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Uncountable Frames for Operators

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Abstract. In this paper, the concepts of an uncountable K-atomic system and an uncountable K-frame for a bounded linear operator in a non-separable Hilbert space H are introduced. Necessary and sufficient conditions for the existence of an uncountable K-atomic system and an uncountable K-frame are found. The stability of an uncountable K-frame is studied. The concept of an uncountable atomic decomposition for subspaces of non-separable Hilbert spaces is also introduced. It is established that a system forms an uncountable atomic decomposition with respect to a subspace of a non-separable Hilbert space if and only if it forms an uncountable frame for the orthogonal projector of this subspace.

Key Words and Phrases: non-separable Hilbert space, uncountable orthonormal basis, uncountable K-frame, uncountable K-atomic decomposition.

2010 Mathematics Subject Classifications: 42C40, 42C15, 46C07

1. Introduction

The concept of a frame for a perturbed sequence of exponentials in the space L_2 was introduced by R.J. Duffin and R.J. Schaeffer [1] in 1952 while studying the properties of non-harmonic Fourier series. Later, the concept of a frame was also given in abstract Hilbert spaces. Namely, a sequence $\{x_n\}_{n\in N}$ is called a frame in a Hilbert space H if there exist constants $A, B > 0$, such that, for each $x \in H$, the inequality

$$A\|x\|^2 \leq \sum_{n=1}^{\infty} |(x, x_n)|^2 \leq B\|x\|^2,$$

holds, where $(\cdot,\cdot)$ is the inner product in H, and $\|\cdot\|$ is the norm generated by the inner product in H. Every element of H admits a frame decomposition,

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which is generally not unique. This advantage leads to important applications of frames, for example, in wavelet theory, time-frequency analysis, and Gabor analysis. Frames are also widely used in coding theory, information processing, data transmission and compression, signal processing, etc. For further information, see [2-11]. Various generalizations of frames in Hilbert spaces include g-frames ([12]), p-frames in invariant subspaces of Lebesgue spaces ([13]), continuous frames ([14]), pg-frames ([15]), K-frames for bounded operators ([16]), frames generated by bilinear mappings ([17, 18]), uncountable frames in non-separable Hilbert spaces ([19]). Among the works devoted to this area are, for example, [20–28]. It is therefore of interest to study uncountable K-frames for bounded operators in non-separable Hilbert spaces. Recall that a sequence $\{x_n\}_{n\in N}$ is called a frame in H for a bounded linear operator $K : H \to H$ if there exist constants $A, B > 0$ such that, for each $x \in H$ the inequality

$$A \left\| K^*(x) \right\|^2 \leq \sum_{n=1}^{\infty} \left| (x, x_n) \right|^2 \leq B \left\| x \right\|^2 ,$$

holds.

This article is devoted to the extension of the concepts of atomic systems and frames for bounded linear operators, as well as families of local atoms for subspaces of separable Hilbert spaces, to non-separable Hilbert spaces with an uncountable orthonormal basis. The concepts of an uncountable K-atomic system and an uncountable K-frame associated with bounded linear operator K on a non-separable Hilbert space H are introduced. Criteria for these concepts and the relationships between them are established. The stability of uncountable K-frames is studied. For a pair of systems, the concept of an uncountable atomic decomposition with respect to subspaces of a Hilbert space H is given and its connection with an uncountable frame with respect to a orthogonal projection onto the corresponding subspace is established.

2. Some auxiliary information

Let H, H_1 and H_2 be Hilbert spaces. Denote by $L(H_1, H_2)$ the space of bounded linear operators acting from H_1 to H_2, in particular, $L(H, H) = L(H)$; T^* is the adjoint operator and $R(T)$ is the image of $T \in L(H_1, H_2)$; $\|\cdot\|$ is the norm in the corresponding normed space.

Regarding the embedding of images of operators, the following theorem holds.

Theorem 1. **([29])** *Let $L_1 \in L(H_1, H)$ and $L_2 \in L(H_2, H)$. The following statements are equivalent:*

i) $R(L_1) \subset R(L_2)$;

ii) $L_1 L_1^* \leq \lambda L_2 L_2^*$, *where* $\lambda \geq 0$ *is some number;*
iii) $L_1 = L_2 L$, *where* $L \in L(H_1, H_2)$ *is some operator.*

We first recall some facts about uncountable Bessel systems and uncountable orthonormal bases in non-separable Hilbert spaces.

