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Calderón-Zygmund Operators and Their Commutators on Modified Mixed Morrey Spaces

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Abstract. We study the boundedness of Calderón-Zygmund integral operators in modified mixed Morrey spaces $\widetilde{L}_{\vec{p},\lambda}(\mathbb{R}^n)$. In addition, we investigate the boundedness of their commutators $[b,T]$, where the symbol b belongs to $BMO(\mathbb{R}^n)$. Sufficient conditions are established to ensure the boundedness of both the Calderón-Zygmund operators T and their commutators $[b,T]$ on $\widetilde{L}_{\vec{p},\lambda}(\mathbb{R}^n)$.

Key Words and Phrases: modified mixed Morrey spaces, Calderón-Zygmund operator, commutator, BMO spaces.

2010 Mathematics Subject Classifications: 42B20, 42B25, 42B35

1. Introduction

Classical Morrey spaces $L_{p,\lambda}$ were originally introduced by Morrey in [26] to study the local behavior of solutions of second-order elliptic partial differential equations. The theory of mixed-norm function spaces has undergone significant development over the past few decades. Nevertheless, the standard literature is still the mixed Lebesgue spaces $L_{\vec{p}}(\mathbb{R}^n)$, $0 < \vec{p} \leq \infty$, as a natural generalization of the classical Lebesgue spaces $L_p(\mathbb{R}^n)$, $0 < p \leq \infty$, it is first introduced by Benedek and Panzone [3] in 1961. Mixed-norm function spaces possess a more refined structural framework than their classical counterparts, thereby enabling wider applications in analysis, including potential analysis, harmonic analysis and partial differential equations. In 2019, Nogayama [27] introduced a new Morrey-type space, called mixed Morrey space by generalizing Morrey spaces and mixed Lebesgue spaces, see also [1, 4, 11, 14, 15, 28, 29]. In this paper, we introduce the modified mixed Morrey spaces $\widetilde{L}_{\vec{p},\lambda}(\mathbb{R}^n)$. These spaces generalize the mixed

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Lebesgue spaces in the sense that $\widetilde{L}_{\vec{p},0}(\mathbb{R}^n) \equiv L_{\vec{p}}(\mathbb{R}^n)$ and the modified Morrey spaces in the sense that $\widetilde{L}_{(p,\dots,p),\lambda}(\mathbb{R}^n) = \widetilde{L}_{p,\lambda}(\mathbb{R}^n)$, see [2, 9, 11].

Let $f \in L_1^{\mathrm{loc}}(\mathbb{G})$. The Hardy-Littlewood maximal operator M is defined by

$$Mf(x) = \sup_{t>0} |B(x,t)|^{-1} \int_{B(x,t)} |f(y)| dy.$$

Let T be a Calderón-Zygmund singular integral operator, briefly a Calderón-Zygmund operator, i.e., a linear operator bounded from $L_2(\mathbb{R}^n)$ to $L_2(\mathbb{R}^n)$ taking all infinitely continuously differentiable functions f with compact support to a function $f \in L_1^{\mathrm{loc}}(\mathbb{R}^n)$ represented by

$$Tf(x) = \int_{\mathbb{R}^n} k(x,y) f(y)\, dy \qquad \text{a.e.} \quad \text{off} \ \ \mathrm{supp} f. \tag{1}$$

Here $k(x,y)$ is a continuous function away from the diagonal which satisfies the standard estimates: there exist $c_1 > 0$ and $0 < \varepsilon \le 1$ such that

$$|k(x,y)| \le c_1 |x-y|^{-n} \tag{2}$$

for all $x, y \in \mathbb{R}^n$, $x \neq y$, and

$$|k(x,y) - k(x',y)| + |k(y,x) - k(y,x')| \le c_1 \left(\frac{|x-x'|}{|x-y|} \right)^{\varepsilon} |x-y|^{-n} \tag{3}$$

whenever $2|x-x'| \le |x-y|$. Such operators were introduced in [6].

It is well known that fractional maximal operator, Riesz potential and Calderón-Zygmund operators play an important role in harmonic analysis (see [23, 31]).

Given $b \in BMO(\mathbb{R}^n)$, the commutator $[b,T]$ generated by b and the operator T is defined by

$$[b,T]f(x) = \int_{\mathbb{R}^n} k(x,y) f(y) \left(b(x) - b(y) \right) dy. \tag{4}$$

The purpose of this article is to investigate the boundedness properties of Calderón-Zygmund operators T and their associated commutators $[b,T]$ in the framework of the modified mixed Morrey spaces $\widetilde{L}_{\vec{p},\lambda}(\mathbb{R}^n)$. In particular, we consider symbols b belonging to the space $BMO(\mathbb{R}^n)$ and establish conditions under which these operators act bounded on the aforementioned function spaces, see [5, 12, 13, 16, 17, 18, 19, 22, 24, 30].

By $A \lesssim B$, we mean that $A \le CB$ for some positive constant C independent of the relevant quantities. If $A \lesssim B$ and $B \lesssim A$, we write $A \approx B$ and say that A and B are equivalent.

2. Definition and basic properties of modified mixed Morrey spaces

For any $r > 0$ and $x \in \mathbb{R}^n$, let $B(x,r) = \{y : |y-x| < r\}$ be the ball centered at x with radius r. Let $\mathrm{B} = \{B(x,r) : x \in \mathbb{R}^n, r > 0\}$ denote the set of all such balls. We also use χ_E and $|E|$ to denote the characteristic function and the Lebesgue measure, respectively, of a measurable set E.

The symbol $\vec{p}$ denotes n-tuples of the numbers in $[0, \infty], (n \geq 1), \vec{p} = (p_1, \cdots, p_n)$. By definition, the inequality, for example, $0 < \vec{p} < \infty$ means that $0 < p_i < \infty$ for all i. For $1 \leq \vec{p} \leq \infty$, we denote $\dfrac{1}{P} = \dfrac{1}{n}\sum_{i=1}^{n}\dfrac{1}{p_i}$, $\vec{p}\,' = (p_1', \cdots, p_n')$, where p_i', P' satisfies $\dfrac{1}{p_i} + \dfrac{1}{p_i'} = 1$, $\dfrac{1}{P} + \dfrac{1}{P'} = 1$.

We first recall the definition of the mixed Lebesgue space defined in [3].

Let $\vec{p} = (p_1, \ldots, p_n) \in (0, \infty]^n$. Then the mixed Lebesgue norm $\|\cdot\|_{L_{\vec{p}}}$ or $\|\cdot\|_{L_{(p_1,\ldots,p_n)}}$ is defined by

$$\|f\|_{L_{\vec{p}}} \equiv \|f\|_{L_{(p_1,\ldots,p_n)}} = \Big(\int_{\mathbb{R}} \cdots \Big(\int_{\mathbb{R}} \Big(\int_{\mathbb{R}} |f(x_1, x_2, \ldots, x_n)|^{p_1} dx_1\Big)^{\frac{p_2}{p_1}} dx_2\Big)^{\frac{p_3}{p_2}} \ldots dx_n\Big)^{\frac{1}{p_n}},$$

where $f : \mathbb{R}^n \to \mathbb{R}$ is a measurable function. If $p_j = \infty$ for some $j = 1, \ldots, n$, then we have to make appropriate modifications. We define the mixed Lebesgue space $L_{\vec{p}}(\mathbb{R}^n) = L_{(p_1,\ldots,p_n)}(\mathbb{R}^n)$ to be the set of all $f \in L_0(\mathbb{R}^n)$ with $\|f\|_{L_{\vec{p}}} < \infty$, where $L_0(\mathbb{R}^n)$ denotes the set of measureable functions on $\mathbb{R}^n$.

The following analogue of the Hölder's inequality for $L_{\vec{p}}$ is well known (see, for example, [34]).

Theorem 1. *Let $\Omega \subset \mathbb{R}^n$ be a measurable set, $1 \leq \vec{p} \leq \infty$ and $\frac{1}{\vec{p}} + \frac{1}{\vec{p}'} = 1$. Then for any $f \in L_{\vec{p}}(\Omega)$ and $g \in L_{\vec{p}'}(\Omega)$, the following inequality is valid*

$$\int_{\Omega} |f(x)g(x)| dx \leq \|f\|_{L_{\vec{p}}(\Omega)} \|g\|_{L_{\vec{p}'}(\Omega)}.$$

By elementary calculations we have the following property.

