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Singular Integrals and Commutators in Parabolic Grand Herz Spaces With Variable Exponents

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Abstract. In this paper, we first establish the boundedness in grand Herz spaces with variable exponent on $\mathbb{R}^n$ with a parabolic metric for a large class of sublinear operators and linear commutators generated by BMO functions and Calderón-Zygmund operators. Using these results, we study the regularity of strong solutions to nondivergence parabolic equations with VMO coefficients in the parabolic grand Herz space with variable exponent.

Key Words and Phrases: parabolic grand Herz space, variable exponent, singular integral, commutator.

2010 Mathematics Subject Classifications: 42B20, 46E30, 35B65

1. Introduction

It is well-known that the classical Herz spaces were introduced by Herz [16] in the study of absolutely convergent Fourier transforms. In the past several decades, Herz spaces and the related function spaces have attracted considerable attention because the interesting norm includes explicitly both local and global informations of the function, see [23, 26]. Especially for the study of parabolic Herz spaces, some progress has also been made. For instance, Chen, Yang and Zhou [6] established the boundedness in parabolic Herz spaces for a large class of sublinear operators and discussed the regularity of strong solutions to nondivergence parabolic equations. Ragusa [27] proved regularity results for weak solutions to divergence form parabolic equations having discontinuous coefficients.

On the other hand, after the paper of Kováčik and Rákosník [22] in 1991, the theory of function spaces with variable exponents has attracted a lot of attentions in recent years (see [9, 10] and the references therein). Owing to their wide

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applications in electrorheological fluids [29], image processing [7] and partial differential equations with non-standard growth [15], the theory of function spaces with variable exponents has made great progresses. Particularly, the Herz spaces with variable exponents and the boundedness some related integral operators have been extensively studied, see [18, 19, 30, 31, 32, 33, 36].

Meanwhile, the grand Lebesgue spaces were introduced by Iwaniec and Sbordone [17] to study the integrability of the Jacobian in 1992. The properties of the grand Lebesgue spaces had been deeply investigated in [5, 12], and the grand spaces have been proved to be useful in application to partial differential equations in [1, 2, 3, 13, 21]. The grand Lebesgue sequence spaces were introduced in [25], where various operators in harmonic analysis were studied on these spaces. Furthermore, Nafis, Rafeiro and Zaighum [24] introduced the grand variable Herz spaces. Wang and Liu [35] introduced some anisotropic grand Herz type spaces with variable exponents and obtained some boundedness of some singular integral operators.

Inspired by the references [6, 27, 28, 34], we will establish the boundedness for a large class of sublinear operators and linear commutators and study the regularity of strong solutions to nondivergence parabolic equations with VMO coefficients in the parabolic grand Herz space with variable exponent. Throughout this paper, we assume the letter C always denotes a positive constant that may vary at each occurrence but is independent of the essential variable. In addition, we denote the Lebesgue measure and the characteristic function of a measurable set $A \subset \mathbb{R}^n$ by $|A|$ and χ_A respectively. The notation $f \approx g$ means that there exist constants $C_1, C_2 > 0$ such that $C_1 g \leqslant f \leqslant C_2 g$.

2. Parabolic grand Herz spaces with variable exponents

Firstly, we endow $\mathbb{R}^n$ with the following parabolic metric introduced by Fabes and Rivière in [11]: $\rho(x,y) = \rho(x-y)$, where $\rho(x) = \sqrt{\frac{|x'|^2+\sqrt{|x'|^4+4t^2}}{2}}$ and $x = (x_1, \cdots, x_{n-1}, t) = (x', t) \in \mathbb{R}^n$. We define parabolic cubes and parabolic balls of center $x = (x', t) \in \mathbb{R}^n$ and radius r, respectively, by $Q_r(x) = \{y = (y_1, \cdots, y_{n-1}, s) \in \mathbb{R}^n : |t-s| < r^2, |x_i - y_i| < r \text{ for } i = 1, \cdots, n-1\}$ and $B_r(x) = \{y \in \mathbb{R}^n : \rho(x-y) < r\}$. Also by αQ_r and αB_r we mean the cube and the ball having radius αr and the same center, respectively, as Q_r and B_r. $f \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$ is said to be in the parabolic $\mathrm{BMO}(\mathbb{R}^n)$ if its BMO seminorm

$$\|f\|_* = \sup_Q \frac{1}{|Q|} \int_Q |f(x) - f_Q| dx < \infty,$$

where Q ranges over the class of parabolic cubes in $\mathbb{R}^n$ and $f_Q = |Q|^{-1} \int_Q f(x) dx$.