Let I be an uncountable set of indices, and I^a a set of at most countable subsets of I. Let H be a non-separable Hilbert space with an orthonormal system $\{e_\alpha\}_{\alpha \in I}$. Obviously, for each $x \in H$, we have

$$I(x) = \{\alpha : (x, e_\alpha) \neq 0\} \in I^a$$

and the series $\sum_{\alpha \in I} (x, e_\alpha) e_\alpha$ converges unconditionally. If the system $\{e_\alpha\}_{\alpha \in I}$ is complete in H, then for each $x \in H$, we have

$$x = \sum_{\alpha \in I} (x, e_\alpha) e_\alpha.$$

In this case, the system $\{e_\alpha\}_{\alpha \in I}$ forms an uncountable orthonormal basis in H and Parseval's equality holds:

$$\|x\|^2 = \sum_{\alpha \in I} |(x, e_\alpha)|^2 . \tag{1}$$

Note that the Banach analogue of the concept of an uncountable unconditional basis was studied in [30].

The coordinate realization of the space H is the space $l_2(I)$ of systems of scalars $\lambda = \{\lambda_\alpha\}_{\alpha \in I}$ such that $I(\lambda) = \{\alpha : \lambda_\alpha \neq 0\} \in I^a$ and $\sum_{\alpha \in I} |\lambda_\alpha|^2 < +\infty$, with the scalar product

$$(\lambda, \mu) = \sum_{\alpha \in I} \lambda_\alpha \bar{\mu}_\alpha.$$

Let $\delta_\alpha = \{\delta_{\alpha\beta}\}_{\beta \in I}$, $\alpha \in I$, $\delta_{\alpha\beta}$ be the Kronecker symbol. It is clear that the system $\delta_\alpha = \{\delta_{\alpha\beta}\}_{\beta \in I}$, $\alpha \in I$, is orthonormal in $l_2(I)$. Furthermore, it is obvious that for any $\lambda \in l_2(I)$ the series $\sum_{\alpha \in I} \lambda_\alpha \delta_\alpha$ converges unconditionally in $l_2(I)$ and there is a unique decomposition

$$\lambda = \sum_{\alpha \in I} \lambda_\alpha \delta_\alpha,$$

that is, the system $\delta_\alpha = \{\delta_{\alpha\beta}\}_{\beta \in I}$, $\alpha \in I$, forms an orthonormal uncountable basis in $l_2(I)$.

Let $\{x_\alpha\}_{\alpha \in I}$ be some system of elements of a non-separable Hilbert space H.

We first recall some facts about uncountable Bessel systems in non-separable Hilbert spaces.

Definition 1. ([19]) *A system $\{x_\alpha\}_{\alpha\in I}$ is called uncountable Bessel in H if there exists a number $B>0$ such that for any $\omega \in I^a$ the inequality*

$$\sum_{\alpha\in\omega} |(x, x_\alpha)|^2 \le B\,\|x\|^2\,,\ x\in H,$$

holds. In this case, $I(x)=\{\alpha : (x,x_\alpha)\neq 0\}\in I^a$, $x\in H$. The number B is called the bound of the Bessel system.

We will need the following theorem

Theorem 2. ([19]) *Let the series $\sum_{\alpha\in I}\lambda_\alpha x_\alpha$ converge for each $\forall\lambda=\{\lambda_\alpha\}_{\alpha\in I}\in l_2(I)$. Then there exists a number $M>0$ such that*

$$\left\|\sum_{\alpha\in I}\lambda_\alpha x_\alpha\right\| \le M\,\|\lambda\|\,, \forall\lambda=\{\lambda_\alpha\}_{\alpha\in I}\in l_2(I).$$

The following criterion of uncountable Besselianness for systems is valid.

Theorem 3. ([19]) *A system $\{x_\alpha\}_{\alpha\in I}$ is an uncountable Bessel system in H, if and only if for $\forall\lambda=\{\lambda_\alpha\}_{\alpha\in I}\in l_2(I)$ the series $\sum_{\alpha\in I}\lambda_\alpha x_\alpha$ converges and the linear operator $T: l_2(I)\to H$, defined by the formula*

$$T(\lambda)=\sum_{\alpha\in I}\lambda_\alpha x_\alpha, \tag{2}$$

is bounded. In this case, $\|T\|^2\le B$, where B is a bound of $\{x_\alpha\}_{\alpha\in I}$.

