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Commutators of Multi-sublinear Maximal Operators on Product Total Morrey-Guliyev Spaces

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Abstract. In this paper, we obtain necessary and sufficient conditions for the boundedness of the multi-sublinear maximal commutator operator $\mathcal{M}_{\vec{b}}$ and the commutator of the multi-sublinear maximal operator $[\vec{b}, \mathcal{M}]$ on product total Morrey spaces $L^{p_1,\lambda_1,\mu_1}(\mathbb{R}^n) \times \ldots \times L^{p_m,\lambda_m,\mu_m}(\mathbb{R}^n)$ to the total Morrey spaces $L^{p,\lambda,\mu}(\mathbb{R}^n)$. New characterizations of certain subclasses of $BMO^m(\mathbb{R}^n)$ are obtained.

Key Words and Phrases: total Morrey-Guliyev spaces, multi-sublinear maximal function, commutator, BMO.

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1. Introduction

The classical Morrey spaces were introduced by Morrey [24] for the study of solutions of some quasi-linear elliptic partial differential equations. For more applications of Morrey spaces to partial differential equation, the reader is referred to [4, 29]. The total Morrey-Guliyev spaces $L^{p,\lambda,\mu}(\mathbb{R}^n)$, introduced by Guliyev [8], extend the Morrey space $L^{p,\lambda}(\mathbb{R}^n)$ by including the second parameter μ, which can be seen as the intermediate spaces between Lebesgue spaces and Morrey spaces. The norm in these spaces is defined by a combination of the norms of $L^{p,\lambda}(\mathbb{R}^n)$ and $L^{p,\mu}(\mathbb{R}^n)$, which allows for a wider range of behavior. Let $0 < p < \infty$, $\lambda \in \mathbb{R}$, $\mu \in \mathbb{R}$, $[t]_1 = \min\{1, t\}$, $t > 0$. The total Morrey-Guliyev spaces $L^{p,\lambda,\mu}(\mathbb{R}^n)$ are the set of all locally integrable functions f with finite (quasi-)norm

$$\|f\|_{L^{p,\lambda,\mu}} = \sup_{x\in\mathbb{R}^n,\, t>0} [t]_1^{-\frac{\lambda}{p}} \, [1/t]_1^{\frac{\mu}{p}} \, \|f\|_{L^p(B(x,t))},$$

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where $B(x,t)$ denotes the ball centered at x with radius $t>0$. Here, the norm in the case $\mu \le \lambda$ is equal to the maximum of the norms of $L^{p,\lambda}(\mathbb{R}^n)$ and $L^{p,\mu}(\mathbb{R}^n)$. Total Morrey-Guliyev spaces can be viewed as generalizations of both classical and modified Morrey spaces. In particular, the case where $\lambda=\mu$ corresponds to classical Morrey space, and the case where $\mu=0$ corresponds to the modified Morrey space $\widetilde{L}_{p,\lambda}(\mathbb{R}^n)$, see [2, 3, 7, 9, 11, 12, 13, 14, 21, 23, 25, 27, 28].

Let $(\mathbb{R}^n)^m=\mathbb{R}^n\times\ldots\times\mathbb{R}^n$ be the m-fold product space ($m\in\mathbb{N}$).

For $x\in\mathbb{R}^n$ and $r>0$, we denote by $B(x,r)$ the open ball centered at x of radius r, and by ${}^{\complement}B(x,r)$ its complement. Let $|B(x,r)|$ be the Lebesgue measure of the ball $B(x,r)$. We denote by $\vec{f}$ the m-tuple $(f_1,f_2,\ldots,f_m)$, $\vec{y}=(y_1,\ldots,y_m)$ and $d\vec{y}=dy_1\cdots dy_m$.

In the past twenty years, the multilinear Calderón-Zygmund theory has been extensively developed and studied by many authors. Grafakos and Torres [6] introduced the multilinear Calderón-Zygmund operator and studied the boundedness of such operators. Later, in [20], Lerner et al. introduced the following multi-sublinear maximal function $\mathcal{M}(\vec{f})(x)$ defined by

$$\mathcal{M}(\vec{f})(x)=\sup_{B\ni x}\prod_{i=1}^{m}\frac{1}{|B|}\int_B|f_i(y_i)|dy_i.$$

When $m=1$, we get $M\equiv M_1$, the classical Hardy-Littlewood maximal operator, given by

$$M(f)(x)=\sup_{B\ni x}\frac{1}{|B|}\int_B|f(y)|dy,$$

where f is a locally integrable function. Obviously, it is easy to see $\mathcal{M}(\vec{f})(x)\le\prod_{j=1}^{m}M(f_j)(x)$.

The sharp maximal function of Fefferman and Stein $M^{\sharp}f$, is defined by

$$M^{\sharp}f(x)=\sup_{B\ni x}|B|^{-1}\int_B|f(y)-f_B|dy,$$

where the supremum is taken over all balls $B\subset\mathbb{R}^n$ containing x. For a fixed $q\in(0,1)$, any suitable function h, and $x\in\mathbb{R}^n$, let $M_q^{\sharp}h(x)=\big(M^{\sharp}\big(|h|^q\big)(x)\big)^{1/q}$ and $M_qh(x)=\big(M\big(|h|^q\big)(x)\big)^{1/q}$.

For any ball $B\subset\mathbb{R}^n$, the mean oscillation space $BMO(\mathbb{R}^n)$ is defined by

$$BMO(\mathbb{R}^n)=\Big\{b\in L_{\rm loc}(\mathbb{R}^n):\|f\|_{BMO}:=\sup_{B}|B|^{-1}\int_B|f(y)-f_B|dy<\infty\Big\},$$

where $f_B=|B|^{-1}\int_B f(y)dy$.

In [8], Guliyev gave the necessary and sufficient conditions for the boundedness of the maximal commutator operator

$$M_b(f)(x) = \sup_{B \ni x} |B|^{-1} \int_B |b(x) - b(y)||f(y)|dy$$

and the commutator of Hardy-Littlewood maximal operator

$$[b, M](f)(x) = b(x)M(f)(x) - M(bf)(x)$$

on $L^{p,\lambda,\mu}(\mathbb{R}^n)$, where $b \in BMO(\mathbb{R}^n)$ (see also [9]).

The commutators of singular integral operators play an important role in studying the regularity of solutions of elliptic partial differential equations. In [35, 36], Zhang studied the multi-sublinear maximal commutator operator $\mathcal{M}_{\vec{b}}$ and the commutator of multi-sublinear maximal operator $[\vec{b}, \mathcal{M}]$ generated by the BMO^m vector function defined as follows:

$$\mathcal{M}_{\vec{b}}(\vec{f})(x) = \sum_{j=1}^{m} \mathcal{M}_{b_j}(\vec{f})(x) \quad \text{and} \quad [\vec{b}, \mathcal{M}](\vec{f})(x) = \sum_{j=1}^{m} [\vec{b}, \mathcal{M}]_j(\vec{f})(x).$$

Here

$$\mathcal{M}_{b_j}(\vec{f})(x) = \sup_{B \ni x} \frac{1}{|B|^m} \int_B \left|b_j(x) - b_j(y_j)\right| \prod_{i=1}^{m} |f_i(y_i)| \, d\vec{y}$$

and

$$[\vec{b}, \mathcal{M}]_j(\vec{f})(x) = b_j(x)\mathcal{M}(\vec{f})(x) - \mathcal{M}(f_1, \ldots, f_{j-1}, b_j f_j, f_{j+1}, \ldots, f_m)(x).$$

Obviously, the operators $\mathcal{M}_{\vec{b}}$ and $[\vec{b}, \mathcal{M}]$ are essentially different from each other because $\mathcal{M}_{\vec{b}}$ is positive and sublinear, whereas $[\vec{b}, \mathcal{M}]$ is neither positive nor sublinear.

Recently, Yu, Zhang and Li [37] extended the results of [31, Theorem 7] on the boundedness of $[b, M]$ on $L^{p,\lambda}(\mathbb{R}^n)$ to product Morrey spaces by replacing $[b, M]$ with $[\vec{b}, \mathcal{M}]$ as follows.