For $f \in \mathrm{BMO}(\mathbb{R}^n)$ and $r > 0$, we define the VMO modulus of f by

$$\eta(r) = \sup_{\rho \le r} \frac{1}{|Q_\rho|} \int_{Q_\rho} |f(x) - f_{Q_\rho}| dx,$$

where Q_ρ ranges over the class of parabolic cubes in $\mathbb{R}^n$ of radius ρ. $f \in \mathrm{BMO}(\mathbb{R}^n)$ is said to belong to the space VMO $(\mathbb{R}^n)$ if $\eta(r) \to 0$ as $r \to 0$. Moreover, letting Ω be an open subset of $\mathbb{R}^n$, we then define the spaces $\mathrm{BMO}(\Omega)$ and $\mathrm{VMO}(\Omega)$ by taking $Q \cap \Omega$ and $Q_\rho \cap \Omega$ in place of Q and Q_ρ in the definitions of $\|f\|_*$ and $\eta(r)$, respectively.

Next, we recall some definitions on function spaces with variable exponent. Given an open set $\Omega \subset \mathbb{R}^n$ with $|\Omega| > 0$.

Definition 1. *Let $p(\cdot) : \Omega \longrightarrow [1, \infty)$ be a measurable function.*

(i) The Lebesgue space with variable exponent $L^{p(\cdot)}(\Omega)$ is defined by

$$L^{p(\cdot)}(\Omega) := \{f \text{ is measurable} : \int_\Omega \left(\frac{|f(x)|}{\eta} \right)^{p(x)} dx < \infty \text{ for some constant } \eta > 0\}.$$

(ii) The space $L^{p(\cdot)}_{\mathrm{loc}}(\Omega)$ is defined by

$$L^{p(\cdot)}_{\mathrm{loc}}(\Omega) := \{f : f \in L^{p(\cdot)}(E) \text{ for all compact subsets } E \subset \Omega\}.$$

$L^{p(\cdot)}(\Omega)$ is a Banach function space when it is equipped with the Luxemburg-Nakano norm

$$\|f\|_{L^{p(\cdot)}(\Omega)} = \inf \left\{ \eta > 0 : \int_\Omega \left(\frac{|f(x)|}{\eta} \right)^{p(x)} dx \le 1 \right\}.$$

Define $\mathcal{P}(\Omega)$ to be set of measurable functions $p(\cdot) : \Omega \longrightarrow [1, \infty)$ such that

$$p^- = \operatorname{ess\,inf}\{p(x) : x \in \Omega\} > 1, \quad p^+ = \operatorname{ess\,sup}\{p(x) : x \in \Omega\} < \infty.$$

Denote $p'(x) = p(x)/(p(x) - 1)$.

Let f be a locally integrable function, then the Hardy-Littlewood maximal operator is defined by

$$Mf(x) = \sup_{B \ni x} \frac{1}{|B|} \int_{B \cap \Omega} |f(y)| dy,$$

where the supremum is taken over all balls B containing x. Let $\mathcal{B}(\Omega)$ be the set of $p(\cdot) \in \mathcal{P}(\Omega)$ such that M is bounded on $L^{p(\cdot)}(\Omega)$.

Now we give the definitions of parabolic grand Herz spaces with variable exponent. Note that $B_{2^k}(0) = \{x \in \mathbb{R}^n : \rho(x) < 2^k\}$ and let $A_k = B_{2^k}(0) \setminus B_{2^{k-1}}(0)$ for $k \in \mathbb{Z}$. Denote $\mathbb{Z}_+$ and $\mathbb{N}$ as the sets of all positive and non-negative integers, $\chi_k = \chi_{A_k}$ for $k \in \mathbb{Z}$, $\tilde{\chi}_k = \chi_k$ if $k \in \mathbb{Z}_+$ and $\tilde{\chi}_0 = \chi_{B_0}$.

Definition 2. *Let $\alpha \in \mathbb{R}, 1 \leq p < \infty$, $q(\cdot) \in \mathcal{P}(\mathbb{R}^n)$ and $\theta > 0$. The homogeneous parabolic grand Herz space with variable exponent $\dot{K}^{\alpha,p),\theta}_{q(\cdot)}(\mathbb{R}^n)$ is defined by*

$$\dot{K}^{\alpha,p),\theta}_{q(\cdot)}(\mathbb{R}^n) = \{f \in L^{q(\cdot)}_{\mathrm{loc}}(\mathbb{R}^n \setminus \{0\}) : \|f\|_{\dot{K}^{\alpha,p),\theta}_{q(\cdot)}(\mathbb{R}^n)} < \infty\},$$

where

$$\|f\|_{\dot{K}^{\alpha,p),\theta}_{q(\cdot)}(\mathbb{R}^n)} = \sup_{\varepsilon>0} \left(\varepsilon^{\theta} \sum_{k=-\infty}^{\infty} 2^{k\alpha p(1+\varepsilon)} \|f\chi_k\|^{p(1+\varepsilon)}_{L^{q(\cdot)}(\mathbb{R}^n)} \right)^{\frac{1}{p(1+\varepsilon)}}.$$