If $\{x_\alpha\}_{\alpha\in I}$ is an uncountable Bessel system, then the operator $T^*: H\to l_2(I)$, adjoint to operator (2), is defined by the formula

$$T^*(x)=\{(x,x_\alpha)\}_{\alpha\in I}\,,\ x\in H.$$

3. Uncountable K-frames

Let $\{x_\alpha\}_{\alpha\in I}$ be some system of elements of a non-separable Hilbert space H and $K\in L(H)$.

The following concept is an uncountable analogue of the concept of an atomic system for K.

Definition 2. *We say that $\{x_\alpha\}_{\alpha\in I}$ is an uncountable atomic system in H for K if the following statements hold:*

i) for $\forall\{\lambda_\alpha\}_{\alpha\in I}\in l_2(I)$ the series $\sum_{\alpha\in I}\lambda_\alpha x_\alpha$ converges;

ii) there exists a number $C>0$ such that for every $x\in H$ there exists $\lambda(x)=\{\lambda_\alpha(x)\}_{\alpha\in I}\in l_2(I)$ such that $\|\lambda(x)\|\le C\,\|x\|$ and $K(x)=\sum_{\alpha\in I}\lambda_\alpha(x)x_\alpha$.

The constant C is called atomic bound of $\{x_\alpha\}_{\alpha\in I}$.

The existence theorem for an uncountable atomic system is valid.

Theorem 4. *Let $\{e_\alpha\}_{\alpha\in I}$ be an orthonormal uncountable basis in H. Then $\{K(e_\alpha)\}_{\alpha\in I}$ is an uncountable atomic system for K .*

Proof. It is known that the series $\sum_{\alpha\in I}\lambda_\alpha e_\alpha$ converges for $\forall\,\{\lambda_\alpha\}_{\alpha\in I}\in l_2(I)$. Therefore, the series $\sum_{\alpha\in I}\lambda_\alpha K(e_\alpha)$ converges. Take $x\in H$. Let $\lambda_\alpha(x)=(x,e_\alpha)$, $\alpha\in I$. Then $\lambda(x)=\{\lambda_\alpha(x)\}_{\alpha\in I}\in l_2(I)$ and, based on Parseval's equality (1), we have

$$\|\lambda(x)\|=\left(\sum_{\alpha\in I}|(x,e_\alpha)|^2\right)^{\frac{1}{2}}=\|x\|\,.$$

Further, since $x=\sum_{\alpha\in I}\lambda_\alpha e_\alpha$ we get

$$K(x)=\sum_{\alpha\in I}\lambda_\alpha K(e_\alpha).$$

◄

The following theorem establishes a connection between the concepts of uncountable Bessel systems and uncountable atomic systems.

Theorem 5. *Let $\{e_\alpha\}_{\alpha\in I}$ be an orthonormal uncountable basis in H. Then the following statements are equivalent:*

i) $\{x_\alpha\}_{\alpha\in I}$ is an uncountable Bessel system in H;

ii) there exists an operator $K\in L(H)$, such that $\{x_\alpha\}_{\alpha\in I}$ is an uncountable atomic system for K.

Proof. Let $\{x_\alpha\}_{\alpha\in I}$ be an uncountable Bessel system in H. Then, as is known ([30], Theorem 3.1), there exists an operator $L\in L(l_2(I),H)$ such that $L(\delta_\alpha)=x_\alpha$, $\alpha\in I$. Moreover, $L=U^*$, where $U\in L(H,l_2(I))$: $U(x)=\{(x,x_\alpha)\}_{\alpha\in I}$. Let $F\in L(l_2(I),H)$ be an isomorphism defined by the formula $F(\lambda)=\sum_{\alpha\in I}\lambda_\alpha e_\alpha$, $\forall\lambda=\{\lambda_\alpha\}_{\alpha\in I}\in l_2(I)$. Put $K=LF^{-1}$. Clearly, $K\in L(H)$ and

$$K(e_\alpha)=L(F^{-1}(e_\alpha))=L(\delta_\alpha)=x_\alpha,\ \alpha\in I.$$

Therefore, by Theorem 4, $\{x_\alpha\}_{\alpha\in I}$ is an uncountable atomic system for K.

The converse statement follows from Theorem 2 and 3. ◄

The following definition is an uncountable generalization of the concept of a K-frame in separable Hilbert spaces.