Theorem 1. *[37] Let b_i be a real valued, locally integrable function in $\mathbb{R}^n$ for $i = 1, \ldots, m$. Suppose that $1 < p_i < \infty$ and $1/p = \sum_{i=1}^{m} 1/p_i$ with $p > 1$ and $0 \leq \lambda < n$.*

Then the following assertions are equivalent:

(i) $b_i \in BMO(\mathbb{R}^n)$ such that $b_i^- \in L^\infty(\mathbb{R}^n)$, $i = 1, 2, \ldots, m$.

(ii) The commutator $[\vec{b}, \mathcal{M}]$ is bounded from $L^{p_1,\lambda}(\mathbb{R}^n) \times \ldots \times L^{p_m,\lambda}(\mathbb{R}^n)$ to $L^{p,\lambda}(\mathbb{R}^n)$.

(iii) There is

$$\sup_B \frac{\left\|\left(b_i - M_B(b_i)\right)\chi_B\right\|_{L^p}}{\prod\limits_{j=1}^{m} \|\chi_B\|_{L^{p_j}}} < \infty,$$

where $M_B(b_i)(x) := \sup\limits_{B'\ni x:\, B'\subset B} |B'|^{-1}\int_{B'}|b_i(y)|\,dy$ *for any* $i = 1, 2, \ldots, m$.

In this paper, we obtain necessary and sufficient conditions for the boundedness of the multi-sublinear maximal commutator operator $\mathcal{M}_{\vec{b}}$ and the commutator of multi-sublinear maximal operator $[\vec{b}, \mathcal{M}]$ on product total Morrey spaces $L^{p_1,\lambda_1,\mu_1}(\mathbb{R}^n) \times \ldots \times L^{p_m,\lambda_m,\mu_m}(\mathbb{R}^n)$.

This paper is organized as follows. In Section 2, we present some theorems on the boundedness of the multi-sublinear maximal operator $\mathcal{M}$ on product total Morrey spaces $L^{p,\lambda,\mu}(\mathbb{R}^n)$. In Section 3, we find necessary and sufficient conditions for the boundedness of the multi-sublinear maximal commutator operator $\mathcal{M}_{\vec{b}}$ on product total Morrey space $L^{p_1,\lambda_1,\mu_1}(\mathbb{R}^n) \times \ldots \times L^{p_m,\lambda_m,\mu_m}(\mathbb{R}^n)$ to total Morrey spaces $L^{p,\lambda,\mu}(\mathbb{R}^n)$. In Section 4, we extend Theorem 1 with $[\vec{b}, \mathcal{M}](\vec{f})$ on the product total Morrey spaces. We find necessary and sufficient conditions for the boundedness of the commutator of the multi-sublinear maximal operator $[\vec{b}, \mathcal{M}]$ from the product total Morrey space $L^{p_1,\lambda_1,\mu_1}(\mathbb{R}^n) \times \ldots \times L^{p_m,\lambda_m,\mu_m}(\mathbb{R}^n)$ to the total Morrey spaces $L^{p,\lambda,\mu}(\mathbb{R}^n)$. New characterizations of certain subclasses of $BMO^m(\mathbb{R}^n)$ are obtained, see also [10, 11, 13, 17, 19, 22, 26, 27, 28, 30, 32, 33, 34].

Throughout this paper, the letter C denotes a positive constant that may vary from line to line but is independent of the essential variables.

2. Multi-sublinear maximal operator on product total Morrey spaces

In this section, we investigate the boundedness of the multi-sublinear maximal operator $\mathcal{M}$ on product total Morrey spaces.

Definition 1. *Let* $0 < p < \infty$, $\lambda \in \mathbb{R}$, $\mu \in \mathbb{R}$, $[t]_1 = \min\{1, t\}$, $t > 0$. *We denote by* $L^{p,\lambda}(\mathbb{R}^n)$ *the classical Morrey space, by* $\widetilde{L}^{p,\lambda}(\mathbb{R}^n)$ *the modified Morrey space [7], and by* $L^{p,\lambda,\mu}(\mathbb{R}^n)$ *the total Morrey space the set of all classes of locally integrable functions* f *with the finite quasi-norms*

$$\|f\|_{L^{p,\lambda}} = \sup_{x\in\mathbb{R}^n,\, t>0} t^{-\frac{\lambda}{p}}\,\|f\|_{L^p(B(x,t))},\ \ \|f\|_{\widetilde{L}^{p,\lambda}} = \sup_{x\in\mathbb{R}^n,\, t>0} [t]_1^{-\frac{\lambda}{p}}\,\|f\|_{L^p(B(x,t))},$$

$$\|f\|_{L^{p,\lambda,\mu}} = \sup_{x\in\mathbb{R}^n,\, t>0} [t]_1^{-\frac{\lambda}{p}}\,[1/t]_1^{\frac{\mu}{p}}\,\|f\|_{L^p(B(x,t))},$$

respectively.

Definition 2. *Let $0 < p < \infty$, $\lambda \in \mathbb{R}$ and $\mu \in \mathbb{R}$. We define the weak Morrey space $WL^{p,\lambda}(\mathbb{R}^n)$, the weak modified Morrey space $W\widetilde{L}^{p,\lambda}(\mathbb{R}^n)$ [7] and the weak total Morrey space $WL^{p,\lambda,\mu}(\mathbb{R}^n)$ as the set of all locally integrable functions f with finite quasi-norms*

$$\begin{aligned}\|f\|_{WL^{p,\lambda}} &= \sup_{x\in\mathbb{R}^n,\, t>0} t^{-\frac{\lambda}{p}} \|f\|_{WL^p(B(x,t))}, \; \|f\|_{W\widetilde{L}^{p,\lambda}} \\ &= \sup_{x\in\mathbb{R}^n,\, t>0} [t]_1^{-\frac{\lambda}{p}} \|f\|_{WL^p(B(x,t))}, \\ \|f\|_{WL^{p,\lambda,\mu}} &= \sup_{x\in\mathbb{R}^n,\, t>0} [t]_1^{-\frac{\lambda}{p}} [1/t]_1^{\frac{\mu}{p}} \|f\|_{WL^p(B(x,t))},\end{aligned}$$

respectively.

Lemma 1. *[8, 9] If $0 < p < \infty$, $0 \le \mu \le \lambda \le n$, then*

$$\begin{aligned}L^{p,\lambda,\mu}(\mathbb{R}^n) &= L^{p,\lambda}(\mathbb{R}^n) \cap L^{p,\mu}(\mathbb{R}^n), \\ WL^{p,\lambda,\mu}(\mathbb{R}^n) &= WL^{p,\lambda}(\mathbb{R}^n) \cap WL^{p,\mu}(\mathbb{R}^n)\end{aligned}$$

and

$$\begin{aligned}\|f\|_{L^{p,\lambda,\mu}(\mathbb{R}^n)} &= \max\left\{\|f\|_{L^{p,\lambda}}, \|f\|_{L^{p,\mu}}\right\}, \\ \|f\|_{WL^{p,\lambda,\mu}(\mathbb{R}^n)} &= \max\left\{\|f\|_{WL^{p,\lambda}}, \|f\|_{WL^{p,\mu}}\right\},\end{aligned}$$

respectively.

The following local estimate is valid.

Lemma 2. *[1, Lemma 3.3] Let $1 \le p < \infty$ and $B(x,r)$ be any ball in $\mathbb{R}^n$. Then, for $p > 1$ the inequality*

$$\|Mf\|_{L^p(B(x,r))} \lesssim r^{\frac{n}{p}} \sup_{t>2r} t^{-\frac{n}{p}} \|f\|_{L^p(B(x,t))} \tag{1}$$

holds for all $B(x,r)$ and for all $f \in L^p_{\mathrm{loc}}(\mathbb{R}^n)$.

Moreover, if $p = 1$, then the inequality

$$\|Mf\|_{WL^1(B(x,r))} \lesssim r^n \sup_{t>2r} t^{-n} \|f\|_{L^1(B(x,t))}$$

holds for all $B(x,r)$ and for all $f \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$.