The non-homogeneous parabolic grand Herz space with variable exponent $K^{\alpha,p),\theta}_{q(\cdot)}(\mathbb{R}^n)$ is defined by

$$K^{\alpha,p),\theta}_{q(\cdot)}(\mathbb{R}^n) = \{f \in L^{q(\cdot)}_{\mathrm{loc}}(\mathbb{R}^n) : \|f\|_{K^{\alpha,p),\theta}_{q(\cdot)}(\mathbb{R}^n)} < \infty\},$$

where

$$\|f\|_{K^{\alpha,p),\theta}_{q(\cdot)}(\mathbb{R}^n)} = \sup_{\varepsilon>0} \left(\varepsilon^{\theta} \sum_{k=0}^{\infty} 2^{k\alpha p(1+\varepsilon)} \|f\tilde{\chi}_k\|^{p(1+\varepsilon)}_{L^{q(\cdot)}(\mathbb{R}^n)} \right)^{\frac{1}{p(1+\varepsilon)}}.$$

3. Boundedness of singular integral operators and their commutators

In this section, we will establish the boundedness results about some singular integral operators and their commutators on parabolic grand Herz spaces with variable exponent $\dot{K}^{\alpha,p),\theta}_{q(\cdot)}(\mathbb{R}^n)$ and $K^{\alpha,p),\theta}_{q(\cdot)}(\mathbb{R}^n)$. To give the main theorems, we need the following lemmas in Lebesgue spaces with variable exponent. Given an open set $\Omega \subset \mathbb{R}^n$ with $|\Omega| > 0$.

Lemma 1 ([9]). *If $p(\cdot) \in \mathcal{P}(\Omega)$ and satisfies*

$$|p(x) - p(y)| \leq \frac{C}{-\log(\rho(x-y))}, \quad \rho(x-y) \leq 1/2 \tag{1}$$

and

$$|p(x) - p(y)| \leq \frac{C}{\log(\rho(x) + e)}, \quad \rho(y) \geq \rho(x), \tag{2}$$

then $p(\cdot) \in \mathcal{B}(\Omega)$, that is the Hardy-Littlewood maximal operator $\mathcal{M}$ is bounded on $L^{p(\cdot)}(\Omega)$.

Lemma 2 ([22]). *(Generalized Hölder's inequality) Let $p(\cdot) \in \mathcal{P}(\Omega)$. If $f \in L^{p(\cdot)}(\Omega)$ and $g \in L^{p'(\cdot)}(\Omega)$, then fg is integrable on Ω and*

$$\int_\Omega |f(x)g(x)|dx \leq r_p \|f\|_{L^{p(\cdot)}(\Omega)} \|g\|_{L^{p'(\cdot)}(\Omega)},$$

where

$$r_p = 1 + 1/p^- - 1/p^+.$$

Lemma 3 ([18]). *Let $q(\cdot) \in \mathcal{B}(\mathbb{R}^n)$. Then there exists a positive constant C such that for all balls B in $\mathbb{R}^n$ and all measurable subsets $S \subset B$,*
$\frac{\|\chi_B\|_{L^{q(\cdot)}(\mathbb{R}^n)}}{\|\chi_S\|_{L^{q(\cdot)}(\mathbb{R}^n)}} \leq C\frac{|B|}{|S|}$, $\frac{\|\chi_S\|_{L^{q(\cdot)}(\mathbb{R}^n)}}{\|\chi_B\|_{L^{q(\cdot)}(\mathbb{R}^n)}} \leq C\left(\frac{|S|}{|B|}\right)^{\delta_1}$ *and* $\frac{\|\chi_S\|_{L^{q'(\cdot)}(\mathbb{R}^n)}}{\|\chi_B\|_{L^{q'(\cdot)}(\mathbb{R}^n)}} \leq C\left(\frac{|S|}{|B|}\right)^{\delta_2}$,
where $0 < \delta_1$, $\delta_2 < 1$ depend on $q(\cdot)$.

Throughout this paper δ_1, δ_2 are the same as in Lemma 3.

Lemma 4 ([18]). *Suppose $q(\cdot) \in \mathcal{B}(\mathbb{R}^n)$. Then there exists a positive constant C such that for all balls B in $\mathbb{R}^n$,*

$$\frac{1}{|B|}\|\chi_B\|_{L^{q(\cdot)}(\mathbb{R}^n)}\|\chi_B\|_{L^{q'(\cdot)}(\mathbb{R}^n)} \leq C.$$

In what follows, we set $\lambda = n + 1$. Now, we can state the boundedness theorem for general sublinear operators on parabolic grand Herz spaces with variable exponent as follows.

Theorem 1. *Let $1 \leq p < \infty$, $q(\cdot) \in \mathcal{B}(\mathbb{R}^n)$ and $-\lambda\delta_1 < \alpha < \lambda\delta_2$. If a sublinear operator T satisfies that for any integrable function f with compact support and for some $c_0 \geq 1$,*

$$|Tf(x)| \leq C \int_{\mathbb{R}^n} \frac{|f(y)|}{\rho(x-y)^\lambda} dy, \quad x \notin c_0 \,\mathrm{supp} f, \tag{3}$$

and T is bounded on $L^{q(\cdot)}(\mathbb{R}^n)$, then T is also bounded on $\dot{K}^{\alpha,p),\theta}_{q(\cdot)}(\mathbb{R}^n)$ and $K^{\alpha,p),\theta}_{q(\cdot)}(\mathbb{R}^n)$.