Definition 3. *We say that $\{x_\alpha\}_{\alpha\in I}$ is an uncountable frame for the operator $K \in L(H)$ (an uncountable K-frame) in H if $I_x = \{\alpha \in I : (x, x_\alpha) \neq 0\} \in I^a$, $x \in H$, and there exist constants $A, B > 0$, such that for each $x \in H$ the inequality*

$$A\,\|K^*(x)\|^2 \le \sum_{\alpha\in I} |(x, x_\alpha)|^2 \le B\,\|x\|^2\,, \tag{3}$$

holds. The constants A and B are called uncountable K-frame bounds.

Obviously, every uncountable frame ([19]) in H is an uncountable K-frame.
The following theorem establishes a criterion for an uncountable K-frame.

Theorem 6. *Let $\{x_\alpha\}_{\alpha\in I}$ be a system of elements in H and $K \in L(H)$. The following statements are equivalent:*
i) $\{x_\alpha\}_{\alpha\in I}$ is an uncountable atomic system for K;
ii) $\{x_\alpha\}_{\alpha\in I}$ is an uncountable frame for K;
iii) $\{x_\alpha\}_{\alpha\in I}$ is an uncountable Bessel system and there exists an uncountable Bessel system $\{y_\alpha\}_{\alpha\in I}$ such that $K(x) = \sum_{\alpha\in I}(x, y_\alpha)x_\alpha$, $x \in H$.

Proof. Let us check the validity of statement *i) ⇒ ii)*. Let $\{x_\alpha\}_{\alpha\in I}$ be an uncountable atomic system for K. Then, for $\forall\{\lambda_\alpha\}_{\alpha\in I} \in l_2(I)$, the series $\sum_{\alpha\in I}\lambda_\alpha x_\alpha$ converges. By Theorem 3, $\{x_\alpha\}_{\alpha\in I}$ is an uncountable Bessel system, and therefore, there exists a number $B > 0$ such that

$$\sum_{\alpha\in\omega} |(x, x_\alpha)|^2 \le B\,\|x\|^2\,, x \in X. \tag{4}$$

According to Definition 2, there exists a constant $C > 0$ such that for each element $x \in X$ there exists $\lambda(x) = \{\lambda_\alpha(x)\}_{\alpha\in I} \in l_2(I)$ such that $\|\lambda(x)\| \le C\,\|x\|$ and the equality $K(x) = \sum_{\alpha\in I}\lambda_\alpha(x)x_\alpha$ holds. Then

$$\|K^*(x)\|^2 = \sup_{\|y\|=1} |(K^*(x), y)|^2 = \sup_{\|y\|=1} |(x, K(y))|^2 =$$

$$= \sup_{\|y\|=1} \left|(x, \sum_{\alpha\in I}\lambda_\alpha(y)x_\alpha)\right|^2 = \sup_{\|y\|=1} \left|\sum_{\alpha\in I}\lambda_\alpha(y)(x_\alpha, x)\right|^2 \le$$

$$\le \sup_{\|y\|=1} \left(\sum_{\alpha\in I}|\lambda_\alpha(y)|\,|(x_\alpha, x)|\right)^2 \le$$

$$\le \sup_{\|y\|=1} \sum_{\alpha\in I}|\lambda_\alpha(y)|^2 \sum_{\alpha\in I}|(x_\alpha, x)|^2 \le C\sum_{\alpha\in I}|(x_\alpha, x)|^2\,. \tag{5}$$

So, according to (4) and (5) we obtain

$$\frac{1}{C}\left\|K^*(x)\right\|^2 \leq \sum_{\alpha\in I}\left|(x_\alpha, x)\right|^2 \leq B\left\|x\right\|^2.$$

Let us check the validity of the statement *ii)* $\Rightarrow$ *iii)*. By virtue of Theorem 3, the operator T, defined by formula (2), is bounded. Consequently, we obtain

$$A\left\|K^*(x)\right\|^2 \leq \sum_{\alpha\in I}\left|(x_\alpha, x)\right|^2 = \left\|T^*(x)\right\|^2.$$

It follows that $AKK^* \leq TT^*$. Therefore, by Theorem 1, there exists an operator $F \in L(H, l_2(I))$ such that $K = TF$. Let us take $x \in X$ and set $F(x) = \lambda(x) = \{\lambda_\alpha(x)\}_{\alpha\in I} \in l_2(I)$. For a linear functional $\lambda_\alpha(x)$, we obtain

$$\left|\lambda_\alpha(x)\right| \leq \left(\sum_{\beta\in I}\left|\lambda_\beta(x)\right|^2\right)^{\frac{1}{2}} \leq \left\|F\right\|\left\|x\right\|.$$

