos((\alpha+\beta)\pi/2)} T_{\alpha+\beta}.$$

4. *If both p and q are even, then*

$$T_\alpha * T_\beta = \frac{\cos(\alpha\pi/2)\cos(\beta\pi/2)}{\cos((\alpha+\beta)\pi/2)} T_{\alpha+\beta}.$$

Here,

$$\begin{aligned} A_{\alpha,\beta}(x,m) = -\frac{i}{2}\frac{\sin(\alpha\pi/2)\sin(\beta\pi/2)}{\sin((\alpha+\beta)\pi/2)} \sum_{k=0}^{\infty} m^{2k}(-1)^{(\alpha+\beta)/2+k}\binom{-(\alpha+\beta)/2}{k} \\ \times \left[H^+_{\alpha+\beta+2k}(x) - H^-_{\alpha+\beta+2k}(x)\right] * S_{\alpha+\beta+2k}(x), \end{aligned} \tag{9}$$

and the kernel $T_\alpha(x,m)$ is given by

$$T_\alpha(x,m) = \sum_{r=0}^{\infty} m^{2r}(-1)^{\alpha/2+r}\binom{-\alpha/2}{r} S_{\alpha+2r}(x) * R_{\alpha+2r}(x). \tag{10}$$

Lemma 3 (Explicit form of $A_{\alpha,-\alpha}(x,m)$)**.** *Let $\beta = -\alpha$. Then the correction term $A_{\alpha,-\alpha}(x,m)$ satisfies*

$$A_{\alpha,-\alpha}(x,m) = \begin{cases} \sin^2(\alpha\pi/2)\,\delta(x), & \text{if } p \text{ is odd and } q \text{ is even,} \\ 0, & \text{if both } p \text{ and } q \text{ are odd.} \end{cases}$$

Lemma 4 (Convolution of T_α and $T_{-\alpha}$)**.** *Using Lemmas 2 and 3, the convolution $T_\alpha * T_{-\alpha}$ satisfies*

$$T_\alpha * T_{-\alpha} = \begin{cases} (1+\sin^2(\alpha\pi/2))\,\delta(x), & \text{if } p \text{ is odd and } q \text{ is even},\\ \delta(x), & \text{if both } p \text{ and } q \text{ are odd},\\ \cos^2(\alpha\pi/2)\,\delta(x), & \text{if } p \text{ is even and } q \text{ is odd},\\ \cos^2(\alpha\pi/2)\,\delta(x), & \text{if both } p \text{ and } q \text{ are even}. \end{cases}$$

Lemma 5 ([18])**.** *Let $\mathcal{D}_m^k$ denote the k-times iteration of the operator $\mathcal{D}_m$ (as defined in Equation* (4)*). Consider the equation*

$$\mathcal{D}_m^k\,\mathcal{M}(x,m) = \delta,$$

where δ denotes the Dirac delta distribution, $x \in \mathbb{R}^n$, $m \geq 0$, and k is a non-negative integer. Then the fundamental solution of $\mathcal{D}_m^k$ is given by

$$\mathcal{M}(x,m) = T_{2k}(x,m) * M_{2k}(w) * N_{2k}(z).$$

In particular, when $m = 0$, the equation reduces to

$$\mathcal{D}_0^k\,\mathcal{M}(x,0) = \delta,$$

and its fundamental solution is

$$\mathcal{M}(x,0) = R_{2k}(u) * (-1)^k S_{2k}(v) * M_{2k}(w) * N_{2k}(z).$$

3. Convolution operator and its inversion

We introduce a generalized convolution-type kernel $\mathcal{M}_{\alpha,\gamma,\nu}$ constructed from the fundamental kernels T_α, M_γ, and N_ν defined previously. This kernel serves as a building block for defining the operator $\mathcal{D}_m^{(\alpha,\gamma,\nu)}$ acting on suitable function spaces.

Our main goal is to investigate the algebraic properties of these convolution kernels, particularly focusing on their behavior under convolution and the existence of inverse operators. Understanding the convolution relations among these kernels is crucial for establishing explicit inversion formulas for the operators they generate.