Lemma 1. *Let $0 < \vec{p} < \infty$ and B be balls in $\mathbb{R}^n$. Then*

$$\|\chi_B\|_{L_{\vec{p}}} = \|\chi_B\|_{WL_{\vec{p}}} = |B|^{\frac{1}{P}}.$$

By Theorem 1 and Lemma 1 we get the following estimate.

Lemma 2. *For $1 \le \vec{p} < \infty$ and for the balls $B = B(x,r)$ the following inequality is valid:*

$$\int_B |f(y)|dy \le |B|^{\frac{1}{P'}} \|f\|_{L_{\vec{p}}(B)}.$$

The following lemma shows the Lebesgue differential theorem in mixed-norm Lebesgue spaces as follows.

Lemma 3. [34, Lemma 2.4] *Let $f \in L_1^{\mathrm{loc}}(\mathbb{R}^n)$ and $0 < \vec{p} < \infty$, then*

$$\lim_{r\to 0} \frac{\|f\|_{L_{\vec{p}}(B(x,r))}}{\|\chi_{B(x,r)}\|_{L_{\vec{p}}}} = |f(x)| \quad a.e.\ x \in \mathbb{R}^n.$$

Definition 1. *Let $0 < \vec{p} < \infty$, $\lambda \in \mathbb{R}$, $[t]_1 = \min\{1,t\}$, $t > 0$. We denote by $L_{\vec{p},\lambda}(\mathbb{R}^n)$ the mixed Morrey space [27] and by $\widetilde{L}_{\vec{p},\lambda}(\mathbb{R}^n)$ the modified mixed Morrey space [9] the set of all classes of locally integrable functions f with the finite norms*

$$\|f\|_{L_{\vec{p},\lambda}} = \sup_{x\in\mathbb{R}^n,\, t>0} t^{-\frac{\lambda}{P}} \|f\|_{L_{\vec{p}}(B(x,t))},$$

$$\|f\|_{\widetilde{L}_{\vec{p},\lambda}} = \sup_{x\in\mathbb{R}^n,\, t>0} [t]_1^{-\frac{\lambda}{P}} \|f\|_{L_{\vec{p}}(B(x,t))},$$

respectively.

Note that

$$\widetilde{L}_{\vec{p},0}(\mathbb{R}^n) = L_{\vec{p},0}(\mathbb{R}^n) = L_{\vec{p}}(\mathbb{R}^n),$$

and

$$\widetilde{L}_{\vec{p},\lambda}(\mathbb{R}^n) \subset\!\succ L_{\vec{p},\lambda}(\mathbb{R}^n), \text{ and } \|f\|_{L_{\vec{p},\lambda}} \le \|f\|_{\widetilde{L}_{\vec{p},\lambda}}, \tag{5}$$

$$\widetilde{L}_{\vec{p},\lambda}(\mathbb{R}^n) \subset\!\succ L_{\vec{p}}(\mathbb{R}^n), \text{ and } \|f\|_{L_{\vec{p}}} \le \|f\|_{\widetilde{L}_{\vec{p},\lambda}} \tag{6}$$

and if $\lambda < 0$ or $\lambda > n$, then $L_{\vec{p},\lambda}(\mathbb{R}^n) = \widetilde{L}_{\vec{p},\lambda}(\mathbb{R}^n) = \Theta$, where $\Theta \equiv \Theta(\mathbb{R}^n)$ is the set of all functions equivalent to 0 on $\mathbb{R}^n$.

Remark 1. *If $0 < p < \infty$, and $\lambda > n$, then*

$$L_{\vec{p},\lambda}(\mathbb{R}^n) = \Theta(\mathbb{R}^n),$$

where $\Theta \equiv \Theta(\mathbb{R}^n)$ is the set of all functions equivalent to 0 on $\mathbb{R}^n$.

Let $\lambda > n$ and $f \in \widetilde{L}_{\vec{p},\lambda}(\mathbb{R}^n)$. By the Lebesgue differentiation theorem on integrals for all $f \in L_{\vec{p}}^{\mathrm{loc}}(\mathbb{R}^n)$ (see Lemma 3) and for almost all $x \in \mathbb{R}^n$:

$$\lim_{r\to 0+} \frac{\|f\|_{L_{\vec{p}}(B(x,r))}}{|B(x,r)|^{\frac{1}{p}}} = \lim_{r\to 0+} \frac{\|f\|_{L_{\vec{p}}(B(x,r))}}{\|\chi_{B(x,r)}\|_{L_{\vec{p}}}} = |f(x)|. \tag{7}$$

Then, we have that for all $x \in \mathbb{R}^n$ and $0 < r < 1$

$$\frac{\|f\|_{L_{\vec{p}}(B(x,r))}}{|B(x,r)|^{\frac{1}{P}}} \le v_n^{-\frac{1}{P}} r^{\frac{\lambda-n}{P}} \|f\|_{\widetilde{L}_{\vec{p},\lambda}}.$$

Passing here to the limit as $r \to 0+$ by (7) we get that $f(x) = 0$ for almost all $x \in \mathbb{R}^n$.

Lemma 4. *Let $1 < \vec{p}, \vec{p}' < \infty$, $\frac{1}{\vec{p}} + \frac{1}{\vec{p}'} = 1$, $0 \le \lambda, \lambda' < n$ and $B = B(x,r)$. Suppose that $f\chi_B \in \widetilde{L}_{\vec{p},\lambda}(\mathbb{R}^n)$ and $g\chi_B \in \widetilde{L}_{\vec{p}',\lambda'}(\mathbb{R}^n)$. Then*

$$\|fg\|_{L_1(B)} \le [r]_1^{\frac{\lambda}{P}+\frac{\lambda'}{P'}} \|f\chi_B\|_{\widetilde{L}_{\vec{p},\lambda}(\mathbb{R}^n)} \|g\chi_B\|_{\widetilde{L}_{\vec{p}',\lambda'}(\mathbb{R}^n)}.$$

Proof. Let $1 < \vec{p}, \vec{p}' < \infty$, $\frac{1}{\vec{p}} + \frac{1}{\vec{p}'} = 1$, $0 \le \lambda, \lambda' < n$. Let $B = B(x,r)$ any ball in $\mathbb{R}^n$. Suppose that $f\chi_B \in \widetilde{L}_{\vec{p},\lambda}(\mathbb{R}^n)$ and $g\chi_B \in \widetilde{L}_{\vec{p}',\lambda'}(\mathbb{R}^n)$. By the generalized Hölder's inequality in Lebesgue spaces, we have

$$\begin{aligned}
\|fg\|_{L_1(B)} &\le \|f\chi_B\|_{L_{\vec{p}}(\mathbb{R}^n)} \|g\chi_B\|_{L_{\vec{p}'}(\mathbb{R}^n)} \\
&= [r]^{\frac{\lambda}{P}+\frac{\lambda'}{P'}} \left([r]^{-\frac{\lambda}{P}} \|f\chi_B\|_{L_{\vec{p}}(\mathbb{R}^n)}\right) \left(r^{-\frac{\lambda'}{P'}} \|f\chi_B\|_{L_{\vec{p}}(\mathbb{R}^n)}\right) \\
&\le [r]_1^{\frac{\lambda}{P}+\frac{\lambda'}{P'}} \|f\chi_B\|_{\widetilde{L}_{\vec{p},\lambda}(\mathbb{R}^n)} \|g\chi_B\|_{\widetilde{L}_{\vec{p}',\lambda'}(\mathbb{R}^n)}.
\end{aligned}$$

◄

Lemma 5. *If $0 < \vec{p} < \infty$, $0 \le \lambda \le n$, then*

$$\widetilde{L}_{\vec{p},\lambda}(\mathbb{R}^n) = L_{\vec{p},\lambda}(\mathbb{R}^n) \cap L_{\vec{p}}(\mathbb{R}^n)$$

and

$$\|f\|_{\widetilde{L}_{\vec{p},\lambda}(\mathbb{R}^n)} = \max\left\{\|f\|_{L_{\vec{p},\lambda}(\mathbb{R}^n)}, \|f\|_{L_{\vec{p}}(\mathbb{R}^n)}\right\}.$$

Proof. Let $f \in \widetilde{L}_{\vec{p},\lambda}(\mathbb{R}^n)$. Then from (5) and (6), we have that $f \in L_{\vec{p},\lambda}(\mathbb{R}^n) \cap L_{\vec{p},\mu}(\mathbb{R}^n)$ and $\max\left\{\|f\|_{L_{\vec{p},\lambda}(\mathbb{R}^n)}, \|f\|_{L_{\vec{p}}(\mathbb{R}^n)}\right\} \le \|f\|_{\widetilde{L}_{\vec{p},\lambda}(\mathbb{R}^n)}$.