Theorem 2. [8] *1. If $f \in L^{1,\lambda,\mu}(\mathbb{R}^n)$, $0 \le \lambda < n$ and $0 \le \mu < n$, then $Mf \in WL^{1,\lambda,\mu}(\mathbb{R}^n)$ and*

$$\|Mf\|_{WL^{1,\lambda,\mu}} \le C_{1,\lambda,\mu}\, \|f\|_{L^{1,\lambda,\mu}}, \tag{2}$$

where $C_{1,\lambda,\mu}$ is independent of f.

2. If $f \in L^{p,\lambda,\mu}(\mathbb{R}^n)$, $1 < p < \infty$, $0 \le \lambda < n$ and $0 \le \mu < n$, then $Mf \in L^{p,\lambda,\mu}(\mathbb{R}^n)$ and

$$\|Mf\|_{L^{p,\lambda,\mu}} \le C_{p,\lambda,\mu}\, \|f\|_{L^{p,\lambda,\mu}}, \tag{3}$$

where $C_{p,\lambda,\mu}$ depends only on p,λ,μ and n.

Lemma 3. [16] *Let p be the harmonic mean of $p_1, \ldots, p_m > 1$ and $\vec{f} \in L^1_{\mathrm{loc}}(\mathbb{R}^n) \times \ldots \times L^1_{\mathrm{loc}}(\mathbb{R}^n)$. Then there exists a constant $C > 0$ such that for any $x \in \mathbb{R}^n$*

$$\mathcal{M}\vec{f}(x) \le \prod_{j=1}^{m} \Big[M\big(f_j^{\frac{p_j}{p}}\big)(x) \Big]^{\frac{p}{p_j}}, \tag{4}$$

When $m \ge 2$, we find out $\mathcal{M}$ also has the same properties by providing the following multi-version of the Theorem 2.

Theorem 3. *Let p be the harmonic mean of $p_1, \ldots, p_m > 1$ and*

$$\frac{\lambda}{p} = \sum_{j=1}^{m} \frac{\lambda_j}{p_j} \quad \text{for} \quad 0 \le \lambda_j < n, \quad \frac{\mu}{p} = \sum_{j=1}^{m} \frac{\mu_j}{p_j} \quad \text{for} \quad 0 \le \mu_j < n. \tag{5}$$

(i) If $p > 1$, then the operator $\mathcal{M}$ is bounded from the product total Morrey space $L^{p_1,\lambda_1,\mu_1}(\mathbb{R}^n) \times \ldots \times L^{p_m,\lambda_m,\mu_m}(\mathbb{R}^n)$ to the total Morrey space $L^{p,\lambda,\mu}(\mathbb{R}^n)$. Moreover, there exists a positive constant C such that the following inequality is valid for all $\vec{f} \in L^{p_1,\lambda_1,\mu_1}(\mathbb{R}^n) \times \ldots \times L^{p_m,\lambda_m,\mu_m}(\mathbb{R}^n)$

$$\|\mathcal{M}\vec{f}\|_{L^{p,\lambda,\mu}} \le C \prod_{j=1}^{m} \|f_j\|_{L^{p_j,\lambda_j,\mu_j}}.$$

(ii) If $p = 1$, then the operator $\mathcal{M}$ is bounded from the product total Morrey space $L^{p_1,\lambda_1,\mu_1}(\mathbb{R}^n) \times \ldots \times L^{p_m,\lambda_m,\mu_m}(\mathbb{R}^n)$ to weak total Morrey space $WL^{p,\lambda,\mu}(\mathbb{R}^n)$. Moreover, there exists a positive constant C such that the following inequality is valid for all $\vec{f} \in L^{p_1,\lambda_1,\mu_1}(\mathbb{R}^n) \times \ldots \times L^{p_m,\lambda_m,\mu_m}(\mathbb{R}^n)$

$$\|\mathcal{M}\vec{f}\|_{WL^{p,\lambda,\mu}} \le C \prod_{j=1}^{m} \|f_j\|_{L^{p_j,\lambda_j,\mu_j}}.$$

Proof.

(i) If $p>1$, by (4) and the Hölder inequality, we get

$$\begin{aligned}
&\left([t]_1^{-\lambda}\,[1/t]_1^{\mu}\int_{B(x,t)}|\mathcal{M}\vec{f}(y)|^p dy\right)^{\frac{1}{p}}\\
&\le\left([t]_1^{-\lambda}\,[1/t]_1^{\mu}\int_{B(x,t)}\left(\prod_{j=1}^m\left[M(f_j^{\frac{p_j}{p}})(y)\right]^{\frac{p}{p_j}}\right)^p dy\right)^{\frac{1}{p}}\\
&\le\prod_{j=1}^m\left([t]_1^{-\lambda_j}\,[1/t]_1^{\mu_j}\int_{B(x,t)}\left[M(f_j^{\frac{p_j}{p}})(y)\right]^p dy\right)^{\frac{1}{p_j}}.
\end{aligned}$$

Taking the p-th root of both sides and applying Theorem 2 with $p>1$ and $\left|f_j\right|^{\frac{p_j}{p}}\in L^{p,\lambda_j,\mu_j}(\mathbb{R}^n)$, we get

$$\begin{aligned}
&\|\mathcal{M}\vec{f}\|_{L^{p,\lambda,\mu}}=\sup_{x\in\mathbb{R}^n,\,t>0}\left([t]_1^{-\lambda}\,[1/t]_1^{\mu}\int_{B(x,t)}|\mathcal{M}\vec{f}(y)|^p dy\right)^{\frac{1}{p}}\\
&=\prod_{j=1}^m\left\|M(f_j^{\frac{p_j}{p}})\right\|_{L^{p,\lambda_j,\mu_j}}\le\prod_{j=1}^m\left\|f_j^{\frac{p_j}{p}}\right\|_{L^{p,\lambda_j,\mu_j}}=\prod_{j=1}^m\|f_j\|_{L^{p_j,\lambda_j,\mu_j}},
\end{aligned}$$

which is the desired inequality.

(ii) If $p=1$, for any $\tau>0$, let $\varepsilon_0=\tau$, $\varepsilon_m=1$ and $\varepsilon_1,\varepsilon_2,...,\varepsilon_{m-1}>0$ be arbitrary constants to be chosen later. From the pointwise estimate (4), we get

$$\begin{aligned}
&\{y\in B(x,t):\mathcal{M}\vec{f}(y)>\tau\}\\
&\subset\bigcup_{j=1}^m\left\{y\in B(x,t):\left[M(f_j^{\frac{p_j}{p}})(y)\right]^{\frac{p}{p_j}}>\frac{\varepsilon_{j-1}}{\varepsilon_j\,[t]_1^{\frac{\lambda-\lambda_j}{p_j}}\,[1/t]_1^{-\frac{\mu-\mu_j}{p_j}}}\right\}.
\end{aligned}$$

Let us now take $\varepsilon_1,\varepsilon_2,...,\varepsilon_{m-1}>0$ such that

$$\frac{\varepsilon_j}{\varepsilon_{j=1}}=\frac{\left[\prod_{j=1}^m\|f_j\|_{L^{p_j,\lambda_j,\mu_j}}\right]^{s'/p_j}}{\tau^{s'/p_j}\|f_j\|_{L^{p_j,\lambda_j,\mu_j}}},\qquad j=1,2,...,m.$$