To this end, we first present a key proposition that characterizes the convolution of two generalized kernels $\mathcal{M}_{\alpha,\gamma,\nu}$ and $\mathcal{M}_{\beta,\gamma',\nu'}$ under various parity conditions on the parameters p and q.

The resulting identities enable us to identify correction terms in some cases and multiplicative constants in others, which are essential for determining the precise form of the inverse operator.

Building upon these results, we formulate and prove the main inversion theorem for the operator $\mathcal{D}_m^{(\alpha,\gamma,\nu)}$ in the next section.

Proposition 1. *Let the generalized convolution kernel $\mathcal{M}_{\alpha,\gamma,\nu}$ be defined by*

$$\begin{aligned}\mathcal{M}_{\alpha,\gamma,\nu}(x,m) &= T_\alpha(x,m) * M_\gamma(w) * N_\nu(z)\\ &= \left(\sum_{r=0}^{\infty} m^{2r}(-1)^{\alpha/2+r}\binom{-\alpha/2}{r} S_{\alpha+2r}(x) * R_{\alpha+2r}(x)\right) * M_\gamma(w) * N_\nu(z),\end{aligned} \tag{11}$$

where $\alpha, \gamma, \nu \in \mathbb{C}$ and the convolution is taken with respect to corresponding variables x, w, and z.

Then the convolution of two such kernels satisfies the following identities:

(i) *If p is odd and q is even, then*

$$\mathcal{M}_{\alpha,\gamma,\nu} * \mathcal{M}_{\beta,\gamma',\nu'} = \mathcal{M}_{\alpha+\beta,\gamma+\gamma',\nu+\nu'} + A_{\alpha,\beta}(x,m) * M_{\gamma+\gamma'}(w) * N_{\nu+\nu'}(z),$$

where $A_{\alpha,\beta}(x,m)$ is a correction term arising from the convolution of T_α and T_β.

(ii) *If both p and q are odd, then*

$$\mathcal{M}_{\alpha,\gamma,\nu} * \mathcal{M}_{\beta,\gamma',\nu'} = \mathcal{M}_{\alpha+\beta,\gamma+\gamma',\nu+\nu'} + A_{\alpha,\beta}(x,m) * M_{\gamma+\gamma'}(w) * N_{\nu+\nu'}(z).$$

(iii) *If p is even and q is odd, then*

$$\mathcal{M}_{\alpha,\gamma,\nu} * \mathcal{M}_{\beta,\gamma',\nu'} = \frac{\cos\left(\frac{\alpha\pi}{2}\right)\cos\left(\frac{\beta\pi}{2}\right)}{\cos\left(\frac{(\alpha+\beta)\pi}{2}\right)} \cdot \mathcal{M}_{\alpha+\beta,\gamma+\gamma',\nu+\nu'}.$$

(iv) *If both p and q are even, then*

$$\mathcal{M}_{\alpha,\gamma,\nu} * \mathcal{M}_{\beta,\gamma',\nu'} = \frac{\cos\left(\frac{\alpha\pi}{2}\right)\cos\left(\frac{\beta\pi}{2}\right)}{\cos\left(\frac{(\alpha+\beta)\pi}{2}\right)} \cdot \mathcal{M}_{\alpha+\beta,\gamma+\gamma',\nu+\nu'}.$$

Proof. We prove each case separately, according to the parity of p and q:

(i) If p is odd and q is even, then by Lemmas 1 and 2, we obtain

$$\begin{aligned}
\mathcal{M}_{\alpha,\gamma,\nu} * \mathcal{M}_{\beta,\gamma',\nu'} &= T_\alpha(x,m) * T_\beta(x,m) * M_\gamma(w) * M_{\gamma'}(w) * N_\nu(z) * N_{\nu'}(z) \\
&= (T_{\alpha+\beta}(x,m) + A_{\alpha,\beta}(x,m)) * M_{\gamma+\gamma'}(w) * N_{\nu+\nu'}(z) \\
&= T_{\alpha+\beta}(x,m) * M_{\gamma+\gamma'}(w) * N_{\nu+\nu'}(z) \\
&\quad + A_{\alpha,\beta}(x,m) * M_{\gamma+\gamma'}(w) * N_{\nu+\nu'}(z) \\
&= \mathcal{M}_{\alpha+\beta,\gamma+\gamma',\nu+\nu'} + A_{\alpha,\beta}(x,m) * M_{\gamma+\gamma'}(w) * N_{\nu+\nu'}(z),
\end{aligned}$$

where $A_{\alpha,\beta}(x,m)$ is defined in Equation (9).