Moreover

$$\begin{aligned}
\|f\|_{\widetilde{L}_{\vec{p},\lambda}} &= \sup_{x\in\mathbb{R}^n, t>0} [t]_1^{-\frac{\lambda}{P}} \|f\|_{L_{\vec{p}}(B(x,t))} \\
&= \max\left\{\sup_{x\in\mathbb{R}^n, 0<t\le 1} t^{-\frac{\lambda}{P}} \|f\|_{L_{\vec{p}}(B(x,t))}, \sup_{x\in\mathbb{R}^n, t>1} \|f\|_{L_{\vec{p}}(B(x,t))}\right\}
\end{aligned}$$

$$\leq \max\left\{\|f\|_{L_{\vec{p},\lambda}}, \|f\|_{L_{\vec{p}}}\right\}.$$

Therefore, $f \in \widetilde{L}_{\vec{p},\lambda}(\mathbb{R}^n)$ and the embedding $L_{\vec{p},\lambda}(\mathbb{R}^n) \cap L_{\vec{p}}(\mathbb{R}^n) \subset\!\!\succ \widetilde{L}_{\vec{p},\lambda}(\mathbb{R}^n)$ is valid.

Thus, $\widetilde{L}_{\vec{p},\lambda}(\mathbb{R}^n) = L_{\vec{p},\lambda}(\mathbb{R}^n) \cap L_{\vec{p}}(\mathbb{R}^n)$ and $\max\left\{\|f\|_{L_{\vec{p},\lambda}}, \|f\|_{L_{\vec{p}}}\right\} = \|f\|_{\widetilde{L}_{\vec{p},\lambda}}$. ◀

Lemma 6. *If $0 < \vec{p} < \infty$, then*

$$\widetilde{L}_{\vec{p},n}(\mathbb{R}^n) = L_\infty(\mathbb{R}^n)$$

and

$$\|f\|_{L_{\vec{p},n}} = v_n^{\frac{1}{P}} \|f\|_{L_\infty}.$$

Proof. Let $f \in L_\infty(\mathbb{R}^n)$. Then for all $x \in \mathbb{R}^n$ and $0 < t \leq 1$

$$t^{-\frac{n}{P}} \|f\|_{L_{\vec{p}}(B(x,t))} \leq v_n^{\frac{1}{P}} \|f\|_{L_\infty}$$

and for all $x \in \mathbb{R}^n$ and $t \geq 1$

$$t^{-\frac{n}{P}} \|f\|_{L_{\vec{p}}(B(x,t))} \leq v_n^{\frac{1}{P}} \|f\|_{L_\infty}.$$

Therefore, $f \in \widetilde{L}_{\vec{p},\lambda}(\mathbb{R}^n)$ and

$$\|f\|_{\widetilde{L}_{\vec{p},n}} \leq v_n^{\frac{1}{P}} \|f\|_{L_\infty}.$$

Let $f \in \widetilde{L}_{\vec{p},n}(\mathbb{R}^n)$. By the Lebesgue's differentiation theorem we have (see Lemma 3)

$$\lim_{t\to 0} |B(x,t)|^{-\frac{1}{P}} \|f\|_{L_{\vec{p}}(B(x,t))} = |f(x)| \quad \text{for} \quad a.e.\ x \in \mathbb{R}^n.$$

Then for a.e. $x \in \mathbb{R}^n$

$$\begin{aligned} |f(x)| &= |B(x,t)|^{-\frac{1}{P}} \|f\|_{L_{\vec{p}}(B(x,t))} \\ &\leq v_n^{-\frac{1}{P}} \sup_{x\in\mathbb{R}^n,\, 0<t\leq 1} t^{-\frac{n}{P}} \|f\|_{L_{\vec{p}}(B(x,t))} \leq v_n^{-\frac{1}{P}} \|f\|_{\widetilde{L}_{\vec{p},n}}. \end{aligned}$$

Therefore, $f \in L_\infty(\mathbb{R}^n)$ and

$$\|f\|_{L_\infty} \leq v_n^{-\frac{1}{P}} \|f\|_{\widetilde{L}_{\vec{p},n}}.$$

◀

Lemma 7. *If $0 \le \lambda < n$, $0 \le \alpha < n-\lambda$, then for $\frac{n-\lambda}{\alpha} \le P \le \frac{n}{\alpha}$*

$$\widetilde{L}_{\vec{p},\lambda}(\mathbb{R}^n) \subset\!\!\succ L_{1,n-\alpha}(\mathbb{R}^n)$$

and for $f \in \widetilde{L}_{\vec{p},\lambda}(\mathbb{R}^n)$ the following inequality

$$\|f\|_{L_{1,n-\alpha}} \le v_n^{\frac{1}{P'}} \|f\|_{\widetilde{L}_{\vec{p},\lambda}}$$

is valid.

Proof. Let $0 < \alpha < n$, $0 \le \lambda < n$, $f \in \widetilde{L}_{\vec{p},\lambda}(\mathbb{R}^n)$ and $\frac{n-\lambda}{\alpha} \le P \le \frac{n}{\alpha}$. By the Hölder's inequality (see Theorem 1) we have

$$\begin{aligned}
&\|f\|_{L_{1,n-\alpha}} = \sup_{x\in\mathbb{R}^n,\, t>0} t^{\alpha-n} \|f\|_{L_1(B(x,t))} \\
&\le \sup_{x\in\mathbb{R}^n,\, t>0} [t]_1^{\alpha-n} [1/t]_1^{n-\alpha} \|f\|_{L_{\vec{p}}(B(x,t))} \|1\|_{L_{\vec{p}'}(B(x,t))} \\
&= v_n^{\frac{1}{P'}} \sup_{x\in\mathbb{R}^n,\, t>0} \left([t]_1 t^{-1}\right)^{-\frac{n}{P'}} [t]_1^{\alpha-\frac{n-\lambda}{P}} [1/t]_1^{n-\alpha} [t]_1^{-\frac{\lambda}{P}} \|f\|_{L_{\vec{p}}(B(x,t))} \\
&\le v_n^{\frac{1}{P'}} \|f\|_{\widetilde{L}_{\vec{p},\lambda}} \sup_{t>0} \left([t]_1 t^{-1}\right)^{\frac{n}{P}-\alpha} [t]_1^{\alpha-\frac{n-\lambda}{P}}.
\end{aligned}$$

Note that

$$\sup_{t>0} \left([t]_1 t^{-1}\right)^{\frac{n}{P}-\alpha} [t]_1^{\alpha-\frac{n-\lambda}{P}} = \max\left\{ \sup_{0<t\le 1} t^{\alpha-\frac{n-\lambda}{P}}, \sup_{t>1} t^{\alpha-\frac{n}{P}} \right\} < \infty$$
$$\Longleftrightarrow \frac{n-\lambda}{\alpha} \le P \le \frac{n}{\alpha}.$$

Therefore, $f \in \widetilde{L}_{\vec{1},n-\alpha}(\mathbb{R}^n)$ and

$$\|f\|_{L_{1,n-\alpha}} \le v_n^{\frac{1}{P'}} \|f\|_{\widetilde{L}_{\vec{p},\lambda}}.$$

◀

Lemma 8. *Let $1 \le \vec{p} \le \vec{q} < \infty$ and $0 \le \lambda_1, \lambda_2 \le n$. Let us assume that $\frac{\lambda_1}{P} - \frac{\lambda_2}{Q} \le \frac{n}{P} - \frac{n}{Q} \le 0$. Then, the following inequality*

$$\|f\|_{\widetilde{L}_{\vec{p},\lambda_1}} \lesssim \|f\|_{\widetilde{L}_{\vec{q},\lambda_2}}$$

is valid.