Proof. We only prove the homogeneous case. The proof of non-homogenous case is similar. For convenience, we suppose $c_0 = 1$ and write

$$f(x) = \sum_{l=-\infty}^{\infty} f(x)\chi_l(x) \equiv \sum_{l=-\infty}^{\infty} f_l(x).$$

So, we have

$$
\begin{aligned}
&\|T(f)\|_{\dot{K}_{q(\cdot)}^{\alpha,p),\theta}(\mathbb{R}^n)}\\
&= \sup_{\varepsilon>0}\left(\varepsilon^{\theta}\sum_{k=-\infty}^{\infty}2^{k\alpha p(1+\varepsilon)}\|T(f)\chi_k\|_{L^{q(\cdot)}(\mathbb{R}^n)}^{p(1+\varepsilon)}\right)^{\frac{1}{p(1+\varepsilon)}}\\
&\leq \sup_{\varepsilon>0}\left(\varepsilon^{\theta}\sum_{k=-\infty}^{\infty}2^{k\alpha p(1+\varepsilon)}\left\|\left(\sum_{l=-\infty}^{\infty}|T(f_l)|\right)\chi_k\right\|_{L^{q(\cdot)}(\mathbb{R}^n)}^{p(1+\varepsilon)}\right)^{\frac{1}{p(1+\varepsilon)}}\\
&\leq C\sup_{\varepsilon>0}\left(\varepsilon^{\theta}\sum_{k=-\infty}^{\infty}2^{k\alpha p(1+\varepsilon)}\left\|\left(\sum_{l=-\infty}^{k-2}|T(f_l)|\right)\chi_k\right\|_{L^{q(\cdot)}(\mathbb{R}^n)}^{p(1+\varepsilon)}\right)^{\frac{1}{p(1+\varepsilon)}}\\
&+C\sup_{\varepsilon>0}\left(\varepsilon^{\theta}\sum_{k=-\infty}^{\infty}2^{k\alpha p(1+\varepsilon)}\left\|\left(\sum_{l=k-1}^{k+1}|T(f_l)|\right)\chi_k\right\|_{L^{q(\cdot)}(\mathbb{R}^n)}^{p(1+\varepsilon)}\right)^{\frac{1}{p(1+\varepsilon)}}\\
&+C\sup_{\varepsilon>0}\left(\varepsilon^{\theta}\sum_{k=-\infty}^{\infty}2^{k\alpha p(1+\varepsilon)}\left\|\left(\sum_{l=k+2}^{\infty}|T(f_l)|\right)\chi_k\right\|_{L^{q(\cdot)}(\mathbb{R}^n)}^{p(1+\varepsilon)}\right)^{\frac{1}{p(1+\varepsilon)}}\\
&\equiv C(I_1+I_2+I_3).
\end{aligned}
$$

For I_2, using the $L^{q(\cdot)}(\mathbb{R}^n)$-boundedness of T, we have

$$
\begin{aligned}
I_2 &\leq C\sup_{\varepsilon>0}\left(\varepsilon^{\theta}\sum_{k=-\infty}^{\infty}2^{k\alpha p(1+\varepsilon)}\left(\sum_{l=k-1}^{k+1}\||T(f_l)|\|_{L^{q(\cdot)}(\mathbb{R}^n)}\right)^{p(1+\varepsilon)}\right)^{\frac{1}{p(1+\varepsilon)}}\\
&\leq C\sup_{\varepsilon>0}\left(\varepsilon^{\theta}\sum_{k=-\infty}^{\infty}2^{k\alpha p(1+\varepsilon)}\left(\sum_{l=k-1}^{k+1}\||f_l|\|_{L^{q(\cdot)}(\mathbb{R}^n)}\right)^{p(1+\varepsilon)}\right)^{\frac{1}{p(1+\varepsilon)}}\\
&\leq C\sup_{\varepsilon>0}\left(\varepsilon^{\theta}\sum_{k=-\infty}^{\infty}2^{k\alpha p(1+\varepsilon)}\|f\chi_k\|_{L^{q(\cdot)}(\mathbb{R}^n)}^{p(1+\varepsilon)}\right)^{\frac{1}{p(1+\varepsilon)}}\\
&= C\|f\|_{\dot{K}_{q(\cdot)}^{\alpha,p),\theta}(\mathbb{R}^n)}.
\end{aligned}
$$

Now we estimate I_1. Using the facts that $l \leq k-2$ and $x \in A_k$, by (3) and generalized Hölder's inequality we have

$$\begin{aligned} |T(f_l)(x)| &\leq C2^{-k\lambda}\int_{\mathbb{R}^n}|f_l(y)|dy \\ &\leq C2^{-k\lambda}\|\chi_l\|_{L^{q'(\cdot)}(\mathbb{R}^n)}\|f_l\|_{L^{q(\cdot)}(\mathbb{R}^n)}. \end{aligned}$$