Then, applying Theorem 2 with $p=1$ and the fact $\left|f_j\right|^{p_j}\in L^{1,\lambda_j,\mu_j}(\mathbb{R}^n)$, we get

$$\left|\left\{y\in B(x,t):\mathcal{M}\vec{f}(y)\right|>\tau\right\}$$

$$\lesssim \sum_{j=1}^{m} \Big| \Big\{ y \in B(x,t) : M(f_j^{p_j})(y) > \Big(\frac{\varepsilon_{j-1}}{\varepsilon_j \, [t]_1^{\frac{\lambda-\lambda_j}{p_j}} \, [1/t]_1^{-\frac{\mu-\mu_j}{p_j}}} \Big)^{p_j} \Big\} \Big|$$

$$\leq \sum_{j=1}^{m} [t]_1^{\lambda_j} \, [1/t]_1^{-\mu_j} \Big(\frac{\varepsilon_j \, [t]_1^{\frac{\lambda-\lambda_j}{p_j}} \, [1/t]_1^{-\frac{\mu-\mu_j}{p_j}}}{\varepsilon_{j-1}} \Big)^{p_j} \| f_j^{p_j} \|_{L^{1,\lambda_j,\mu_j}}$$

$$= \sum_{j=1}^{m} [t]_1^{\lambda} \, [1/t]_1^{-\mu} \Big(\frac{\varepsilon_j}{\varepsilon_{j-1}} \Big)^{p_j} \| f_j \|_{L^{p_j,\lambda_j,\mu_j}}^{p_j} = \sum_{j=1}^{m} [t]_1^{\lambda} \, [1/t]_1^{-\mu} \Big[\Big(\frac{\varepsilon_j}{\varepsilon_{j-1}} \Big) \| f_j \|_{L^{p_j,\lambda_j,\mu_j}} \Big]^{p_j}$$

$$= \sum_{j=1}^{m} [t]_1^{\lambda} \, [1/t]_1^{-\mu} \Big(\frac{1}{\tau} \prod_{j=1}^{m} \| f_j \|_{L^{p_j,\lambda_j,\mu_j}} \Big)^{s'} = [t]_1^{\lambda} \, [1/t]_1^{-\mu} \Big(\frac{1}{\tau} \prod_{j=1}^{m} \| f_j \|_{L^{p_j,\lambda_j,\mu_j}} \Big)^{p}.$$

Hence, we obtain the following inequality

$$\begin{aligned} &\| \mathcal{M} \vec{f} \|_{WL^{p,\lambda,\mu}} \\ &= \sup_{\tau>0} \tau \sup_{x \in \mathbb{R}^n, t>0} \Big([t]_1^{-\lambda} \, [1/t]_1^{\mu} \Big| \Big\{ y \in B(x,t) : \mathcal{M} \vec{f}(y) > \tau \Big\} \Big| \Big)^{\frac{1}{p}} \\ &\lesssim \prod_{j=1}^{m} \| f_j \|_{L^{p_j,\lambda_j,\mu_j}}. \end{aligned}$$

This is the conclusion (ii) of Theorem 3.

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In the case $\lambda = \mu$, $\lambda_j = \mu_j$, $j = 1, \dots, m$ from Theorem 3 we get the following corollary

Corollary 1. [*15, Theorem* 2] *Let p be the harmonic mean of $p_1, \dots, p_m > 1$ and*

$$\frac{\lambda}{p} = \sum_{j=1}^{m} \frac{\lambda_j}{p_j} \quad \text{for} \quad 0 \leq \lambda_j < n. \tag{6}$$

(i) If $p > 1$, then the operator $\mathcal{M}$ is bounded from product Morrey space $L^{p_1,\lambda_1}(\mathbb{R}^n) \times \ldots \times L^{p_m,\lambda_m}(\mathbb{R}^n)$ to Morrey space $L^{p,\lambda}(\mathbb{R}^n)$. Moreover, there exists a positive constant C such that the following inequality is valid for all $\vec{f} \in L^{p_1,\lambda_1}(\mathbb{R}^n) \times \ldots \times L^{p_m,\lambda_m}(\mathbb{R}^n)$

$$\| \mathcal{M} \vec{f} \|_{L^{p,\lambda}} \leq C \prod_{j=1}^{m} \| f_j \|_{L^{p_j,\lambda_j}}.$$

(ii) If $p = 1$, then the operator $\mathcal{M}$ is bounded from product Morrey space $L^{p_1,\lambda_1}(\mathbb{R}^n) \times \ldots \times L^{p_m,\lambda_m}(\mathbb{R}^n)$ to weak Morrey space $WL^{p,\lambda}(\mathbb{R}^n)$. Moreover, there exists a positive constant C such that the following inequality is valid for all $\vec{f} \in L^{p_1,\lambda_1}(\mathbb{R}^n) \times \ldots \times L^{p_m,\lambda_m}(\mathbb{R}^n)$

$$\|\mathcal{M}\vec{f}\|_{WL^{p,\lambda}} \le C \prod_{j=1}^{m} \|f_j\|_{L^{p_j,\lambda_j}}.$$

In the case $\mu = \mu_j = 0$, $j = 1, \ldots, m$ from Theorem 3 we get the following corollary

Corollary 2. [*15, Theorem* 4] *Let p be the harmonic mean of $p_1, \ldots, p_m > 1$ and satisfy* (6).

(i) If $p > 1$, then the operator $\mathcal{M}$ is bounded from product modified Morrey space $\widetilde{L}^{p_1,\lambda_1}(\mathbb{R}^n) \times \ldots \times \widetilde{L}^{p_m,\lambda_m}(\mathbb{R}^n)$ to modified Morrey space $\widetilde{L}^{p,\lambda}(\mathbb{R}^n)$. Moreover, there exists a positive constant C such that the following inequality is valid for all $\vec{f} \in \widetilde{L}^{p_1,\lambda_1}(\mathbb{R}^n) \times \ldots \times \widetilde{L}^{p_m,\lambda_m}(\mathbb{R}^n)$

$$\|\mathcal{M}\vec{f}\|_{\widetilde{L}^{p,\lambda}} \le C \prod_{j=1}^{m} \|f_j\|_{\widetilde{L}^{p_j,\lambda_j}}.$$

(ii) If $p = 1$, then the operator $\mathcal{M}$ is bounded from product modified Morrey space $\widetilde{L}^{p_1,\lambda_1}(\mathbb{R}^n) \times \ldots \times \widetilde{L}^{p_m,\lambda_m}(\mathbb{R}^n)$ to weak modified Morrey space $W\widetilde{L}^{p,\lambda}(\mathbb{R}^n)$. Moreover, there exists a positive constant C such that the following inequality is valid for all $\vec{f} \in \widetilde{L}^{p_1,\lambda_1}(\mathbb{R}^n) \times \ldots \times \widetilde{L}^{p_m,\lambda_m}(\mathbb{R}^n)$

$$\|\mathcal{M}\vec{f}\|_{W\widetilde{L}^{p,\lambda}} \le C \prod_{j=1}^{m} \|f_j\|_{\widetilde{L}^{p_j,\lambda_j}}.$$

3. Multi-sublinear maximal commutator operator on product total Morrey spaces

In this section we find necessary and sufficient conditions for the boundedness of the multi-sublinear maximal commutator $\mathcal{M}_{\vec{b}}$ from $L^{p_1,\lambda_1,\mu_1}(\mathbb{R}^n) \times \ldots \times L^{p_m,\lambda_m,\mu_m}(\mathbb{R}^n)$ to $L^{p,\lambda,\mu}(\mathbb{R}^n)$ spaces.

Theorem 4. [*18, Lemma* 1] *If $b \in BMO(\mathbb{G})$, then for any $q \in (0,1)$, there exists a positive constant C such that*

$$M_q^{\sharp}\big(M_b f\big)(x) \le C\|b\|_* M^2 f(x) \tag{7}$$

for every $x \in \mathbb{G}$ and for all $f \in L^1_{\mathrm{loc}}(\mathbb{G})$.

Lemma 4. *If $\vec{b} \in BMO^m(\mathbb{R}^n)$, then there exists a positive constant C such that*

$$\mathcal{M}_{\vec{b}}(f)(x) \lesssim \sum_{k=1}^{m} \|b_k\|_{BMO}\, M^2(f_i)(x) \prod_{j\neq i} M(f_j)(x) \tag{8}$$

for almost every $x \in \mathbb{R}^n$ and for all $\vec{f} \in L^1_{\rm loc}(\mathbb{R}^n) \times \ldots \times L^1_{\rm loc}(\mathbb{R}^n)$.