Then, by the Riesz representation theorem of there exists $y_\alpha \in X$ such that $\lambda_\alpha(x) = (x, y_\alpha)$. We have

$$\sum_{\alpha\in I}\left|(x, y_\alpha)\right|^2 \leq \left\|F\right\|^2\left\|x\right\|^2,$$

that is, $\{y_\alpha\}_{\alpha\in I}$ forms an uncountable Bessel system in H. On the other hand, the equality

$$K(x) = T(F(x)) = \sum_{\alpha\in I}(x, y_\alpha)x_\alpha,$$

holds.

Let us check the validity of statement *iii)* $\Rightarrow$ *i)*. Let $\{x_\alpha\}_{\alpha\in I}$ and $\{y_\alpha\}_{\alpha\in I}$ be uncountable Bessel systems in H such that $K(x) = \sum_{\alpha\in I}(x, y_\alpha)x_\alpha$, $x \in H$. By Theorem 3, the series $\sum_{\alpha\in I}\lambda_\alpha x_\alpha$ converges for each $\lambda = \{\lambda_\alpha\}_{\alpha\in I} \in l_2(I)$. For each $x \in H$, we define $\lambda_\alpha(x) = (x, y_\alpha)$, $\alpha \in I$. Since $\{y_\alpha\}_{\alpha\in I}$ is an uncountable Bessel system in H, then $\lambda(x) = \{\lambda_\alpha(x)\}_{\alpha\in I} \in l_2(I)$ and the inequality

$$\left\|\lambda(x)\right\| = \left(\sum_{\alpha\in I}\left|\lambda_\alpha(x)\right|^2\right)^{\frac{1}{2}} = \left(\sum_{\alpha\in I}\left|(x, y_\alpha)\right|^2\right)^{\frac{1}{2}} \leq \sqrt{B_1}\left\|x\right\|,$$

holds, where B_1 is the bound of $\{y_\alpha\}_{\alpha\in I}$. Thus, $\{x_\alpha\}_{\alpha\in I}$ is an uncountable atomic system for K. ◄

The following criterion is valid.

Theorem 7. *Let $\{e_\alpha\}_{\alpha\in I}$ be an orthonormal uncountable basis in H and $\{x_\alpha\}_{\alpha\in I}$ be some system of elements in H. The following statements are equivalent:*

i) $\{x_\alpha\}_{\alpha\in I}$ is an uncountable K-frame in H;

ii) there exists an operator $L \in L(H)$ such that $x_\alpha = Le_\alpha$ and $R(K) \subset R(L)$.

Proof. Let $\{x_\alpha\}_{\alpha\in I}$ form an uncountable K-frame in H. Then there exist constants $A, B > 0$ such that

$$A\left\|K^*(x)\right\|^2 \le \sum_{\alpha\in I} |(x, x_\alpha)|^2 \le B\left\|x\right\|^2, x \in H. \tag{6}$$

Let us consider the operator $U : H \to H$ according to the formula

$$U(x) = \sum_{\alpha\in I}(x, x_\alpha)e_\alpha,\ x \in H.$$

Taking into account (1), according to (6), we obtain

$$\|U(x)\|^2 = \sum_{\alpha\in I} |(x, x_\alpha)|^2 \le B\left\|x\right\|^2.$$

It follows that the operator U is bounded. Let us find the adjoint operator to U. For $x, y \in H$, we obtain

$$(U(x), y) = (\sum_{\alpha\in I}(x, x_\alpha)e_\alpha, y) = \sum_{\alpha\in I}(x, x_\alpha)(e_\alpha, y) = (x, \sum_{\alpha\in I}(y, e_\alpha)x_\alpha).$$

Thus, the adjoint operator U^* is expressed by the formula

$$U^*(y) = \sum_{\alpha\in I}(y, e_\alpha)x_\alpha,\ y \in H.$$

Hence, in particular, $U^*(e_\alpha) = x_\alpha$. Let $L = U^*$. Using (6), we have

$$A\left\|K^*(x)\right\|^2 \le \sum_{\alpha\in I} |(x, x_\alpha)|^2 = \|U(x)\|^2 = \left\|L^*(x)\right\|^2,$$

that is, the inequality $AKK^* \le LL^*$ holds. Then, by Theorem 1, we obtain $R(K) \subset R(L)$.