By Lemma 3 and Lemma 4, we have

$$\begin{aligned} I_1 &\leq C\sup_{\varepsilon>0}\left(\varepsilon^\theta\sum_{k=-\infty}^{\infty}2^{k\alpha p(1+\varepsilon)}\left(\sum_{l=-\infty}^{k-2}2^{-k\lambda}\|\chi_l\|_{L^{q'(\cdot)}(\mathbb{R}^n)}\|f_l\|_{L^{q(\cdot)}(\mathbb{R}^n)}\|\chi_k\|_{L^{q(\cdot)}(\mathbb{R}^n)}\right)^{p(1+\varepsilon)}\right)^{\frac{1}{p(1+\varepsilon)}} \\ &\leq C\sup_{\varepsilon>0}\left(\varepsilon^\theta\sum_{k=-\infty}^{\infty}\left(\sum_{l=-\infty}^{k-2}2^{k\alpha}\|f_l\|_{L^{q(\cdot)}(\mathbb{R}^n)}\frac{\|\chi_l\|_{L^{q'(\cdot)}(\mathbb{R}^n)}}{\|\chi_k\|_{L^{q'(\cdot)}(\mathbb{R}^n)}}\right)^{p(1+\varepsilon)}\right)^{\frac{1}{p(1+\varepsilon)}} \\ &\leq C\sup_{\varepsilon>0}\left(\varepsilon^\theta\sum_{k=-\infty}^{\infty}\left(\sum_{l=-\infty}^{k-2}2^{k\alpha}\|f_l\|_{L^{q(\cdot)}(\mathbb{R}^n)}2^{(l-k)\lambda\delta_2}\right)^{p(1+\varepsilon)}\right)^{\frac{1}{p(1+\varepsilon)}} \\ &= C\sup_{\varepsilon>0}\left(\varepsilon^\theta\sum_{k=-\infty}^{\infty}\left(\sum_{l=-\infty}^{k-2}2^{l\alpha}\|f_l\|_{L^{q(\cdot)}(\mathbb{R}^n)}2^{(k-l)(\alpha-\lambda\delta_2)}\right)^{p(1+\varepsilon)}\right)^{\frac{1}{p(1+\varepsilon)}}. \end{aligned}$$

Take $\frac{1}{p(1+\varepsilon)}+\frac{1}{[p(1+\varepsilon)]'}=1$. Since $\alpha<\lambda\delta_2$, by Hölder's inequality we have

$$\begin{aligned} I_1 &\leq C\sup_{\varepsilon>0}\left\{\varepsilon^\theta\sum_{k=-\infty}^{\infty}\left(\sum_{l=-\infty}^{k-2}2^{l\alpha p(1+\varepsilon)}\|f_l\|^{p(1+\varepsilon)}_{L^{q(\cdot)}(\mathbb{R}^n)}2^{(k-l)(\alpha-\lambda\delta_2)p(1+\varepsilon)/2}\right)\right. \\ &\qquad\left.\times\left(\sum_{l=-\infty}^{k-2}2^{(k-l)(\alpha-\lambda\delta_2)[p(1+\varepsilon)]'/2}\right)^{p(1+\varepsilon)/[p(1+\varepsilon)]'}\right\}^{\frac{1}{p(1+\varepsilon)}} \\ &\leq C\sup_{\varepsilon>0}\left\{\varepsilon^\theta\sum_{l=-\infty}^{\infty}2^{l\alpha p(1+\varepsilon)}\|f_l\|^{p(1+\varepsilon)}_{L^{q(\cdot)}(\mathbb{R}^n)}\left(\sum_{k=l+2}^{\infty}2^{(k-l)(\alpha-\lambda\delta_2)p(1+\varepsilon)/2}\right)\right\}^{\frac{1}{p(1+\varepsilon)}} \\ &\leq C\sup_{\varepsilon>0}\left\{\varepsilon^\theta\sum_{l=-\infty}^{\infty}2^{l\alpha p(1+\varepsilon)}\|f_l\|^{p(1+\varepsilon)}_{L^{q(\cdot)}(\mathbb{R}^n)}\right\}^{\frac{1}{p(1+\varepsilon)}} \\ &= C\|f\|_{\dot{K}^{\alpha,p),\theta}_{q(\cdot)}(\mathbb{R}^n)}. \end{aligned}$$

For I_3. Note that for $l \geq k+2$ and $x \in A_k$, from the condition (3) it follows that

$$\begin{aligned} |T(f_l)(x)| &\leq C2^{-l\lambda}\int_{\mathbb{R}^n}|f_l(y)|dy \\ &\leq C2^{-l\lambda}\|\chi_l\|_{L^{q'(\cdot)}(\mathbb{R}^n)}\|f_l\|_{L^{q(\cdot)}(\mathbb{R}^n)}. \end{aligned}$$