Proof. The claim follows from the classical embedding theorem for Morrey spaces. We recall that if $1 \le \vec{p} \le \vec{q} < \infty$, then we have

$$\|f\|_{L_{\vec{p}}(B(x,t))} \lesssim t^{\frac{n}{P}-\frac{n}{Q}} \|f\|_{L_{\vec{q}}(B(x,t))}.$$

Therefore, we obtain

$$\|f\|_{\widetilde{L}_{\vec{p},\lambda_1}} = \sup_{x\in\mathbb{R}^n,\, t>0} [t]_1^{-\frac{\lambda_1}{P}} \|f\|_{L_{\vec{p}}(B(x,t))}$$

$$= \max\left\{ \sup_{x\in\mathbb{R}^n,\, 0<t\le 1} t^{-\frac{\lambda_1}{P}} \|f\|_{L_{\vec{p}}(B(x,t))}, \sup_{x\in\mathbb{R}^n,\, t>1} \|f\|_{L_{\vec{p}}(B(x,t))} \right\}$$

$$\lesssim \max\left\{ \sup_{x\in\mathbb{R}^n,\, 0<t\le 1} t^{\frac{n-\lambda_1}{P}} t^{-\frac{n}{Q}} \|f\|_{L_{\vec{q}}(B(x,t))}, \sup_{x\in\mathbb{R}^n,\, t>1} t^{\frac{n}{P}-\frac{n}{Q}} \|f\|_{L_{\vec{q}}(B(x,t))} \right\}$$

$$= \max\left\{ \sup_{x\in\mathbb{R}^n,\, 0<t\le 1} t^{\frac{n-\lambda_1}{P}-\frac{n-\lambda_2}{Q}} t^{-\frac{\lambda_2}{Q}} \|f\|_{L_{\vec{q}}(B(x,t))}, \sup_{x\in\mathbb{R}^n,\, t>1} t^{\frac{n}{P}-\frac{n}{Q}} \|f\|_{L_{\vec{q}}(B(x,t))} \right\}$$

$$\lesssim \max\left\{ \sup_{x\in\mathbb{R}^n,\, 0<t\le 1} t^{-\frac{\lambda_2}{Q}} \|f\|_{L_{\vec{q}}(B(x,t))}, \sup_{x\in\mathbb{R}^n,\, t>1} \|f\|_{L_{\vec{q}}(B(x,t))} \right\} = \|f\|_{\widetilde{L}_{\vec{q},\lambda_2}}.$$

◄

By Lemma 8, in the case $\vec{p} = \vec{q}$, we get the following corollary.

Corollary 1. *If $1 \le \vec{p} < \infty$ and $0 \le \lambda_1 \le \lambda_2 \le n$. Then, the following inequality*

$$\|f\|_{\widetilde{L}_{\vec{p},\lambda_1}} \lesssim \|f\|_{\widetilde{L}_{\vec{p},\lambda_2}}.$$

is valid.

3. $\widetilde{L}_{\vec{p},\lambda,\mu}$-boundedness of the Calderón-Zygmund singular integral operators

In this section we study the $\widetilde{L}_{\vec{p},\lambda,\mu}$-boundedness of the Calderón-Zygmund singular integral operators T.

Theorem 2. *[27] Let T be the Calderón-Zygmund singular integral operator defined by* (1)*, whose kernel k satisfies conditions* (2)*,* (3) *and $1 < \vec{p} < \infty$. Then the operator T is bounded on $L_{\vec{p}}(\mathbb{R}^n)$.*

The following local estimate holds. The special case $\vec{p} = (p, \dots, p)$ was proved in [8, Theorem 6.1], see also [7].

Lemma 9. *Let T be the Calderón-Zygmund singular integral operator defined by* (1)*, whose kernel k satisfies conditions* (2), (3)*, $1<\vec{p}<\infty$ and for any $x_0\in\mathbb{R}^n$*

$$\int_1^\infty t^{-\frac{n}{P}-1}\,\|f\|_{L_{\vec{p}}(B(x_0,t))}\,dt<\infty.$$

Then Calderón-Zygmund singular integral $Tf(x)$ exists for a.e. $x\in\mathbb{R}^n$ and the inequality

$$\|Tf\|_{L_{\vec{p}}(B(x_0,r))}\lesssim r^{\frac{n}{P}}\int_{2r}^\infty t^{-\frac{n}{P}-1}\,\|f\|_{L_{\vec{p}}(B(x_0,t))}\,dt \tag{8}$$

holds for any ball $B(x_0,r)$ and for all $f\in L^{\text{loc}}_{\vec{p}}(\mathbb{R}^n)$.

Proof. Let $B=B(x_0,r)$ be the ball centered at x_0, with the radius $r>0$, $2B=B(x_0,2r)$, ${}^{\complement}(2B)=\mathbb{R}^n\setminus B(x_0,2r)$, $1<\vec{p}<\infty$. We set $f=f_1+f_2$, where $f_1=f\chi_{2B}$ and $f_2=f\chi_{{}^{\complement}(2B)}$.

We remark that, due to the lack of density of smooth functions in the mixed Morrey spaces, the singular integral operators need to be defined in a convenient way. To avoid confusion, we denote by T_0 the singular integral operator T acting on $L_{\vec{p}}(\mathbb{R}^n)$.

For all $f\in L^{\text{loc}}_{\vec{p}}(\mathbb{R}^n)$ for the singular integral operator we define

$$Tf(x):=T_0f_1(x)+\int_{\mathbb{R}^n}k(x,y)f_2(y)dy. \tag{9}$$

First, we show that $Tf(x)$ is well defined for almost all x and is independent of the choice of the ball B containing x. Since T_0 is bounded on $L_{\vec{p}}(\mathbb{R}^n)$ and $f_1\in L_{\vec{p}}(\mathbb{R}^n)$, T_0f_1 is well defined. Next, we show that the second-term of the right-hand side defining $Tf(x)$ converges absolutely for any $f\in L^{\text{loc}}_{\vec{p}}(\mathbb{R}^n)$ and almost every $x\in\mathbb{R}^n$.

Finally, it remains to show that the definition is independent of the choice of B. Let $\mathcal{B}=\{B(x,r):x\in\mathbb{R}^n,\ r>0\}$. That is, if $B_1,B_2\in\mathcal{B}$ and $x\in B_1\cap B_2$, then

$$T_0(f\chi_{2B_1})(x)+\int_{\mathbb{R}^n\setminus 2B_1}k(x,y)f(y)dy=$$

$$=T_0(f\chi_{2B_2})(x)+\int_{\mathbb{R}^n\setminus 2B_2}k(x,y)f(y)dy. \tag{10}$$

To this end, let $B_3\in\mathcal{B}$ be chosen so that $2B_1\cup 2B_2\subset B_3$.

Since $f\chi_{2B_1},f\chi_{B_3\setminus 2B_1}\in L_{\vec{p}}(\mathbb{R}^n)$, the linearity of T_0 on $L_{\vec{p}}(\mathbb{R}^n)$ yields

$$T_0(f\chi_{2B_1})(x)+\int_{\mathbb{R}^n\setminus 2B_1}k(x,y)f(y)dy$$

$$
\begin{aligned}
&= T_0(f\chi_{2B_1})(x) + \int_{B_3\setminus 2B_1} k(x,y)f(y)dy + \int_{\mathbb{R}^n\setminus B_3} k(x,y)f(y)dy \\
&= T_0(f\chi_{2B_1})(x) + T_0(f\chi_{B_3\setminus 2B_1})(x) + \int_{\mathbb{R}^n\setminus B_3} k(x,y)f(y)dy \\
&= T_0(f\chi_{B_3})(x) + \int_{\mathbb{R}^n\setminus B_3} k(x,y)f(y)dy. \qquad (11)
\end{aligned}
$$

Similarly, we also have

$$
\begin{aligned}
T_0(f\chi_{2B_2})(x) + \int_{\mathbb{R}^n\setminus 2B_2} k(x,y)f(y)dy = \\
= T_0(f\chi_{B_3})(x) + \int_{\mathbb{R}^n\setminus B_3} k(x,y)f(y)dy. \qquad (12)
\end{aligned}
$$

Thus, combining (11) and (12), we obtain (10).