Proof. Note that

$$\mathcal{M}_{b_i}(\vec{f})(x) \le M_{b_i}(f_i)(x) \prod_{j\neq i} M(f_j)(x)$$

and

$$M_{b_i}(f_i)(x) \overset{[18, Lemma\ 1]}{\lesssim} \|b_i\|_{BMO}\, M^2(f_i)(x) \prod_{j\neq i} M(f_j)(x),$$

then

$$\mathcal{M}_{\vec{b}}(\vec{f})(x) = \sum_{i=1}^{m} \mathcal{M}_{b_i}(\vec{f})(x) \le \sum_{i=1}^{m} M_{b_i}(f_i)(x) \prod_{j\neq i} M(f_j)(x)$$
$$\lesssim \sum_{i=1}^{m} \|b_i\|_{BMO}\, M^2(f_i)(x) \prod_{j\neq i} M(f_j)(x).$$

◀

Theorem 5. *Let b_i be a real valued, locally integrable function in $\mathbb{R}^n$, $1 < p_i < \infty$, $0 \le \lambda_i \le n$ and $0 \le \mu_i \le n$ with $i = 1, \ldots, m$. Let also $\frac{1}{p} = \frac{1}{p_1} + \ldots + \frac{1}{p_m}$, $\frac{\lambda}{p} = \frac{\lambda_1}{p_1} + \ldots + \frac{\lambda_m}{p_m}$ and $\frac{\mu}{p} = \frac{\mu_1}{p_1} + \ldots + \frac{\mu_m}{p_m}$.*
The following assertions are equivalent:
(i) $\vec{b} \in BMO^m(\mathbb{R}^n)$.
(ii) The operator $\mathcal{M}_{\vec{b}}$ is bounded from $L^{p_1,\lambda_1,\mu_1}(\mathbb{R}^n) \times \ldots \times L^{p_m,\lambda_m,\mu_m}(\mathbb{R}^n)$ to $L^{p,\lambda,\mu}(\mathbb{R}^n)$.

Proof. *(i)* ⇒ *(ii)*. Suppose that $\vec{b} \in BMO^m(\mathbb{R}^n)$. Combining Theorem 3 and Lemma 4, applying Hölder's inequality (see [5, Exercise 1.1.2, p. 10]), we get

$$\|\mathcal{M}_{\vec{b}}(\vec{f})\|_{L^{p,\lambda,\mu}} = \sup_{x\in\mathbb{R}^n,\, t>0} [t]_1^{-\frac{\lambda}{p}}\, [1/t]_1^{\frac{\mu}{p}}\, \|\mathcal{M}_{\vec{b}}(\vec{f})\|_{L^p(B(x,t))}$$
$$\le \sup_{x\in\mathbb{R}^n,\, t>0} [t]_1^{-\frac{\lambda}{p}}\, [1/t]_1^{\frac{\mu}{p}} \sum_{i=1}^{m} \|b_i\|_{BMO} \Big\|M^2(f_i) \prod_{j\neq i} M(f_j)\Big\|_{L^p(B(x,t))}$$

$$\leq \sup_{x\in\mathbb{R}^n,\, t>0} [t]_1^{-\frac{\lambda}{p}} [1/t]_1^{\frac{\mu}{p}} \sum_{i=1}^m \|b_i\|_{BMO} \left\|M^2(f_i)\right\|_{L^{p_i}(B(x,t))} \prod_{j\neq i} \left\|M(f_j)\right\|_{L^{p_j}(B(x,t))}.$$

From the inequality (1), we get

$$\begin{aligned}
&\left\|M^2(f_i)\right\|_{L^{p_i}(B(x,t))} \lesssim t^{\frac{n}{p_i}} \sup_{\tau>2t} \tau^{-\frac{n}{p_i}} \|M(f_i)\|_{L^p(B(x,\tau))} \\
&\lesssim t^{\frac{n}{p_i}} \sup_{\tau>2t} \tau^{-\frac{n}{p_i}} \tau^{\frac{n}{p_i}} \sup_{\rho>2\tau} \rho^{-\frac{n}{p_i}} \|M(f_i)\|_{L^{p_i}(B(x,\rho))} \\
&= t^{\frac{n}{p_i}} \sup_{\tau>2t} \sup_{\rho>2\tau} \rho^{-\frac{n}{p_i}} \|f_i\|_{L^{p_i}(B(x,\rho))} \leq t^{\frac{n}{p_i}} \sup_{\rho>4t} \rho^{-\frac{n}{p_i}} \|f_i\|_{L^{p_i}(B(x,\rho))} \\
&\leq \|f_i\|_{L^{p_i,\lambda_i,\mu_i}} t^{\frac{n}{p_i}} \sup_{\rho>4t} \rho^{-\frac{n}{p_i}} [\rho]_1^{\frac{\lambda_i}{p_i}} [1/\rho]_1^{-\frac{\mu_i}{p_i}} = \|f_i\|_{L^{p_i,\lambda_i,\mu_i}} t^{\frac{n}{p_i}} \sup_{\rho>4t} [\rho]_1^{\frac{\lambda_i-n}{p_i}} [1/\rho]_1^{-\frac{\mu_i-n}{p_i}} \\
&= \|f_i\|_{L^{p_i,\lambda_i,\mu_i}} [t]_1^{\frac{\lambda_i}{p_i}} [1/t]_1^{-\frac{\mu_i}{p_i}}
\end{aligned}$$

and

$$\begin{aligned}
&\left\|M(f_j)\right\|_{L^{p_j}(B(x,t))} \lesssim t^{\frac{n}{p_j}} \sup_{\tau>2t} \tau^{-\frac{n}{p_j}} \|f_j\|_{L^{p_j}(B(x,\tau))} \\
&\leq \|f_j\|_{L^{p_j,\lambda_j,\mu_j}} t^{\frac{n}{p_j}} \sup_{\rho>4t} \rho^{-\frac{n}{p_j}} [\rho]_1^{\frac{\lambda_j}{p_j}} [1/\rho]_1^{-\frac{\mu_j}{p_j}} = \|f_j\|_{L^{p_j,\lambda_j,\mu_j}} t^{\frac{n}{p_j}} \sup_{\rho>4t} [\rho]_1^{\frac{\lambda_j-n}{p_j}} [1/\rho]_1^{-\frac{\mu_j-n}{p_j}} \\
&= \|f_j\|_{L^{p_j,\lambda_j,\mu_j}} [t]_1^{\frac{\lambda_j}{p_j}} [1/t]_1^{-\frac{\mu_j}{p_j}}.
\end{aligned}$$

Then

$$\begin{aligned}
\|\mathcal{M}_{\vec{b}}(\vec{f})\|_{L^{p,\lambda,\mu}} &\leq \sup_{x\in\mathbb{R}^n,\, t>0} [t]_1^{-\frac{\lambda}{p}} [1/t]_1^{\frac{\mu}{p}} \sum_{i=1}^m \|b_i\|_{BMO} \left\|M^2(f_i)\right\|_{L^{p_i}(B(x,t))} \\
&\times \prod_{j\neq i} \left\|M(f_j)\right\|_{L^{p_j}(B(x,t))} \\
&\leq \sup_{x\in\mathbb{R}^n,\, t>0} [t]_1^{-\frac{\lambda}{p}} [1/t]_1^{\frac{\mu}{p}} \sum_{i=1}^m \|b_i\|_{BMO} \|f_i\|_{L^{p_i,\lambda_i,\mu_i}} [t]_1^{\frac{\lambda_i}{p_i}} [1/t]_1^{-\frac{\mu_i}{p_i}} \\
&\times \prod_{j\neq i} \|f_j\|_{L^{p_j,\lambda_j,\mu_j}} [t]_1^{\frac{\lambda_j}{p_j}} [1/t]_1^{-\frac{\mu_j}{p_j}} \\
&\leq \sup_{x\in\mathbb{R}^n,\, t>0} [t]_1^{-\frac{\lambda}{p}} [1/t]_1^{\frac{\mu}{p}} \sum_{i=1}^m \|b_i\|_{BMO} \prod_{j=1}^m \|f_j\|_{L^{p_j,\lambda_j,\mu_j}} [t]_1^{\frac{\lambda_j}{p_j}} [1/t]_1^{-\frac{\mu_j}{p_j}}
\end{aligned}$$

$$= \sum_{i=1}^{m} \|b_i\|_{BMO} \prod_{j=1}^{m} \|f_j\|_{L^{p_j,\lambda_j,\mu_j}}.$$