By Lemma 3 and Lemma 4, we have

$$\begin{aligned} I_3 &\leq \sup_{\varepsilon>0}\left(\varepsilon^\theta\sum_{k=-\infty}^{\infty}2^{k\alpha p(1+\varepsilon)}\left(\sum_{l=k+2}^{\infty}2^{-l\lambda}\|\chi_l\|_{L^{q'(\cdot)}(\mathbb{R}^n)}\|f_l\|_{L^{q(\cdot)}(\mathbb{R}^n)}\|\chi_k\|_{L^{q(\cdot)}(\mathbb{R}^n)}\right)^{p(1+\varepsilon)}\right)^{\frac{1}{p(1+\varepsilon)}} \\ &\leq C\sup_{\varepsilon>0}\left(\varepsilon^\theta\sum_{k=-\infty}^{\infty}\left(\sum_{l=k+2}^{\infty}2^{k\alpha}\|f_l\|_{L^{q(\cdot)}(\mathbb{R}^n)}\frac{\|\chi_k\|_{L^{q(\cdot)}(\mathbb{R}^n)}}{\|\chi_l\|_{L^{q(\cdot)}(\mathbb{R}^n)}}\right)^{p(1+\varepsilon)}\right)^{\frac{1}{p(1+\varepsilon)}} \\ &\leq C\sup_{\varepsilon>0}\left(\varepsilon^\theta\sum_{k=-\infty}^{\infty}\left(\sum_{l=k+2}^{\infty}2^{k\alpha}\|f_l\|_{L^{q(\cdot)}(\mathbb{R}^n)}2^{(k-l)\lambda\delta_1}\right)^{p(1+\varepsilon)}\right)^{\frac{1}{p(1+\varepsilon)}} \\ &= C\sup_{\varepsilon>0}\left(\varepsilon^\theta\sum_{k=-\infty}^{\infty}\left(\sum_{l=k+2}^{\infty}2^{l\alpha}\|f_l\|_{L^{q(\cdot)}(\mathbb{R}^n)}2^{(k-l)(\alpha+\lambda\delta_1)}\right)^{p(1+\varepsilon)}\right)^{\frac{1}{p(1+\varepsilon)}}. \end{aligned}$$

Take $\frac{1}{p(1+\varepsilon)}+\frac{1}{[p(1+\varepsilon)]'}=1$. Since $\alpha>-\lambda\delta_1$, by Hölder's inequality we have

$$\begin{aligned} I_3 &\leq C\sup_{\varepsilon>0}\left\{\varepsilon^\theta\sum_{k=-\infty}^{\infty}\left(\sum_{l=k+2}^{\infty}2^{l\alpha p(1+\varepsilon)}\|f_l\|^{p(1+\varepsilon)}_{L^{q(\cdot)}(\mathbb{R}^n)}2^{(k-l)(\alpha+\lambda\delta_1)p(1+\varepsilon)/2}\right)\right. \\ &\qquad\left.\times\left(\sum_{l=k+2}^{\infty}2^{(k-l)(\alpha+\lambda\delta_1)[p(1+\varepsilon)]'/2}\right)^{p(1+\varepsilon)/[p(1+\varepsilon)]'}\right\}^{\frac{1}{p(1+\varepsilon)}} \\ &\leq C\sup_{\varepsilon>0}\left\{\varepsilon^\theta\sum_{l=-\infty}^{\infty}2^{l\alpha p(1+\varepsilon)}\|f_l\|^{p(1+\varepsilon)}_{L^{q(\cdot)}(\mathbb{R}^n)}\left(\sum_{k=-\infty}^{l-2}2^{(k-l)(\alpha+\lambda\delta_1)p(1+\varepsilon)/2}\right)\right\}^{\frac{1}{p(1+\varepsilon)}} \\ &\leq C\sup_{\varepsilon>0}\left\{\varepsilon^\theta\sum_{l=-\infty}^{\infty}2^{l\alpha p(1+\varepsilon)}\|f_l\|^{p(1+\varepsilon)}_{L^{q(\cdot)}(\mathbb{R}^n)}\right\}^{\frac{1}{p(1+\varepsilon)}} \\ &= C\|f\|_{\dot{K}^{\alpha,p),\theta}_{q(\cdot)}(\mathbb{R}^n)}. \end{aligned}$$

This finishes the proof of Theorem 1. ◀

For linear commutators $[a,T]$ of parabolic BMO($\mathbb{R}^n$) function a with linear operator T, defined by $[a,T]f(x)=a(x)Tf(x)-T(af)(x)$, we also have the following general theorem.

Theorem 2. *Let $1 \le p < \infty$, $q(\cdot) \in \mathcal{B}(\mathbb{R}^n)$, $-\lambda\delta_1 < \alpha < \lambda\delta_2$ and $a \in \mathrm{BMO}(\mathbb{R}^n)$. If the linear operator T satisfies (3) and $[a,T]$ is bounded on $L^{q(\cdot)}(\mathbb{R}^n)$, then $[a,T]$ is also bounded on $\dot{K}_{q(\cdot)}^{\alpha,p),\theta}(\mathbb{R}^n)$ and $K_{q(\cdot)}^{\alpha,p),\theta}(\mathbb{R}^n)$.*

Similar to the method of Lemma 3 in [19], we can establish some relations between Lebesgue spaces with variable exponent and parabolic BMO spaces.