Since T is a linear operator, we have

$$\|Tf\|_{L_{\vec{p}}(B)} = \|Tf_1 + Tf_2\|_{L_{\vec{p}}(B)} \le \|Tf_1\|_{L_{\vec{p}}(\mathbb{R}^n)} + \|Tf_2\|_{L_{\vec{p}}(B)}.$$

Note that $f_1 \in L_{\vec{p}}(\mathbb{R}^n)$ and T is bounded in $L_{\vec{p}}(\mathbb{R}^n)$, we have

$$\|Tf_1\|_{L_{\vec{p}}(B)} \le \|Tf_1\|_{L_{\vec{p}}(\mathbb{R}^n)} \lesssim \|f_1\|_{L_{\vec{p}}(\mathbb{R}^n)} = \|f\|_{L_{\vec{p}}(2B)}.$$

It is clear that $x \in B$, $y \in {}^{\complement}(2B)$ imply $\frac{1}{2}|x_0 - y| \le |x - y| \le \frac{3}{2}|x_0 - y|$, which further yields

$$\left|Tf_2(x)\right| \lesssim \int_{{}^{\complement}(2B)} \frac{|f(y)|}{|x_0 - y|^n}\, dy.$$

Fubini's theorem, we have

$$
\begin{aligned}
\int_{{}^{\complement}(2B)} \frac{|f(y)|}{|x_0 - y|^n}\, dy &\approx \int_{{}^{\complement}(2B)} |f(y)| \int_{|x_0-y|}^{\infty} \frac{dt}{t^{n+1}}\, dy \\
&\approx \int_{2r}^{\infty} \int_{B(x_0,t)\setminus 2B} |f(y)|\, dy\, \frac{dt}{t^{n+1}} \\
&\lesssim \int_{2r}^{\infty} \int_{B(x_0,t)} |f(y)|\, dy\, \frac{dt}{t^{n+1}}.
\end{aligned}
$$

Applying Hölder's inequality on mixed Lebesgue spaces (see Theorem 1 and Lemma 2), we obtain

$$\int_{{}^{\complement}(2B)} \frac{|f(y)|}{|x_0 - y|^n}\, dy \lesssim \int_{2r}^{\infty} \|f\|_{L_1(B(x_0,t))} \frac{dt}{t^{n+1}}$$

$$\leq \int_{2r}^{\infty} |B(x_0,t)|^{\frac{1}{P'}} \, \|f\|_{L_{\vec{p}}(B(x_0,t))} \, dt$$
$$\lesssim \int_{2r}^{\infty} t^{-\frac{n}{P}-1} \, \|f\|_{L_{\vec{p}}(B(x_0,t))} \, dt. \qquad (13)$$

Therefore, by (13), the second term on the right-hand side $\int_{\mathbb{R}^n} k(x,y)f_2(y)dy$ converges absolutely for any $f \in L^{\rm loc}_{\vec{p}}(\mathbb{R}^n)$ and almost every $x \in \mathbb{R}^n$, and therefore, we get the right-hand side of (9) is finite.

Therefore, the singular integral $Tf(x)$ is well defined for almost all x and independent of the choice B containing x.

Then from Lemma 1, we have

$$\|Tf_2\|_{L_{\vec{p}}(B)} \lesssim |B|^{\frac{1}{P}} \int_{2r}^{\infty} t^{-\frac{n}{P}-1} \, \|f\|_{L_{\vec{p}}(B(x_0,t))} \, dt \approx r^{\frac{n}{P}} \int_{2r}^{\infty} t^{-\frac{n}{P}-1} \, \|f\|_{L_{\vec{p}}(B(x_0,t))} \, dt.$$

Therefore, we get

$$\|Tf\|_{L_{\vec{p}}(B)} \lesssim \|f\|_{L_{\vec{p}}(2B)} + r^{\frac{n}{P}} \int_{2r}^{\infty} t^{-\frac{n}{P}-1} \, \|f\|_{L_{\vec{p}}(B(x_0,t))} \, dt.$$

On the other hand,

$$\|f\|_{L_{\vec{p}}(2B)} \approx r^{\frac{n}{P}} \, \|f\|_{L_{\vec{p}}(2B)} \int_{2r}^{\infty} t^{-\frac{n}{P}-1} \, dt$$
$$\leq r^{\frac{n}{P}} \int_{2r}^{\infty} t^{-\frac{n}{P}-1} \, \|f\|_{L_{\vec{p}}(B(x_0,t))} \, dt. \qquad (14)$$

Thus

$$\|Tf\|_{L_{\vec{p}}(B)} \lesssim r^{\frac{n}{P}} \int_{2r}^{\infty} t^{-\frac{n}{P}-1} \, \|f\|_{L_{\vec{p}}(B(x_0,t))} \, dt.$$

◀

Theorem 3. *Let T be the Calderón-Zygmund singular integral operator defined by* (1) *with kernel k satisfying conditions* (2), (3), $1 < \vec{p} < \infty$. *If $f \in \widetilde{L}_{\vec{p},\lambda}(\mathbb{R}^n)$, $0 \leq \lambda < n$, then $Tf \in \widetilde{L}_{\vec{p},\lambda}(\mathbb{R}^n)$ and*

$$\|Tf\|_{\widetilde{L}_{\vec{p},\lambda}} \lesssim \|f\|_{\widetilde{L}_{\vec{p},\lambda}}. \qquad (15)$$

Proof. From the inequality (8) we get

$$\|Tf\|_{\widetilde{L}_{\vec{p},\lambda}} = \sup_{x\in\mathbb{R}^n,\, t>0} [t]_1^{-\frac{\lambda}{P}} \, \|Tf\|_{L_{\vec{p}}(B(x,t))}$$

$$\lesssim \sup_{x\in\mathbb{R}^n,\, t>0} [t]_1^{-\frac{\lambda}{P}}\, t^{\frac{n}{P}} \int_{2t}^{\infty} \tau^{-\frac{n}{P}}\, \|f\|_{L_{\vec{p}}(B(x,\tau))}\, \frac{d\tau}{\tau}$$

$$\lesssim \|f\|_{\widetilde{L}_{\vec{p},\lambda}} \sup_{t>0} [t]_1^{-\frac{\lambda}{P}}\, t^{\frac{n}{P}} \int_{t}^{\infty} \tau^{-\frac{n}{P}}\, [\tau]_1^{\frac{\lambda}{P}}\, \frac{d\tau}{\tau}$$

$$= \|f\|_{\widetilde{L}_{\vec{p},\lambda}} \sup_{t>0} [t]_1^{-\frac{\lambda}{P}} \int_{1}^{\infty} \tau^{-\frac{n}{P}}\, [t\tau]_1^{\frac{\lambda}{P}}\, \frac{d\tau}{\tau}$$

$$= \|f\|_{\widetilde{L}_{\vec{p},\lambda}} \max\Big\{ \sup_{0<t\le 1} t^{-\frac{\lambda}{P}} \int_{1}^{\infty} \tau^{-\frac{n}{P}}\, [t\tau]_1^{\frac{\lambda}{P}}\, \frac{d\tau}{\tau},\ \ \sup_{t>1} \int_{1}^{\infty} \tau^{-\frac{n}{P}}\, \frac{d\tau}{\tau} \Big\}$$

$$= \|f\|_{\widetilde{L}_{\vec{p},\lambda}} \max\Big\{ \sup_{0<t\le 1} t^{-\frac{\lambda}{P}} \Big(\int_{1}^{1/t} \tau^{-\frac{n}{P}}\, (t\tau)^{\frac{\lambda}{P}}\, \frac{d\tau}{\tau} + \int_{1/t}^{\infty} \tau^{-\frac{n}{P}}\, \frac{dt}{t} \Big),\ \int_{1}^{\infty} \tau^{-\frac{n}{P}}\, \frac{d\tau}{\tau} \Big\}$$

$$= \|f\|_{\widetilde{L}_{\vec{p},\lambda}} \max\Big\{ \sup_{0<t\le 1} \Big(\int_{1}^{1/t} \tau^{\frac{\lambda-n}{P}}\, \frac{d\tau}{\tau} + t^{-\frac{\lambda}{P}} \int_{1/t}^{\infty} \tau^{-\frac{n}{P}}\, \frac{d\tau}{\tau} \Big),\ \int_{1}^{\infty} t^{-\frac{n}{P}}\, \frac{d\tau}{\tau} \Big\}$$

$$\le \|f\|_{\widetilde{L}_{\vec{p},\lambda}} \max\Big\{ \sup_{0<t\le 1} \Big(\int_{1}^{1/t} \tau^{-\frac{n-\lambda}{P}}\, \frac{d\tau}{\tau} + \int_{1/t}^{\infty} \tau^{-\frac{n-\lambda}{P}}\, \frac{d\tau}{\tau} \Big),\ \int_{1}^{\infty} \tau^{-\frac{n}{P}}\, \frac{d\tau}{\tau} \Big\}$$

$$= \|f\|_{\widetilde{L}_{\vec{p},\lambda}} \max\Big\{ \int_{1}^{\infty} \tau^{-\frac{n-\lambda}{P}}\, \frac{d\tau}{\tau},\ \int_{1}^{\infty} \tau^{-\frac{n}{P}}\, \frac{d\tau}{\tau} \Big\} \approx \|f\|_{\widetilde{L}_{\vec{p},\lambda}},$$

which implies that the operator T is bounded in $\widetilde{L}_{\vec{p},\lambda}(\mathbb{R}^n)$. ◀