$(ii) \Rightarrow (i)$. Assume that $\mathcal{M}_{\vec{b}}$ is bounded from $L^{p_1,\lambda_1,\mu_1}(\mathbb{R}^n)\times\ldots\times L^{p_m,\lambda_m,\mu_m}(\mathbb{R}^n)$ to $L^{p,\lambda,\mu}(\mathbb{R}^n)$. Let $B = B(x,r)$ be a fixed ball. We consider $\vec{f} = \vec{\chi}_B$, where $\vec{\chi}_B = (\underbrace{\chi_B,\ldots,\chi_B}_{m \text{ times}})$. It is easy to compute that

$$\begin{aligned}
\|\chi_B\|_{L^{p_i,\lambda_i,\mu_i}} &\approx \sup_{y\in\mathbb{R}^n,t>0} \Big([t]_1^{-\lambda_i}\ [1/t]_1^{\mu_i} \int_{B(y,t)} \chi_B(z)dz \Big)^{\frac{1}{p_i}} \\
&= \sup_{y\in\mathbb{R}^n,t>0} \Big(|B(y,t)\cap B|\ [t]_1^{-\lambda_i}\ [1/t]_1^{\mu_i} \Big)^{\frac{1}{p_i}} \qquad (9)\\
&= \sup_{B(y,t)\subseteq B} \Big(|B(y,t)|\ [t]_1^{-\lambda_i}\ [1/t]_1^{\mu_i} \Big)^{\frac{1}{p_i}} = r^{\frac{n}{p_i}}\ [r]_1^{-\frac{\lambda_i}{p_i}}\ [1/r]_1^{\frac{\mu_i}{p_i}}.
\end{aligned}$$

On the other hand, since

$$\mathcal{M}_{\vec{b}}(\vec{\chi}_B)(x) \gtrsim \sum_{i=1}^{m} \frac{1}{|B|} \int_B |b_i(z) - (b_i)_B| dz \quad \text{for all} \quad x \in B,$$

we have

$$\begin{aligned}
\|\mathcal{M}_{\vec{b}}(\vec{\chi}_B)\|_{L^{p,\lambda,\mu}} &\approx \sup_{B(y,t)} \Big(\ [t]_1^{-\lambda}\ [1/t]_1^{\mu} \int_{B(y,t)} |\mathcal{M}_{\vec{b}}(\vec{\chi}_B)(z)|^p dz \Big)^{\frac{1}{p}} \\
&\gtrsim r^{\frac{n}{p}}\ [r]_1^{-\frac{\lambda}{p}}\ [1/r]_1^{\frac{\mu}{p}} \sum_{i=1}^{m} \frac{1}{|B|} \int_B |b_i(z) - (b_i)_B| dz. \qquad (10)
\end{aligned}$$

Since, by assumption,

$$\|\mathcal{M}_{\vec{b}}(\vec{\chi}_B)\|_{L^{p,\lambda,\mu}} \lesssim \prod_{i=1}^{m} \|\chi_B\|_{L^{p_i,\lambda_i,\mu_i}},$$

it follows from (9) and (10), that

$$\sum_{i=1}^{m} \frac{1}{|B|} \int_B |b_i(z) - (b_i)_B| dz \lesssim 1.$$

◀

From Theorem 5 in the case $\lambda = \mu$ and $\lambda_i = \mu_i$ with $i = 1,\ldots,m$ we get the following new result.

Corollary 3. *Let b_i be a real valued, locally integrable function in $\mathbb{R}^n$, $1 < p_i < \infty$ and $0 \le \lambda_i \le n$ with $i = 1, \dots, m$. Let also $\frac{1}{p} = \frac{1}{p_1} + \dots + \frac{1}{p_m}$ and $\frac{\lambda}{p} = \frac{\lambda_1}{p_1} + \dots + \frac{\lambda_m}{p_m}$. Then the following assertions are equivalent:*

(i) $\vec{b} \in BMO^m(\mathbb{R}^n)$.

(ii) The operator $\mathcal{M}_{\vec{b}}$ *is bounded from* $L^{p_1,\lambda_1}(\mathbb{R}^n) \times \dots \times L^{p_m,\lambda_m}(\mathbb{R}^n)$ *to* $L^{p,\lambda}(\mathbb{R}^n)$.

From Theorem 5 in the case $\mu = \mu_1 = \dots = \mu_m = 0$ we get the following new result.

Corollary 4. *Let b_i be a real valued, locally integrable function in $\mathbb{R}^n$, $1 < p_i < \infty$ and $0 \le \lambda_i \le n$ with $i = 1, \dots, m$. Let also $\frac{1}{p} = \frac{1}{p_1} + \dots + \frac{1}{p_m}$ and $\frac{\lambda}{p} = \frac{\lambda_1}{p_1} + \dots + \frac{\lambda_m}{p_m}$. Then the following assertions are equivalent:*

(i) $\vec{b} \in BMO^m(\mathbb{R}^n)$.

(ii) The operator $\mathcal{M}_{\vec{b}}$ *is bounded from* $\widetilde{L}^{p_1,\lambda_1}(\mathbb{R}^n) \times \dots \times \widetilde{L}^{p_m,\lambda_m}(\mathbb{R}^n)$ *to* $\widetilde{L}^{p,\lambda}(\mathbb{R}^n)$.

From Theorem 5 in the case $\lambda_1 = \dots = \lambda_m = \lambda$ and $\mu_1 = \dots = \mu_m = \mu$ we get the following corollary.

Corollary 5. *Let b_i be a real valued, locally integrable function in $\mathbb{R}^n$, $1 < p_i < \infty$ with $i = 1, \dots, m$. Let also $\frac{1}{p} = \frac{1}{p_1} + \dots + \frac{1}{p_m}$, $0 \le \lambda \le n$ and $0 \le \mu \le n$.*

The following assertions are equivalent:

(i) $\vec{b} \in BMO^m(\mathbb{R}^n)$.

(ii) The operator $\mathcal{M}_{\vec{b}}$ *is bounded from* $L^{p_1,\lambda,\mu}(\mathbb{R}^n) \times \dots \times L^{p_m,\lambda,\mu}(\mathbb{R}^n)$ *to* $L^{p,\lambda,\mu}(\mathbb{R}^n)$.

From Corollary 5 in the case $\lambda_1 = \dots = \lambda_m = \lambda = \mu_1 = \dots = \mu_m = \mu$ we get the following corollary, proved in [37].

Corollary 6. [*37, Lemma* 3.2] *Let b_i be a real valued, locally integrable function in $\mathbb{R}^n$, $1 < p_i < \infty$ with $i = 1, \dots, m$. Let also $\frac{1}{p} = \frac{1}{p_1} + \dots + \frac{1}{p_m}$, $0 \le \lambda \le n$. Then the following assertions are equivalent:*

(i) $\vec{b} \in BMO^m(\mathbb{R}^n)$.

(ii) The operator $\mathcal{M}_{\vec{b}}$ *is bounded from* $L^{p_1,\lambda}(\mathbb{R}^n) \times \dots \times L^{p_m,\lambda}(\mathbb{R}^n)$ *to* $L^{p,\lambda}(\mathbb{R}^n)$.

From Corollary 5 in the case $\mu_1 = \dots = \mu_m = \mu = 0$ we get the following new result.

Corollary 7. *Let b_i be a real valued, locally integrable function in $\mathbb{R}^n$, $1 < p_i < \infty$ with $i = 1, \dots, m$. Let also $\frac{1}{p} = \frac{1}{p_1} + \dots + \frac{1}{p_m}$, $0 \le \lambda \le n$. Then the following assertions are equivalent:*

(i) $\vec{b} \in BMO^m(\mathbb{R}^n)$.

(ii) The operator $\mathcal{M}_{\vec{b}}$ *is bounded from* $\widetilde{L}^{p_1,\lambda}(\mathbb{R}^n) \times \dots \times \widetilde{L}^{p_m,\lambda}(\mathbb{R}^n)$ *to* $\widetilde{L}^{p,\lambda}(\mathbb{R}^n)$.

Remark 1. *Note that Corollaries 3, 4, 5 and 7 are new.*

4. Commutator of multi-sublinear maximal operator on product total Morrey spaces

In this section, we establish necessary and sufficient conditions for the boundedness of the commutator of the multi-sublinear maximal operator $[\vec{b}, \mathcal{M}]$ on product total Morrey spaces $L^{p,\lambda,\mu}(\mathbb{R}^n)$.