Conversely, suppose there exists an operator $L \in L(H)$ such that $x_\alpha = Le_\alpha$ and $R(K) \subset R(L)$. Then, for any $x \in H$, we obtain

$$\sum_{\alpha\in I} |(x, x_\alpha)|^2 = \sum_{\alpha\in I} |(x, L(e_\alpha))|^2 = \sum_{\alpha\in I} |(L^*(x), e_\alpha)|^2 = \left\|L^*(x)\right\|^2 \le \|L\|^2\left\|x\right\|^2.$$

Furthermore, by Theorem 1, there exists a number $A > 0$ such that $AKK^* \leq LL^*$. Then, based on Parseval's equality (1), we have

$$A\left\|K^*(x)\right\|^2 \leq \left\|L^*(x)\right\|^2 = \sum_{\alpha\in I}\left|(L^*(x), e_\alpha)\right|^2 = \sum_{\alpha\in I}\left|(x, x_\alpha)\right|^2.$$

Therefore, the system $\{x_\alpha\}_{\alpha\in I}$ forms an uncountable K-frame in H. ◄

The following theorems on the stability of uncountable operator frames are valid.

Theorem 8. *Let $K_1, K_2 \in L(H)$, $\{x_\alpha\}_{\alpha\in I}$ be an uncountable K_1-frame in H and $R(K_2) \subset R(K_1)$. Then $\{x_\alpha\}_{\alpha\in I}$ forms an uncountable K_2-frame in H.*

Proof. By Theorem 1, there exists a number $\lambda \geq 0$ such that $K_2K_2^* \leq \lambda K_1K_1^*$. Case $\lambda = 0$ is obvious. Let $\lambda > 0$. Then for every $x \in H$, we have

$$\left\|K_2^*(x)\right\| \leq \lambda\left\|K_1^*(x)\right\|.$$

Therefore, if A and B are bounds of an K_1-frame $\{x_\alpha\}_{\alpha\in I}$, then

$$\frac{A}{\lambda^2}\left\|K_2^*(x)\right\|^2 \leq A\left\|K_1^*(x)\right\|^2 \leq \sum_{\alpha\in I}\left|(x, x_\alpha)\right|^2 \leq B\left\|x\right\|^2.$$

Thus, $\{x_\alpha\}_{\alpha\in I}$ forms an uncountable K_2-frame in H. ◄

Theorem 9. *Let $K_1, K_2 \in L(H)$, $\{x_\alpha\}_{\alpha\in I}$ be an uncountable K_1-frame in H and let there exist numbers $\lambda_1 \geq 0$ and $0 \leq \lambda_2 < 1$ such that*

$$\left\|K_1^*(x) - K_2^*(x)\right\| \leq \lambda_1\left\|K_1^*(x)\right\| + \lambda_2\left\|K_2^*(x)\right\|, x \in H.$$

Then $\{x_\alpha\}_{\alpha\in I}$ forms an uncountable K_2-frame in H.

Proof. For $x \in H$, we have

$$\left\|K_2^*(x)\right\| \leq \left\|K_1^*(x) - K_2^*(x)\right\| + \left\|K_1^*(x)\right\| \leq (1+\lambda_1)\left\|K_1^*(x)\right\| + \lambda_2\left\|K_2^*(x)\right\|.$$

As a result of equivalent transformations, the last inequality reduces to the inequality

$$\left\|K_2^*(x)\right\| \leq \frac{1+\lambda_1}{1-\lambda_2}\left\|K_1^*(x)\right\|.$$

Therefore, by Theorems 1 and 8, the system $\{x_\alpha\}_{\alpha\in I}$ forms an uncountable K_2-frame in H. ◄

Corollary 1. *Let $K \in L(H)$ and $\{x_\alpha\}_{\alpha\in I}$ be an uncountable K-frame in H. Suppose that there exist numbers $\lambda_1 \geq 0$ and $0 \leq \lambda_2 < 1$ such that*

$$\|K^*(x) - x\| \leq \lambda_1 \|K^*(x)\| + \lambda_2 \|x\|, x \in H.$$

Then $\{x_\alpha\}_{\alpha\in I}$ is an uncountable frame in H.

The following theorem is also true.