4. $\widetilde{L}_{\vec{p},\lambda}$-boundedness of the commutator of Calderón-Zygmund operators $[b,T]$

In this section we study the $\widetilde{L}_{\vec{p},\lambda}$-boundedness of the commutator of Calderón-Zygmund operators $[b,T]$ with $b \in BMO(\mathbb{R}^n)$.

Definition 2. *The space $BMO(\mathbb{R}^n)$ is defined as the set of all locally integrable functions f with finite norm*

$$\|b\|_{BMO} = \sup_{x\in\mathbb{R}^n, t>0} |B(x,t)|^{-1} \int_{B(x,t)} |b(y) - b_{B(x,t)}| dy < \infty,$$

where $b_{B(x,t)} = |B(x,t)|^{-1} \int_{B(x,t)} b(y)dy$.

The John-Nirenberg ineuqalitiy for $BMO(\mathbb{R}^n)$ yields that for any $1 < p < \infty$ and $b \in BMO(\mathbb{R}^n)$, the $BMO(\mathbb{R}^n)$ norm of b is equivalent to

$$\|b\|_{BMO^p} = \sup_{x\in\mathbb{R}^n, t>0} \Big(|B(x,t)|^{-1} \int_{B(x,t)} |b(y) - b_{B(x,t)}|^p dy \Big)^{\frac{1}{p}}.$$

Recall that for any $\vec{p} = (p_1, \ldots, p_n) \in (1, \infty)^n$, the John-Nirenberg inequality for mixed norm spaces [20] shows that the $BMO(\mathbb{R}^n)$ norm of all $b \in BMO(\mathbb{R}^n)$ is also equivalent to

$$\|b\|_{BMO^{\vec{p}}} = \sup_{x \in \mathbb{R}^n, t>0} \frac{\|(b - b_{B(x,t)})\, \chi_{B(x,t)}\|_{L_{\vec{p}}(\mathbb{R}^n)}}{\|\chi_{B(x,t)}\|_{L_{\vec{p}}(\mathbb{R}^n)}}. \tag{16}$$

The following property for $BMO(\mathbb{R}^n)$ functions is valid.

Lemma 10. *Let $b \in BMO(\mathbb{R}^n)$. Then for all $0 < 2r < t < \infty$, we have*

$$\left|b_{B(x,r)} - b_{B(x,t)}\right| \lesssim \|b\|_{BMO} \ln \frac{t}{r}. \tag{17}$$

Theorem 4. *[33] Let T be the Calderón-Zygmund singular integral operator defined by* (1) *with kernel k satisfying conditions* (2), (3), *$b \in BMO(\mathbb{R}^n)$ and $1 < \vec{p} < \infty$. Then the operator $[b, T]$ is bounded on $L_{\vec{p}}(\mathbb{R}^n)$ and*

$$\|[b, T]f\|_{L_{\vec{p}}(\mathbb{R}^n)} \lesssim \|b\|_{BMO}\, \|f\|_{L_{\vec{p}}(\mathbb{R}^n)}.$$

.

The following local estimate is valid (see [10]).

Lemma 11. *Let T be the Calderón-Zygmund singular integral operator defined by* (1) *with kernel k satisfying conditions* (2), (3), *$b \in BMO(\mathbb{R}^n)$ and $1 < \vec{p} < \infty$.*

Then the commutator of the Calderón-Zygmund singular integral operator $[b, T]$ exists for a.e. $x \in \mathbb{R}^n$ and the inequality

$$\|[b, T]f\|_{L_{\vec{p}}(B(x_0,r))} \lesssim \|b\|_{BMO}\, r^{\frac{n}{P}} \int_{2r}^{1} \Big(1 + \ln \frac{t}{r}\Big) t^{-\frac{n}{P}-1} \|f\|_{L_{\vec{p}}(B(x_0,t))} dt \tag{18}$$

holds for any ball $B(x_0, r)$ and for all $f \in L^{\rm loc}_{\vec{p}}(\mathbb{R}^n)$.

Proof. Let $B = B(x_0, r)$ be the ball centered at x_0, with the radius $r > 0$, $2B = B(x_0, 2r)$, ${}^{\complement}(2B) = \mathbb{R}^n \setminus B(x_0, 2r)$, $1 < \vec{p} < \infty$. We set $f = f_1 + f_2$, where $f_1 = f\chi_{2B}$ and $f_2 = f\chi_{{}^{\complement}(2B)}$.

Since smooth functions are not dense in the modified mixed Morrey spaces, the commutators of singular integral operators require a careful and appropriate definition. For the moment, for the commutator of the singular integral operator $[b, T]$ on $L_{\vec{p}}(\mathbb{R}^n)$ by $[b, T_0]$ to avoid confusion.

For all $f \in L^{\rm loc}_{\vec{p}}(\mathbb{R}^n)$, we define the commutator of the singular integral operator by

$$[b,T]f(x) := [b,T_0]f_1(x) + \int_{\mathbb{R}^n} k(x,y)\,(b(x)-b(y))\,f_2(y)dy. \tag{19}$$

First, we show that $[b,T]f(x)$ is well defined for almost all x and independent of the choice B containing x. As $[b,T_0]$ is bounded on $L_{\vec{p}}(\mathbb{R}^n)$ and $f_1 \in L_{\vec{p}}(\mathbb{R}^n)$, $[b,T_0]f_1$ is well defined. Next, we show that the second-term of the right-hand side defining $[b,T]f(x)$ converges absolutely for any $f \in L^{\mathrm{loc}}_{\vec{p}}(\mathbb{R}^n)$ and almost every $x \in \mathbb{R}^n$.

Finally, it remains to show that the definition is independent of the choice of B. Let $\mathcal{B} = \{B(x,r) : x \in \mathbb{R}^n,\ r>0\}$. That is, if $B_1, B_2 \in \mathcal{B}$ and $x \in B_1 \cap B_2$, then

$$\begin{aligned}
&[b,T_0](f\chi_{2B_1})(x) + \int_{\mathbb{R}^n\setminus 2B_1} k(x,y)\,(b(x)-b(y))\,f(y)dy \\
&= [b,T_0](f\chi_{2B_2})(x) + \int_{\mathbb{R}^n\setminus 2B_2} k(x,y)\,(b(x)-b(y))\,f(y)dy.
\end{aligned} \tag{20}$$

Actually, let $B_3 \in \mathcal{B}$ be selected so that $2B_1 \cup 2B_2 \subset B_3$.

Since $f\chi_{2B_1}, f\chi_{B_3\setminus 2B_1} \in L_{\vec{p}}(\mathbb{R}^n)$, the linearity of $[b,T_0]$ on $L_{\vec{p}}(\mathbb{R}^n)$ yields

$$\begin{aligned}
&[b,T_0](f\chi_{2B_1})(x) + \int_{\mathbb{R}^n\setminus 2B_1} k(x,y)\,(b(x)-b(y))\,f(y)dy \\
&= [b,T_0](f\chi_{2B_1})(x) + \int_{B_3\setminus 2B_1} k(x,y)\,(b(x)-b(y))\,f(y)dy \\
&+ \int_{\mathbb{R}^n\setminus B_3} k(x,y)\,(b(x)-b(y))\,f(y)dy = [b,T_0](f\chi_{2B_1})(x) \\
&+ [b,T_0](f\chi_{B_3\setminus 2B_1})(x) + \int_{\mathbb{R}^n\setminus B_3} k(x,y)\,(b(x)-b(y))\,f(y)dy \\
&= [b,T_0](f\chi_{B_3})(x) + \int_{\mathbb{R}^n\setminus B_3} k(x,y)\,(b(x)-b(y))\,f(y)dy.
\end{aligned} \tag{21}$$

Similarly, we also have

$$\begin{aligned}
&[b,T_0](f\chi_{2B_2})(x) + \int_{\mathbb{R}^n\setminus 2B_2} k(x,y)\,(b(x)-b(y))\,f(y)dy \\
&= [b,T_0](f\chi_{B_3})(x) + \int_{\mathbb{R}^n\setminus B_3} k(x,y)\,(b(x)-b(y))\,f(y)dy.
\end{aligned} \tag{22}$$

Thus, combining (21) and (22) we obtain (20).