(ii) If both p and q are odd, the same computation as in case (i) applies:

$$\mathcal{M}_{\alpha,\gamma,\nu} * \mathcal{M}_{\beta,\gamma',\nu'} = \mathcal{M}_{\alpha+\beta,\gamma+\gamma',\nu+\nu'} + A_{\alpha,\beta}(x,m) * M_{\gamma+\gamma'}(w) * N_{\nu+\nu'}(z).$$

(iii) If p is even and q is odd, then by Lemmas 1 and 2, we have

$$\begin{aligned}
\mathcal{M}_{\alpha,\gamma,\nu} * \mathcal{M}_{\beta,\gamma',\nu'} &= T_\alpha(x,m) * T_\beta(x,m) * M_\gamma(w) * M_{\gamma'}(w) * N_\nu(z) * N_{\nu'}(z) \\
&= \frac{\cos\left(\frac{\alpha\pi}{2}\right)\cos\left(\frac{\beta\pi}{2}\right)}{\cos\left(\frac{(\alpha+\beta)\pi}{2}\right)} T_{\alpha+\beta}(x,m) * M_{\gamma+\gamma'}(w) * N_{\nu+\nu'}(z) \\
&= \frac{\cos\left(\frac{\alpha\pi}{2}\right)\cos\left(\frac{\beta\pi}{2}\right)}{\cos\left(\frac{(\alpha+\beta)\pi}{2}\right)} \cdot \mathcal{M}_{\alpha+\beta,\gamma+\gamma',\nu+\nu'}.
\end{aligned}$$

(iv) If both p and q are even, then again by Lemmas 1 and 2,

$$\mathcal{M}_{\alpha,\gamma,\nu} * \mathcal{M}_{\beta,\gamma',\nu'} = \frac{\cos\left(\frac{\alpha\pi}{2}\right)\cos\left(\frac{\beta\pi}{2}\right)}{\cos\left(\frac{(\alpha+\beta)\pi}{2}\right)} \cdot \mathcal{M}_{\alpha+\beta,\gamma+\gamma',\nu+\nu'}.$$

To conclude, we verify the fundamental identity:

$$\mathcal{M}_{\alpha,\gamma,\nu} * \mathcal{M}_{-\alpha,-\gamma,-\nu}.$$

Case A: If p is odd and q is even,

$$\begin{aligned}
\mathcal{M}_{\alpha,\gamma,\nu} * \mathcal{M}_{-\alpha,-\gamma,-\nu} &= (T_0(x,m) + A_{\alpha,-\alpha}(x,m)) * \delta(w) * \delta(z) \\
&= \delta(x) + \sin^2\left(\frac{\alpha\pi}{2}\right)\delta(x) \\
&= \left(1 + \sin^2\left(\frac{\alpha\pi}{2}\right)\right)\delta(x). \qquad (12)
\end{aligned}$$

Case B: If both p and q are odd,

$$\mathcal{M}_{\alpha,\gamma,\nu} * \mathcal{M}_{-\alpha,-\gamma,-\nu} = T_0(x,m) * \delta(w) * \delta(z) = \delta(x). \tag{13}$$

Case C: If p is even and q is odd,

$$\mathcal{M}_{\alpha,\gamma,\nu} * \mathcal{M}_{-\alpha,-\gamma,-\nu} = \cos^2\left(\frac{\alpha\pi}{2}\right)\delta(x). \tag{14}$$

Case D: If both p and q are even,

$$\mathcal{M}_{\alpha,\gamma,\nu} * \mathcal{M}_{-\alpha,-\gamma,-\nu} = \cos^2\left(\frac{\alpha\pi}{2}\right)\delta(x). \tag{15}$$

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4. Main results

In this section, we establish the explicit form of the inverse operator $L^{(\alpha,\gamma,\nu)}$ corresponding to the convolution-type operator $\mathcal{D}_m^{(\alpha,\gamma,\nu)}$. Based on the convolution identities derived earlier, we provide conditions under which the inverse exists and determine its precise expression depending on the parity of p and q. This result forms the core of our investigation into the invertibility of generalized convolution operators associated with the kernel $\mathcal{M}_{\alpha,\gamma,\nu}$.