Theorem 10. *Let $K \in L(H)$, $\{x_\alpha\}_{\alpha\in I}$ be an uncountable K-frame in H with a lower bound A, and let $\{y_\alpha\}_{\alpha\in I}$ be an uncountable Bessel system in H such that*

$$\left(\sum_{\alpha\in I} |(x, x_\alpha - y_\alpha)|^2\right)^{\frac{1}{2}} \leq \lambda \left(\sum_{\alpha\in I} |(x, x_\alpha)|^2\right)^{\frac{1}{2}} +$$

$$+\mu \left(\sum_{\alpha\in I} |(x, y_\alpha)|^2\right)^{\frac{1}{2}} + \gamma \|K^*(x)\|, x \in H, \tag{7}$$

where $0 \leq \gamma < \sqrt{A}$, $0 \leq \mu$ and $0 \leq \lambda < 1 - \frac{\gamma}{\sqrt{A}}$ are some numbers. Then $\{y_\alpha\}_{\alpha\in I}$ forms an uncountable K-frame in H.

Proof. For $x \in H$, using (7), we obtain

$$\left(\sum_{\alpha\in I} |(x, x_\alpha)|^2\right)^{\frac{1}{2}} \leq \left(\sum_{\alpha\in I} |(x, x_\alpha - y_\alpha)|^2\right)^{\frac{1}{2}} + \left(\sum_{\alpha\in I} |(x, y_\alpha)|^2\right)^{\frac{1}{2}} \leq$$

$$\leq \lambda \left(\sum_{\alpha\in I} |(x, x_\alpha)|^2\right)^{\frac{1}{2}} + (1+\mu) \left(\sum_{\alpha\in I} |(x, y_\alpha)|^2\right)^{\frac{1}{2}} + \gamma \|K^*(x)\|.$$

It follows that

$$\left(\sum_{\alpha\in I} |(x, x_\alpha)|^2\right)^{\frac{1}{2}} \leq \frac{1+\mu}{1-\lambda} \left(\sum_{\alpha\in I} |(x, y_\alpha)|^2\right)^{\frac{1}{2}} + \frac{\gamma}{1-\lambda} \|K^*(x)\|. \tag{8}$$

Comparing (6) and (8), we obtain

$$\sqrt{A} \|K^*(x)\| \leq \frac{1+\mu}{1-\lambda} \left(\sum_{\alpha\in I} |(x, y_\alpha)|^2\right)^{\frac{1}{2}} + \frac{\gamma}{1-\lambda} \|K^*(x)\|.$$

The last inequality is equivalent to the following inequality:

$$\frac{(1-\lambda)\sqrt{A}-\gamma}{1-\lambda}\left\|K^{*}(x)\right\| \leq \frac{1+\mu}{1-\lambda}\left(\sum_{\alpha\in I}\left|(x,y_{\alpha})\right|^{2}\right)^{\frac{1}{2}}. \tag{9}$$

Dividing both sides of (9) by $\frac{1+\mu}{1-\lambda}$ and squaring the resulting inequality, we obtain

$$A_{1}\left\|K^{*}(x)\right\|^{2} \leq \sum_{\alpha\in I}\left|(x,y_{\alpha})\right|^{2},$$

where $A_1 = \left(\frac{(1-\lambda)\sqrt{A}-\gamma}{1+\mu}\right)^2$. Thus, $\{y_\alpha\}_{\alpha\in I}$ forms an uncountable K-frame in H with a lower bound A_1. ◄

4. Uncountable atomic decomposition of subspaces

Let H_0 be a subspace of the Hilbert space of H. Suppose that $P_{H_0} \in L(H, H_0)$ is the orthogonal projection on H_0.

The following concept generalizes the concept of atomic decomposition for subspaces ([16]).

Definition 4. *Let $\{x_\alpha\}_{\alpha\in I}$ be an uncountable Bessel system in H and $\{f_\alpha\}_{\alpha\in I}$ be a system of linear bounded functionals in H. We say that the pair $\left(\{x_\alpha\}_{\alpha\in I}, \{f_\alpha\}_{\alpha\in I}\right)$ is an uncountable atomic decomposition for H_0, if*

i) there exists a number $C > 0$ such that for every $x \in H$ and $\omega \in I^a$ the equality

$$\sum_{\alpha\in\omega}\left|f_{\alpha}(x)\right|^{2} \leq C\left\|x\right\|^{2},$$

holds;

ii) for every $x \in H_0$, the equality $x = \sum_{\alpha\in I} f_\alpha(x)x_\alpha$ holds.