The following relations between $[\vec{b}, \mathcal{M}]$ and $\mathcal{M}_{\vec{b}}$ are valid:

Let b_i be any non-negative locally integrable function with $i = 1, \dots, m$. Then for all $f \in L^1_{\rm loc}(\mathbb{R}^n)$ and $x \in \mathbb{R}^n$ the following inequality is valid

$$\begin{aligned}
&\big|[\vec{b}, \mathcal{M}](\vec{f})(x)\big| = \Big|\sum_{j=1}^{m}[\vec{b}, \mathcal{M}]_j(\vec{f})(x)\Big| \le \sum_{j=1}^{m}\Big|[\vec{b}, \mathcal{M}]_j(\vec{f})(x)\Big| \\
&= \sum_{j=1}^{m}\Big|b_j(x)\mathcal{M}(\vec{f})(x) - \mathcal{M}(f_1, \dots, f_{j-1}, b_j f_j, f_{j+1}, \dots, f_m)(x)\Big| \\
&= \sum_{j=1}^{m}\Big|\mathcal{M}(f_1, ..., f_{j-1}, b_j(x) f_j, f_{j+1}, ..., f_m)(x) - \mathcal{M}(f_1, ..., f_{j-1}, b_j f_j, f_{j+1}, ..., f_m)(x)\Big| \\
&\le \sum_{j=1}^{m}\mathcal{M}(f_1, \dots, f_{j-1}, \big|b_j(x) - b_j\big| f_j, f_{j+1}, \dots, f_m)(x) \\
&= \sum_{j=1}^{m}\mathcal{M}_{b_j}(\vec{f})(x) = \mathcal{M}_{\vec{b}}(\vec{f})(x).
\end{aligned}$$

If b_i is any locally integrable function on $\mathbb{R}^n$ with $i = 1, \dots, m$, then

$$|[\vec{b}, \mathcal{M}](\vec{f})(x)| \le \mathcal{M}_{\vec{b}}(\vec{f})(x) + 2\Big(\sum_{j=1}^{m} b_j^-(x)\Big)\mathcal{M}(\vec{f})(x), \qquad x \in \mathbb{R}^n \tag{11}$$

holds for all $\vec{f} \in L^1_{\rm loc}(\mathbb{R}^n) \times \dots \times L^1_{\rm loc}(\mathbb{R}^n)$ (see, for example [35]).

Applying Theorem 5, we obtain the following result.

Theorem 6. *Let b_i be a real valued, locally integrable function in $\mathbb{R}^n$, $1 < p_i < \infty$, $0 \le \lambda_i \le n$ and $0 \le \mu_i \le n$ with $i = 1, \dots, m$. Let also $\frac{1}{p} = \frac{1}{p_1} + \dots + \frac{1}{p_m}$, $\frac{\lambda}{p} = \frac{\lambda_1}{p_1} + \dots + \frac{\lambda_m}{p_m}$ and $\frac{\mu}{p} = \frac{\mu_1}{p_1} + \dots + \frac{\mu_m}{p_m}$.*

Then the following assertions are equivalent:

(i) $b_i \in BMO(\mathbb{R}^n)$ such that $b_i^- \in L^\infty(\mathbb{R}^n)$ with $i = 1, \dots, m$.

(ii) The operator $[\vec{b}, \mathcal{M}]$ is bounded from $L^{p_1,\lambda_1,\mu_1}(\mathbb{R}^n) \times \dots \times L^{p_m,\lambda_m,\mu_m}(\mathbb{R}^n)$ to $L^{p,\lambda,\mu}(\mathbb{R}^n)$.

(iii) For $p \in (0,\infty)$, there is

$$\sup_B \frac{\left\|\left(b_i - M_B(b_i)\right)\chi_B\right\|_{L^p}}{\prod\limits_{j=1}^{m}\|\chi_B\|_{L^{p_i}}} < \infty, \; i = 1,\dots,m.$$

Proof. $(i) \Rightarrow (ii)$. Suppose that $\vec{b} \in BMO^m(\mathbb{R}^n)$. Combining Theorems 2 and 5, and the inequality (11), we get

$$\begin{aligned}
\|[\vec{b},\mathcal{M}](\vec{f})\|_{L^{p,\lambda,\mu}} &\le \|\mathcal{M}_{\vec{b}}(\vec{f}) + 2\Big(\sum_{j=1}^{m} b_j^-(x)\Big)\,\mathcal{M}(\vec{f})\|_{L^{p,\lambda,\mu}} \\
&\le \|\mathcal{M}_{\vec{b}}f\|_{L^{p,\lambda,\mu}} + 2\Big(\sum_{j=1}^{m}\|b_j^-\|_{L^\infty}\Big)\,\|\mathcal{M}(\vec{f})\|_{L^{p,\lambda,\mu}} \\
&\lesssim \sum_{j=1}^{m}\left(\|b_j\|_* + \|b_j^-\|_{L^\infty}\right)\prod_{j=1}^{m}\|f_j\|_{L^{p_j,\lambda_j,\mu_j}}. \qquad (12)
\end{aligned}$$

$(ii) \Rightarrow (iii)$. Assume that $[\vec{b},\mathcal{M}]$ is bounded from $L^{p_1,\lambda_1,\mu_1}(\mathbb{R}^n)\times\ldots\times L^{p_m,\lambda_m,\mu_m}(\mathbb{R}^n)$ to $L^{p,\lambda,\mu}(\mathbb{R}^n)$. Let $B = B(x,r)$ be a fixed ball. Denote by $M_B f$ the local maximal function of b_i:

$$M_B(b_i)(x) := \sup_{B'\ni x\colon B'\subset B}\frac{1}{|B'|}\int_{B'}|b_i(y)|dy, \; x\in\mathbb{R}^n.$$

Since

$$\begin{aligned}
&M(b_i\chi_B)\chi_B = M_B(b_i) \quad\text{and}\quad M(\chi_B)\chi_B = \chi_B, \\
&\sum_{j=1}^{m}\left(M_B(b_i)(x) - b_i(x)\right) = [\vec{b},\mathcal{M}]\,(\vec{\chi_B})(x) \text{ for any } x\in B.
\end{aligned}$$

we have

$$0 \le \left(M_B(b_i) - b_i\right)\chi_B \le \sum_{j=1}^{m}\left(M_B(b_i) - b_i\right)\chi_B = [\vec{b},\mathcal{M}]\,(\vec{\chi_B}).$$

Hence

$$\begin{aligned}
\|\left(M_B(b_i) - b_i\right)\chi_B\|_{L^p(\mathbb{R}^n)} &\le \Big\|\sum_{j=1}^{m}\left(M_B(b_i) - b_i\right)\chi_B\Big\|_{L^p(\mathbb{R}^n)} \\
&\le \|[\vec{b},\mathcal{M}]\,(\vec{\chi_B})\chi_B\|_{L^p(\mathbb{R}^n)}.
\end{aligned}$$

Thus from (9) we get

$$\frac{\left\|\left(b_i - M_B(b_i)\right)\chi_B\right\|_{L^p}}{\prod\limits_{j=1}^{m}\|\chi_B\|_{L^{p_i}}} \approx \frac{\left\|\left(b_i - M_B(b_i)\right)\chi_B\right\|_{L^p}}{|B|^{\frac{1}{p_1}+\ldots+\frac{1}{p_m}}} \lesssim \frac{\left\|[\vec{b},\mathcal{M}]\,(\vec{\chi_B})\right\|_{L^p}}{|B|^{\frac{1}{p}}}$$

$$\lesssim r_B^{-\frac{n}{p}}\,[r_B]_1^{\frac{\lambda}{p}}\,[1/r_B]_1^{-\frac{\mu}{p}}\,\|[\vec{b},\mathcal{M}]\,(\vec{\chi_B})\|_{L^{p,\lambda,\mu}(\mathbb{R}^n)} \tag{13}$$

$$\lesssim r_B^{-\frac{n}{p}}\,[r_B]_1^{\frac{\lambda}{p}}\,[1/r_B]_1^{-\frac{\mu}{p}}\,\|\chi_B\|_{L^{p,\lambda,\mu}} \approx 1.$$

Thus, by (13), the proof of $(ii) \Rightarrow (iii)$ is complete.