Let $\mathcal{D}_m^{(\alpha,\gamma,\nu)}(f)$ be the $\mathcal{D}_m$ operator of order (α,γ,ν) acting on the function f, defined by

$$\mathcal{D}_m^{(\alpha,\gamma,\nu)}(f) = \mathcal{M}_{\alpha,\gamma,\nu} * f, \tag{16}$$

where $\mathcal{M}_{\alpha,\gamma,\nu}$ is defined by (11), $\alpha,\gamma,\nu \in \mathbb{C}$, and $f \in S$.

Recall that our objective is to obtain the operator $L^{(\alpha,\gamma,\nu)} = \left[\mathcal{D}_m^{(\alpha,\gamma,\nu)}\right]^{-1}$ such that if $\mathcal{D}_m^{(\alpha,\gamma,\nu)}(f) = \varphi$, then $L^{(\alpha,\gamma,\nu)}\varphi = f$ for all $\alpha,\gamma,\nu \in \mathbb{C}$.

We are now ready to state our main theorem.

Theorem 1. *Let $f \in \mathcal{S}$ and suppose that*

$$\mathcal{D}_m^{(\alpha,\gamma,\nu)}(f) = \varphi,$$

where $\mathcal{D}_m^{(\alpha,\gamma,\nu)}$ is defined by (16)*. Then the inverse operator $L^{(\alpha,\gamma,\nu)} = \left[\mathcal{D}_m^{(\alpha,\gamma,\nu)}\right]^{-1}$ is given by*

$$L^{(\alpha,\gamma,\nu)} = \begin{cases} \left[1+\sin^2\left(\frac{\alpha\pi}{2}\right)\right]^{-1}\mathcal{M}_{-\alpha,-\gamma,-\nu}, & \text{if } p \text{ is odd and } q \text{ is even;} \\ \mathcal{M}_{-\alpha,-\gamma,-\nu}, & \text{if both } p \text{ and } q \text{ are odd;} \\ \sec^2\left(\frac{\alpha\pi}{2}\right)\mathcal{M}_{-\alpha,-\gamma,-\nu}, & \text{if } p \text{ is even and } \alpha \neq 2s+1, \end{cases}$$

s is a non-negative integer, such that $L^{(\alpha,\gamma,\nu)}(\varphi) = f$.

Proof. From (16), we have

$$\mathcal{D}_m^{(\alpha,\gamma,\nu)}(f) = \mathcal{M}_{\alpha,\gamma,\nu} * f = \varphi,$$

where $\alpha, \gamma, \nu \in \mathbb{C}$, and $f \in \mathcal{S}$.

Case 1: If p is odd and q is even, then by (12),

$$\begin{aligned}
&\left[1+\sin^2\left(\frac{\alpha\pi}{2}\right)\right]^{-1} \mathcal{M}_{-\alpha,-\gamma,-\nu} * (\mathcal{M}_{\alpha,\gamma,\nu} * f)\\
&= \left[1+\sin^2\left(\frac{\alpha\pi}{2}\right)\right]^{-1} (\mathcal{M}_{-\alpha,-\gamma,-\nu} * \mathcal{M}_{\alpha,\gamma,\nu}) * f\\
&= \left[1+\sin^2\left(\frac{\alpha\pi}{2}\right)\right]^{-1} \left[1+\sin^2\left(\frac{\alpha\pi}{2}\right)\right] \delta(x) * f\\
&= \delta(x) * f = f.
\end{aligned}$$