The constant C is called an atomic bound of $\{f_\alpha\}_{\alpha\in I}$.

The following theorem gives equivalent conditions for an uncountable atomic decomposition for subspaces.

Theorem 11. *Let $\{e_\alpha\}_{\alpha\in I}$ be an orthonormal uncountable basis in H and $\{x_\alpha\}_{\alpha\in I}$ an uncountable Bessel system in H. The following statements are equivalent:*

i) there exists a system $\{f_\alpha\}_{\alpha\in I} \subset H^*$, such that the pair $\left(\{x_\alpha\}_{\alpha\in I}, \{f_\alpha\}_{\alpha\in I}\right)$ is an uncountable atomic decomposition of H_0;

ii) $\{x_\alpha\}_{\alpha\in I}$ is an uncountable frame for P_{H_0};

iii) there exists an uncountable Bessel system $\{y_\alpha\}_{\alpha\in I}\subset H$ *such that* $P_{H_0}x = \sum_{\alpha\in I}(x,y_\alpha)x_\alpha$, $x\in H$.

iv) there exists an operator $L\in L(l_2(I),H)$, *such that* $x_\alpha = L\delta_\alpha$ *and* $H_0\subset R(L)$.

Proof. Let us check the validity of statement *i)* $\Leftrightarrow$ *ii)*. Let the pair $(\{x_\alpha\}_{\alpha\in I}, \{f_\alpha\}_{\alpha\in I})$ form an uncountable atomic decomposition for H_0 and let C be an atomic bound of $\{f_\alpha\}_{\alpha\in I}$. Then for any $\lambda = \{\lambda_\alpha\}_{\alpha\in I}\in l_2(I)$ the series $\sum_{\alpha\in I}\lambda_\alpha x_\alpha$ converges and for any $x\in H$ inequality holds

$$\sum_{\alpha\in I}|f_\alpha(x)|^2\le C\,\|x\|^2. \tag{10}$$

Let $\lambda_\alpha(x) = f_\alpha(P_{H_0}(x))$, $x\in H$. It is clear that $\lambda(x) = \{\lambda_\alpha(x)\}_{\alpha\in I}\in l_2(I)$, $x\in H$. On the other hand, by virtue of (10), we get

$$\sum_{\alpha\in I}|\lambda_\alpha(x)|^2 = \sum_{\alpha\in I}|f_\alpha(P_{H_0}(x))|^2\le C\,\|P_{H_0}(x)\|^2\le C\,\|x\|^2.$$

Finally, for each $x\in H$ we obtain

$$P_{H_0}x = \sum_{\alpha\in I}f_\alpha(P_{H_0}(x))x_\alpha = \sum_{\alpha\in I}\lambda_\alpha(x)x_\alpha.$$

So, $\{x_\alpha\}_{\alpha\in I}$ is an uncountable atomic system for P_{H_0}. Then, by Theorem 6, the system $\{x_\alpha\}_{\alpha\in I}$ is an uncountable frame for P_{H_0}.

Conversely, let $\{x_\alpha\}_{\alpha\in I}$ be an uncountable frame for P_{H_0}. Then, by Theorem 6, it is an uncountable atomic system for P_{H_0}. Therefore, for every $x\in H$ there exists $\lambda(x) = \{\lambda_\alpha(x)\}_{\alpha\in I}\in l_2(I)$ and there exists a number $C>0$ such that for every $x\in H$ we have

$$\sum_{\alpha\in I}|\lambda_\alpha(x)|^2\le C\,\|x\|^2.$$

It is clear that $f_\alpha(x) = \lambda_\alpha(x)$, $x\in H$, is a system of linear bounded functionals in H satisfying the condition

$$\sum_{\alpha\in I}|f_\alpha(x)|^2\le C\,\|x\|^2.$$

Moreover, for each $x\in H_0$ we have

$$x = \sum_{\alpha\in I}\lambda_\alpha(x)x_\alpha = \sum_{\alpha\in I}f_\alpha(x)x_\alpha.$$

Therefore, the pair $\left(\{x_\alpha\}_{\alpha\in I}, \{f_\alpha\}_{\alpha\in I}\right)$ forms an uncountable atomic decomposition for H_0.

The validity of *ii)* $\Leftrightarrow$ *iii)* and *iii)* $\Leftrightarrow$ *iv)* follows from Theorems 6 and 7, respectively. ◄

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