Since $[b,T]$ is a linear operator, we have

$$\|[b,T]f\|_{L_{\vec{p}}(B)} = \|[b,T]f_1 + [b,T]f_2\|_{L_{\vec{p}}(B)} \le \|[b,T]f_1\|_{L_{\vec{p}}(\mathbb{R}^n)} + \|[b,T]f_2\|_{L_{\vec{p}}(B)}.$$

Note that $f_1 \in L_{\vec{p}}(\mathbb{R}^n)$ and $[b,T]$ is bounded in $L_{\vec{p}}(\mathbb{R}^n)$, we have

$$\|[b,T]f_1\|_{L_{\vec{p}}(B)} \le \|[b,T]f_1\|_{L_{\vec{p}}(\mathbb{R}^n)} \lesssim$$

$$\lesssim \|b\|_{BMO}\, \|f_1\|_{L_{\vec{p}}(\mathbb{R}^n)} = \|b\|_{BMO}\, \|f\|_{L_{\vec{p}}(2B)}.$$

Since $x \in B$, $y \in {}^{\complement}(2B)$ imply $\frac{1}{2}\,|x_0 - y| \le |x-y| \le \frac{3}{2}\,|x_0-y|$, we get

$$\big|[b,T]f_2(x)\big| \lesssim \int_{^{\complement}(2B)} \frac{|b(x)-b(y)|}{|x_0-y|^n}\,|f(y)|\,dy.$$

By using generalized Minkowski inequality on mixed Lebesgue spaces, we have

$$\|[b,T]f_2\|_{L_{\vec{p}}(B)} \lesssim \Big\| \int_{^{\complement}(2B)} \frac{|b(\cdot)-b(y)|}{|x_0-y|^n}\,|f(y)|\,dy \Big\|_{L_{\vec{p}}(B)}$$

$$\le \Big\| \int_{^{\complement}(2B)} \frac{|b(y)-b_B|}{|x_0-y|^n}\,|f(y)|\,dy \Big\|_{L_{\vec{p}}(B)} + \Big\| \int_{^{\complement}(2B)} \frac{|b(\cdot)-b_B|}{|x_0-y|^n}\,|f(y)|\,dy \Big\|_{L_{\vec{p}}(B)}$$

$$= I_1 + I_2.$$

For the term I_1, by Fubini's theorem, we have

$$\begin{aligned}
I_1 &\approx |B|^{\frac{1}{P}} \int_{^{\complement}(2B)} \frac{|b(y)-b_B|}{|x_0-y|^n}\,|f(y)|\,dy \\
&\approx r^{\frac{n}{P}} \int_{^{\complement}(2B)} |b(y)-b_B|\,|f(y)| \int_{|x_0-y|}^{\infty} \frac{dt}{t^{n+1}}\,dy \\
&\approx r^{\frac{n}{P}} \int_{2r}^{\infty} \int_{B(x_0,t)\setminus 2B} |b(y)-b_B|\,|f(y)|\,dy\,\frac{dt}{t^{n+1}} \\
&\lesssim r^{\frac{n}{P}} \int_{2r}^{\infty} \int_{B(x_0,t)} |b(y)-b_B|\,|f(y)|\,dy\,\frac{dt}{t^{n+1}}.
\end{aligned}$$

Applying Hölder's inequality and (16), (17), we get

$$\begin{aligned}
I_1 &\lesssim r^{\frac{n}{P}} \int_{2r}^{\infty} \int_{B(x_0,t)} |b(y)-b_B|\,|f(y)|\,dy\,\frac{dt}{t^{n+1}} \\
&\lesssim r^{\frac{n}{P}} \int_{2r}^{\infty} \int_{B(x_0,t)} |b(y)-b_{B(x_0,t)}|\,|f(y)|\,dy\,\frac{dt}{t^{n+1}}
\end{aligned}$$

$$+\, r^{\frac{n}{P}} \int_{2r}^{\infty} \int_{B(x_0,t)} |b_{B(x_0,r)} - b_{B(x_0,t)}|\, |f(y)|\, dy\, \frac{dt}{t^{n+1}}$$
$$\lesssim r^{\frac{n}{P}} \int_{2r}^{\infty} \|(b - b_{B(x_9,t)})\, \chi_{B(x_0,t)}\|_{L_{\vec{p}'}(\mathbb{R}^n)}\, \|f\|_{L_{\vec{p}}(B(x_0,t))}\, \frac{dt}{t^{n+1}}$$
$$+\, r^{\frac{n}{P}} \int_{2r}^{\infty} |b_{B(x_0,r)} - b_{B(x_0,t)}|\, |B(x_0,t)|^{\frac{1}{P'}}\, \|f\|_{L_{\vec{p}}(B(x_0,t))}\, \frac{dt}{t^{n+1}}$$
$$\lesssim \|b\|_{BMO}\, r^{\frac{n}{P}} \int_{2r}^{\infty} \left(1 + \ln \frac{t}{r}\right) t^{-\frac{n}{P}-1} \|f\|_{L_{\vec{p}}(B(x_0,t))} dt.$$

In order to estimate I_2, note that

$$I_2 = \Big\| \int_{\complement_{(2B)}} \frac{|b(\cdot) - b_B|}{|x_0 - y|^n}\, |f(y)|\, dy \Big\|_{L_{\vec{p}}(B)}$$

$$= \Big\| b(\cdot) - b_B \Big\|_{L_{\vec{p}}(B)} \times \int_{\complement_{(2B)}} \frac{|f(y)|}{|x_0 - y|^n}\, dy.$$

It follows from (16) that

$$I_2 \lesssim \|b\|_{BMO}\, r^{\frac{n}{P}} \int_{\complement_{(2B)}} \frac{|f(y)|}{|x_0 - y|^n}\, dy.$$

Thus, by (13), we get

$$I_2 \lesssim \|b\|_{BMO}\, r^{\frac{n}{P}} \int_{2r}^{\infty} t^{-\frac{n}{P}-1} \|f\|_{L_{\vec{p}}(B(x_0,t))} dt.$$

Summing up I_1 and I_2, we get

$$\|[b,T]f_2\|_{L_{\vec{p}}(B)} \lesssim \|b\|_{BMO}\, r^{\frac{n}{P}} \int_{2r}^{\infty} \left(1 + \ln \frac{t}{r}\right) t^{-\frac{n}{P}-1} \|f\|_{L_{\vec{p}}(B(x_0,t))} dt.$$

Therefore, by (14), holds

$$\|[b,T]f\|_{L_{\vec{p}}(B)} \lesssim$$

$$\lesssim \|b\|_{BMO} \left(\|f\|_{L_{\vec{p}}(2B)} + r^{\frac{n}{P}} \int_{2r}^{\infty} \left(1 + \ln \frac{t}{r}\right) t^{-\frac{n}{P}-1} \|f\|_{L_{\vec{p}}(B(x_0,t))} dt \right)$$
$$\lesssim \|b\|_{BMO}\, r^{\frac{n}{P}} \int_{2r}^{\infty} \left(1 + \ln \frac{t}{r}\right) t^{-\frac{n}{P}-1} \|f\|_{L_{\vec{p}}(B(x_0,t))} dt.$$

The proof was completed. ◄

Theorem 5. *Let $1 < \vec{p} < \infty$, $0 < \lambda < n$ and $b \in BMO(\mathbb{R}^n)$, and T be the Calderón-Zygmund singular integral operator defined by* (1) *with kernel k satisfying conditions* (2), (3).