Hence,

$$\left[\mathcal{D}_m^{(\alpha,\gamma,\nu)}\right]^{-1} = \left[1+\sin^2\left(\frac{\alpha\pi}{2}\right)\right]^{-1} \mathcal{M}_{-\alpha,-\gamma,-\nu}. \tag{17}$$

Case 2: If both p and q are odd, then by (13),

$$\begin{aligned}
\mathcal{M}_{-\alpha,-\gamma,-\nu} * (\mathcal{M}_{\alpha,\gamma,\nu} * f) &= (\mathcal{M}_{-\alpha,-\gamma,-\nu} * \mathcal{M}_{\alpha,\gamma,\nu}) * f\\
&= \delta(x) * f = f.
\end{aligned}$$

Therefore,

$$\left[\mathcal{D}_m^{(\alpha,\gamma,\nu)}\right]^{-1} = \mathcal{M}_{-\alpha,-\gamma,-\nu}. \tag{18}$$

Case 3: If p is even, then from (14) and (15), and provided that $\alpha \neq 2s+1$ for $s \in \mathbb{N}_0$, we have

$$\begin{aligned}
\sec^2\left(\frac{\alpha\pi}{2}\right) \mathcal{M}_{-\alpha,-\gamma,-\nu} * (\mathcal{M}_{\alpha,\gamma,\nu} * f) &= \sec^2\left(\frac{\alpha\pi}{2}\right) (\mathcal{M}_{-\alpha,-\gamma,-\nu} * \mathcal{M}_{\alpha,\gamma,\nu}) * f\\
&= \sec^2\left(\frac{\alpha\pi}{2}\right) \cos^2\left(\frac{\alpha\pi}{2}\right) \delta(x) * f\\
&= \delta(x) * f = f.
\end{aligned}$$

Therefore,

$$\left[\mathcal{D}_m^{(\alpha,\gamma,\nu)}\right]^{-1} = \sec^2\left(\frac{\alpha\pi}{2}\right) \mathcal{M}_{-\alpha,-\gamma,-\nu}. \tag{19}$$

Combining (17), (18), and (19) completes the proof. ◀

Example 1. *Consider the parameters*

$$\alpha = 1, \quad \gamma = 0, \quad \nu = 0,$$

and the dimension parameters

$$p = 3 \quad (odd), \quad q = 2 \quad (even).$$

Let $f \in \mathcal{S}(\mathbb{R}^5)$ *be a Schwartz function, and define the operator*

$$\mathcal{D}_m^{(1,0,0)}(f) = \mathcal{M}_{1,0,0} * f = \varphi.$$

By the theorem, the inverse operator in this case (Case 1: p odd, q even) is given by

$$L^{(1,0,0)} = \left[1 + \sin^2\left(\frac{\pi}{2}\right)\right]^{-1} \mathcal{M}_{-1,0,0} = \frac{1}{1+1}\mathcal{M}_{-1,0,0} = \frac{1}{2}\mathcal{M}_{-1,0,0}.$$

Hence, we can recover the original function f from φ *by*

$$f = L^{(1,0,0)}(\varphi) = \frac{1}{2}\mathcal{M}_{-1,0,0} * \varphi.$$

Remark 1. *This example illustrates how the inverse operator depends on the dimension parameters* p, q *and the parameter* α*. Here, the scaling constant is* $\frac{1}{2}$ *due to* $\sin^2(\pi/2) = 1$*.*

5. Conclusions

In this work, we have established explicit formulas for the inverse operators associated with the generalized convolution-type operators $\mathcal{D}_m^{(\alpha,\gamma,\nu)}$ on Schwartz spaces. The inversion formulas depend critically on the parity of the dimension parameters p and q, as well as the parameter α, which affects the scaling factors in the inverse operators.

Our results generalize known fundamental solutions and provide a clear convolution framework for solving operator equations involving $\mathcal{D}_m^{(\alpha,\gamma,\nu)}$. These findings lay the groundwork for further analysis of partial differential equations in ultra-hyperbolic and elliptic settings, with potential applications in mathematical physics and related fields.

Future work may extend these inversion formulas to broader function spaces or investigate their numerical implementations in solving practical problems.

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