Lemma 5. *Let $q(\cdot) \in \mathcal{B}(\mathbb{R}^n)$, k be a positive integer and B be a parabolic ball in $\mathbb{R}^n$. Then we have that for all $b \in \mathrm{BMO}(\mathbb{R}^n)$ and all $j, i \in \mathbb{Z}$ with $j > i$,*

$$\frac{1}{C}\|b\|_*^k \le \sup_B \frac{1}{\|\chi_B\|_{L^{q(\cdot)}(\mathbb{R}^n)}} \|(b-b_B)^k \chi_B\|_{L^{q(\cdot)}(\mathbb{R}^n)} \le C\|b\|_*^k,$$

$$\|(b-b_{B_{2^i}(0)})^k \chi_{B_{2^j}(0)}\|_{L^{q(\cdot)}(\mathbb{R}^n)} \le C(j-i)^k \|b\|_*^k \|\chi_{B_{2^j}(0)}\|_{L^{q(\cdot)}(\mathbb{R}^n)},$$

where $B_{2^i}(0) = \{x \in \mathbb{R}^n : \rho(x) \le 2^i\}$ and $B_{2^j}(0) = \{x \in \mathbb{R}^n : \rho(x) \le 2^j\}$.

Proof of Theorem 2. Similar to the proof of Theorem 1, we only show the homogeneous case and write

$$f(x) = \sum_{j=-\infty}^{\infty} f(x)\chi_j(x) \equiv \sum_{j=-\infty}^{\infty} f_j(x).$$

So, we have

$$\begin{aligned}
\|[a,T](f)\|_{\dot{K}_{q(\cdot)}^{\alpha,p),\theta}(\mathbb{R}^n)} &= \sup_{\varepsilon>0}\left(\varepsilon^\theta \sum_{k=-\infty}^{\infty} 2^{k\alpha p(1+\varepsilon)} \|[a,T](f)\chi_k\|_{L^{q(\cdot)}(\mathbb{R}^n)}^{p(1+\varepsilon)}\right)^{\frac{1}{p(1+\varepsilon)}} \\
&\le \sup_{\varepsilon>0}\left(\varepsilon^\theta \sum_{k=-\infty}^{\infty} 2^{k\alpha p(1+\varepsilon)} \left\|\left(\sum_{j=-\infty}^{\infty} |[a,T](f_j)|\right)\chi_k\right\|_{L^{q(\cdot)}(\mathbb{R}^n)}^{p(1+\varepsilon)}\right)^{\frac{1}{p(1+\varepsilon)}} \\
&\le C\sup_{\varepsilon>0}\left(\varepsilon^\theta \sum_{k=-\infty}^{\infty} 2^{k\alpha p(1+\varepsilon)} \left(\sum_{j=-\infty}^{k-3} \||[a,T](f_j)|\chi_k\|_{L^{q(\cdot)}(\mathbb{R}^n)}\right)^{p(1+\varepsilon)}\right)^{\frac{1}{p(1+\varepsilon)}} \\
&\quad + C\sup_{\varepsilon>0}\left(\varepsilon^\theta \sum_{k=-\infty}^{\infty} 2^{k\alpha p(1+\varepsilon)} \left(\sum_{j=k-2}^{k+2} \||[a,T](f_j)|\chi_k\|_{L^{q(\cdot)}(\mathbb{R}^n)}\right)^{p(1+\varepsilon)}\right)^{\frac{1}{p(1+\varepsilon)}}
\end{aligned}$$

$$+C\sup_{\varepsilon>0}\left(\varepsilon^{\theta}\sum_{k=-\infty}^{\infty}2^{k\alpha p(1+\varepsilon)}\left(\sum_{j=k+3}^{\infty}\||[a,T](f_j)|\chi_k\|_{L^{q(\cdot)}(\mathbb{R}^n)}\right)^{p(1+\varepsilon)}\right)^{\frac{1}{p(1+\varepsilon)}}$$

$$\equiv C(J_1+J_2+J_3).$$

For J_2, by the $L^{q(\cdot)}(\mathbb{R}^n)$-boundedness of $[a,T]$, we obtain

$$\begin{aligned} J_2 &\le C\sup_{\varepsilon>0}\left(\varepsilon^{\theta}\sum_{k=-\infty}^{\infty}2^{k\alpha p(1+\varepsilon)}\left(\sum_{j=k-2}^{k+2}\|f_j\|_{L^{q(\cdot)}(\mathbb{R}^n)}\right)^{p(1+\varepsilon)}\right)^{\frac{1}{p(1+\varepsilon)}} \\ &\le C\|f\|_{\dot{K}^{\alpha,p),\theta}_{q(\cdot)}(\mathbb{R}^n)}. \end{aligned}$$