If $f \in \widetilde{L}_{\vec{p},\lambda}(\mathbb{R}^n)$, then $[b,T]f \in \widetilde{L}_{\vec{p},\lambda}(\mathbb{R}^n)$ and

$$\|[b,T]f\|_{\widetilde{L}_{\vec{p},\lambda}} \lesssim \|b\|_{BMO}\, \|f\|_{\widetilde{L}_{\vec{p},\lambda}}.$$

Proof. Let $1 < p < \infty$, $0 < \lambda < n$ and $b \in BMO(\mathbb{R}^n)$. From the inequality (18) we get

$$\begin{aligned}
&\|[b,T]f\|_{\widetilde{L}_{\vec{p},\lambda}} = \sup_{x\in\mathbb{R}^n,\, r>0} [r]_1^{-\frac{\lambda}{P}}\, \|[b,T]f\|_{L_{\vec{p}}(B(x,r))} \\
&\lesssim \|b\|_{BMO} \sup_{x\in\mathbb{R}^n,\, r>0} [r]_1^{-\frac{\lambda}{P}}\, r^{\frac{n}{P}} \int_{2r}^{\infty} \Big(1+\ln\frac{t}{r}\Big)\, t^{-\frac{n}{P}}\, \|f\|_{L_{\vec{p}}(B(x,t))}\, \frac{dt}{t} \\
&\lesssim \|b\|_{BMO}\, \|f\|_{\widetilde{L}_{\vec{p},\lambda}} \sup_{r>0} [r]_1^{-\frac{\lambda}{P}}\, r^{\frac{n}{P}} \int_{r}^{\infty} \Big(1+\ln\frac{t}{r}\Big)\, t^{-\frac{n}{P}}\, [t]_1^{\frac{\lambda}{P}}\, \frac{dt}{t} \\
&= \|b\|_{BMO}\, \|f\|_{\widetilde{L}_{\vec{p},\lambda}} \sup_{r>0}\, [r]_1^{-\frac{\lambda}{P}} \int_{1}^{\infty} \big(1+\ln\tau\big)\, \tau^{-\frac{n}{P}}\, [r\tau]_1^{\frac{\lambda}{P}}\, \frac{d\tau}{\tau} \\
&= \|b\|_{BMO}\, \|f\|_{\widetilde{L}_{\vec{p},\lambda}} \max\Big\{ \sup_{0<r\le 1} r^{-\frac{\lambda}{P}} \int_{1}^{\infty} \big(1+\ln\tau\big)\, \tau^{-\frac{n}{P}}\, [r\tau]_1^{\frac{\lambda}{P}}\, \frac{d\tau}{\tau}, \\
&\quad \sup_{r>1} \int_{1}^{\infty} \big(1+\ln\tau\big)\, \tau^{-\frac{n}{P}}\, \frac{d\tau}{\tau} \Big\} \\
&= \|b\|_{BMO}\, \|f\|_{\widetilde{L}_{\vec{p},\lambda}} \max\Big\{ \sup_{0<r\le 1} r^{-\frac{\lambda}{P}} \Big(\int_{1}^{1/r} \big(1+\ln\tau\big)\, \tau^{-\frac{n}{P}}\, (r\tau)^{\frac{\lambda}{P}}\, \frac{d\tau}{\tau} \\
&+ \int_{1/r}^{\infty} \big(1+\ln\tau\big)\, \tau^{-\frac{n}{P}}\, \frac{dt}{t} \Big),\ \int_{1}^{\infty} \big(1+\ln\tau\big)\, \tau^{-\frac{n}{P}}\, \frac{d\tau}{\tau} \Big\} \\
&= \|b\|_{BMO}\, \|f\|_{\widetilde{L}_{\vec{p},\lambda}} \max\Big\{ \sup_{0<r\le 1} \Big(\int_{1}^{1/r} \big(1+\ln\tau\big)\, \tau^{\frac{\lambda-n}{P}}\, \frac{d\tau}{\tau} \\
&+ r^{-\frac{\lambda}{P}} \int_{1/r}^{\infty} \big(1+\ln\tau\big)\, \tau^{-\frac{n}{P}}\, \frac{d\tau}{\tau} \Big),\ \int_{1}^{\infty} \big(1+\ln\tau\big)\, \tau^{-\frac{n}{P}}\, \frac{d\tau}{\tau} \Big\} \\
&\le \|b\|_{BMO}\, \|f\|_{\widetilde{L}_{\vec{p},\lambda}} \max\Big\{ \sup_{0<r\le 1} \Big(\int_{1}^{1/r} \big(1+\ln\tau\big)\, \tau^{-\frac{n-\lambda}{P}}\, \frac{d\tau}{\tau} \\
&+ \int_{1/r}^{\infty} \big(1+\ln\tau\big)\, \tau^{-\frac{n-\lambda}{P}}\, \frac{d\tau}{\tau} \Big),\ \int_{1}^{\infty} \big(1+\ln\tau\big)\, \tau^{-\frac{n}{P}}\, \frac{d\tau}{\tau} \Big\} \\
&= \|b\|_{BMO}\, \|f\|_{\widetilde{L}_{\vec{p},\lambda}} \max\Big\{ \int_{1}^{\infty} \big(1+\ln\tau\big)\, \tau^{-\frac{n-\lambda}{P}}\, \frac{d\tau}{\tau},\ \int_{1}^{\infty} \big(1+\ln\tau\big)\, \tau^{-\frac{n}{P}}\, \frac{d\tau}{\tau} \Big\} \\
&\lesssim \|b\|_{BMO}\, \|f\|_{\widetilde{L}_{\vec{p},\lambda}},
\end{aligned}$$

which implies that the operator $[b, T]$ is bounded on $\widetilde{L}_{\vec{p},\lambda}(\mathbb{R}^n)$. ◄

5. Some applications

In this section, we apply Theorems 3 and 5 to the pseudo-differential operators.

Pseudo-differential operators are generalizations of differential operators and singular integrals. Let m be a real number, $0 \le \delta < 1$ and $0 \le \rho < 1$. Following [21, 32], a symbol in $S^m_{\rho,\delta}$ is a smooth function $\sigma(x, \xi)$ defined on $\mathbb{R}^n \times \mathbb{R}^n$ such that for all multi-indices α and β the following estimate holds:

$$\left| D^\alpha_x D^\beta_\xi \sigma(x, \xi) \right| \le C_{\alpha\beta}(1 + |\xi|)^{m-\rho|\beta|+\delta|\alpha|},$$

where $C_{\alpha\beta} > 0$ is independent of x and ξ. A symbol in $S^{-\infty}_{\rho,\delta}$ is one which satisfies the above estimates for each real number m.

The operator A given by

$$Af(x) = \int_{\mathbb{R}^n} \sigma(x, \xi)\, e^{2\pi i x\xi}\, \hat{f}(\xi) d\xi$$

is called a pseudo-differential operator with symbol $\sigma(x, \xi) \in S^m_{\rho,\delta}$, where f is a Schwartz function and $\hat{f}$ denotes the Fourier transform of f. As usual, $L^m_{\rho,\delta}$ will denote the class of pseudo-differential operators with symbols in $S^m_{\rho,\delta}$.

We consider pseudo-differential operators in the Hörmander class $L^m_{\rho,\delta}$ [21]. Miller [25] showed the boundedness of $L^0_{1,0}$ pseudo-differential operators on weighted L_p $(1 < p < \infty)$ spaces whenever the weight function belongs to Muckenhoupt's class A_p. In [6] it is shown that pseudo-differential operators in $L^0_{1,0}$ are Calderón-Zygmund operators, then from Theorems 3 and 5 we get the following new results.

Corollary 2. *Let $1 < \vec{p} < \infty$ and $0 < \lambda < n$. If A is a pseudo-differential operator of the Hörmander class $L^0_{1,0}$, then the operator A is bounded on $\widetilde{L}_{\vec{p},\lambda}(\mathbb{R}^n)$.*

Corollary 3. *Let $1 < \vec{p} < \infty$, $0 < \lambda < n$ and $b \in BMO(\mathbb{R}^n)$. Let also A be a pseudo-differential operator of the Hörmander class $L^0_{1,0}$, Then the commutator operator $[b, A]$ is bounded on $\widetilde{L}_{\vec{p},\lambda}(\mathbb{R}^n)$.*

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