Finally, we give the proof of $(iii) \Rightarrow (i)$.

Denote by

$$E := \{x \in B:\, b_i(x) \le (b_i)_B\} \quad \text{and} \quad F := \{x \in B:\, b_i(x) > (b_i)_B\}.$$

Since

$$\int_E |b_i(t) - (b_i)_B|\,dt = \int_F |b_i(t) - (b_i)_B|\,dt,$$

in view of the inequality $b_i(x) \le (b_i)_B \le M_B(b_i)$, $x \in E$, we get

$$\frac{1}{|B|}\int_B |b_i(t) - (b_i)_B|dt = \frac{2}{|B|}\int_E |b_i(t) - (b_i)_B|dt$$

$$\le \frac{2}{|B|}\int_E |b_i(t) - M_B(b_i)(t)|dt \le \frac{2}{|B|}\int_B |b_i(t) - M_B(b_i)(t)|dt$$

$$\le \frac{2}{|B|}\left\|\left(b_i - M_B(b_i)\right)\chi_B\right\|_{L^p}\|\chi_B\|_{L^{p'}} = \frac{\left\|\left(b_i - M_B(b_i)\right)\chi_B\right\|_{L^p}}{\prod\limits_{j=1}^{m}\|\chi_B\|_{L^{p_i}}} < \infty.$$

Consequently, $b_i \in BMO(\mathbb{R}^n)$ with $i = 1, \ldots, m$.

In order to show that $b_i^- \in L^\infty(\mathbb{R}^n)$, note that $M_B(b_i) \ge |b_i|$. Hence

$$0 \le b_i^- = |b_i| - b_i^+ \le M_B(b_i) - b_i^+ + b_i^- = M_B(b_i) - b_i.$$

Thus

$$(b_i^-)_B \le c,$$

and by the Lebesgue Differentiation theorem we get that

$$b_i^-(x) \le c \quad \text{for a.e. } x \in \mathbb{R}^n.$$

and finish the proof of Theorem 6. ◄

From Theorem 6 in the case $\lambda = \mu$ or $\mu = 0$ we get the following new corollaries.

Corollary 8. *Let b_i be a real valued, locally integrable function in $\mathbb{R}^n$, $1 < p_i < \infty$ and $0 \le \lambda_i \le n$ with $i = 1, \ldots, m$. Let also $\frac{1}{p} = \frac{1}{p_1} + \ldots + \frac{1}{p_m}$ and $\frac{\lambda}{p} = \frac{\lambda_1}{p_1} + \ldots + \frac{\lambda_m}{p_m}$. Then the following assertions are equivalent:*

(i) $b_i \in BMO(\mathbb{R}^n)$ such that $b_i^- \in L^\infty(\mathbb{R}^n)$ with $i = 1, \ldots, m$.

(ii) The operator $[\vec{b}, \mathcal{M}]$ is bounded from $L^{p_1,\lambda_1}(\mathbb{R}^n) \times \ldots \times L^{p_m,\lambda_m}(\mathbb{R}^n)$ to $L^{p,\lambda}(\mathbb{R}^n)$.

(iii) For $p \in (1, \infty)$, there is

$$\sup_B \Big(\frac{1}{|B|} \int_B \big| b_i(x) - M_B(b_i)(x) \big| \, dx \Big)^{\frac{1}{p}} < \infty, \ i = 1, \ldots, m.$$

Corollary 9. *Let b_i be a real valued, locally integrable function in $\mathbb{R}^n$, $1 < p_i < \infty$ and $0 \le \lambda_i \le n$ with $i = 1, \ldots, m$. Let also $\frac{1}{p} = \frac{1}{p_1} + \ldots + \frac{1}{p_m}$ and $\frac{\lambda}{p} = \frac{\lambda_1}{p_1} + \ldots + \frac{\lambda_m}{p_m}$. Then the following assertions are equivalent:*

(i) $b_i \in BMO(\mathbb{R}^n)$ such that $b_i^- \in L^\infty(\mathbb{R}^n)$ with $i = 1, \ldots, m$.

(ii) The operator $[\vec{b}, \mathcal{M}]$ is bounded from $\widetilde{L}^{p_1,\lambda_1}(\mathbb{R}^n) \times \ldots \times \widetilde{L}^{p_m,\lambda_m}(\mathbb{R}^n)$ to $\widetilde{L}^{p,\lambda}(\mathbb{R}^n)$.

(iii) For $p \in (1, \infty)$, there is

$$\sup_B \Big(\frac{1}{|B|} \int_B \big| b_i(x) - M_B(b_i)(x) \big| \, dx \Big)^{\frac{1}{p}} < \infty, \ i = 1, \ldots, m.$$

From Corollary 8, in the case $\lambda_1 = \ldots = \lambda_m = \lambda = \mu_1 = \ldots = \mu_m = \mu$, we obtain the following corollary, which was proved in [37].

Corollary 10. [*37, Theorem* 1.1] *Let b_i be a real valued, locally integrable function in $\mathbb{R}^n$, $1 < p_i < \infty$ with $i = 1, \ldots, m$. Let also $\frac{1}{p} = \frac{1}{p_1} + \ldots + \frac{1}{p_m}$ and $0 \le \lambda \le n$. Then the following assertions are equivalent:*

(i) $b_i \in BMO(\mathbb{R}^n)$ such that $b_i^- \in L^\infty(\mathbb{R}^n)$ with $i = 1, \ldots, m$.

(ii) The operator $[\vec{b}, \mathcal{M}]$ is bounded from $L^{p_1,\lambda}(\mathbb{R}^n) \times \ldots \times L^{p_m,\lambda}(\mathbb{R}^n)$ to $L^{p,\lambda}(\mathbb{R}^n)$.

(iii) For $p \in (1, \infty)$, there is

$$\sup_B \Big(\frac{1}{|B|} \int_B \big| b_i(x) - M_B(b_i)(x) \big| \, dx \Big)^{\frac{1}{p}} < \infty, \ i = 1, \ldots, m.$$

From Corollary 8, in the case $\lambda_1 = \ldots = \lambda_m = \lambda$ and $\mu_1 = \ldots = \mu_m = \mu = 0$, we obtain the following new result.

Corollary 11. *Let b_i be a real valued, locally integrable function in $\mathbb{R}^n$, $1 < p_i < \infty$ with $i = 1, \ldots, m$. Let also $\frac{1}{p} = \frac{1}{p_1} + \ldots + \frac{1}{p_m}$ and $0 \le \lambda \le n$. Then the following assertions are equivalent:*

(i) $b_i \in BMO(\mathbb{R}^n)$ *such that* $b_i^- \in L^\infty(\mathbb{R}^n)$ *with* $i = 1, \dots, m$.

(ii) *The operator* $[\vec{b}, \mathcal{M}]$ *is bounded from* $\widetilde{L}^{p_1,\lambda}(\mathbb{R}^n) \times \dots \times \widetilde{L}^{p_m,\lambda}(\mathbb{R}^n)$ *to* $\widetilde{L}^{p,\lambda}(\mathbb{R}^n)$.

(iii) *For* $p \in (1, \infty)$, *there is*

$$\sup_B \Big(\frac{1}{|B|} \int_B \big| b_i(x) - M_B(b_i)(x) \big| \, dx \Big)^{\frac{1}{p}} < \infty, \ i = 1, \dots, m.$$

Remark 2. *Note that Corollaries 8, 9, and 11 are new.*

5. Conclusion

This paper has investigated the boundedness properties of the multi-sublinear maximal commutator operator and the commutator of the multi-sublinear maximal operator on product total Morrey spaces. Was obtained necessary and sufficient conditions for the boundedness of the multi-sublinear maximal commutator operator $\mathcal{M}_{\vec{b}}$ and the commutators of the multi-sublinear maximal operator $[\vec{b}, \mathcal{M}]$ on product total Morrey spaces $L^{p_1,\lambda_1,\mu_1}(\mathbb{R}^n) \times \dots \times L^{p_m,\lambda_m,\mu_m}(\mathbb{R}^n)$ to the total Morrey spaces $L^{p,\lambda,\mu}(\mathbb{R}^n)$.

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