For J_1. Note that $j\le k-3$ and $x\in A_k$, by Lemma 5 and generalized Hölder's inequality we have

$$\begin{aligned} |[a,T](f_j)(x)| &\le C\int_{A_j}\frac{|a(y)-a(x)||f(y)|}{\rho(x-y)^{\lambda}}dy \\ &\le C2^{-k\lambda}\int_{A_j}|a(y)-a(x)||f(y)|dy \\ &\le C2^{-k\lambda}\int_{A_j}|a(y)-a_j||f(y)|dy+C2^{-k\lambda}|a_j-a(x)|\int_{A_j}|f(y)|dy \\ &\le C2^{-k\lambda}\|f_j\|_{L^{q(\cdot)}(\mathbb{R}^n)}\|(a(\cdot)-a_j)\chi_j\|_{L^{q'(\cdot)}(\mathbb{R}^n)} \\ &\quad +C2^{-k\lambda}|a_j-a(x)|\|f_j\|_{L^{q(\cdot)}(\mathbb{R}^{n+1})}\|\chi_j\|_{L^{q'(\cdot)}(\mathbb{R}^n)} \\ &\le C2^{-k\lambda}\|\chi_{B_{2^j}(0)}\|_{L^{q'(\cdot)}(\mathbb{R}^n)}\|f_j\|_{L^{q(\cdot)}(\mathbb{R}^{n+1})}(\|a\|_*+|a_j-a(x)|). \end{aligned}$$

Here and subsequently, we set

$$a_j=\frac{1}{|B_{2^j}(0)|}\int_{B_{2^j}(0)}a(y)dy.$$

Thus, for $j\le k-3$, by Lemmas 3–5 we obtain that

$$\begin{aligned} \||[a,T](f_j)|\chi_k\|_{L^{q(\cdot)}(\mathbb{R}^n)} &\le C2^{-k\lambda}\|\chi_{B_{2^j}(0)}\|_{L^{q'(\cdot)}(\mathbb{R}^n)}\|f_j\|_{L^{q(\cdot)}(\mathbb{R}^n)}(\|a\|_*\|\chi_k\|_{L^{q(\cdot)}(\mathbb{R}^n)} \\ &\quad +\|(a_j-a(\cdot))\chi_k\|_{L^{q(\cdot)}(\mathbb{R}^n)}) \\ &\le C2^{-k\lambda}(k-j)\|a\|_*\|\chi_{B_{2^j}(0)}\|_{L^{q'(\cdot)}(\mathbb{R}^n)}\|f_j\|_{L^{q(\cdot)}(\mathbb{R}^n)}\|\chi_{B_{2^k}(0)}\|_{L^{q(\cdot)}(\mathbb{R}^n)} \\ &\le C(k-j)\|a\|_*\|f_j\|_{L^{q(\cdot)}(\mathbb{R}^n)}\frac{\|\chi_{B_{2^j}(0)}\|_{L^{q'(\cdot)}(\mathbb{R}^n)}}{\|\chi_{B_{2^k}(0)}\|_{L^{q'(\cdot)}(\mathbb{R}^n)}} \end{aligned}$$

$$\leq C2^{(j-k)\lambda\delta_2}(k-j)\|a\|_*\|f_j\|_{L^{q(\cdot)}(\mathbb{R}^n)}.$$

Therefore, similar to the method of estimating I_1, by Hölder's inequality and using the fact that $\alpha < \lambda\delta_2$ we have

$$\begin{aligned}
J_1 &\leq C\|a\|_* \sup_{\varepsilon>0}\left\{\varepsilon^\theta \sum_{k=-\infty}^{\infty}\left(\sum_{j=-\infty}^{k-3} 2^{j\alpha}2^{(k-j)(\alpha-\lambda\delta_2)}(k-j)\|f_j\|_{L^{q(\cdot)}(\mathbb{R}^n)}\right)^{p(1+\varepsilon)}\right\}^{\frac{1}{p(1+\varepsilon)}}\\
&\leq C\|a\|_* \sup_{\varepsilon>0}\left\{\varepsilon^\theta \sum_{j=-\infty}^{\infty} 2^{j\alpha p(1+\varepsilon)}\|f_j\|_{L^{q(\cdot)}(\mathbb{R}^n)}^{p(1+\varepsilon)}\right\}^{\frac{1}{p(1+\varepsilon)}}\\
&= C\|a\|_*\|f\|_{\dot{K}_{q(\cdot)}^{\alpha,p),\theta}(\mathbb{R}^n)}.
\end{aligned}$$

To estimate J_3. We note that for $j \geq k+3$ and $x \in A_k$, by Lemma 5 and generalized Hölder's inequality we have

$$\begin{aligned}
|[a,T](f_j)(x)| &\leq C\int_{A_j}\frac{|a(y)-a(x)||f(y)|}{\rho(x-y)^\lambda}dy\\
&\leq C2^{-j\lambda}\int_{A_j}|a(y)-a_j||f(y)|dy + C2^{-j\lambda}|a_j-a(x)|\int_{A_j}|f(y)|dy\\
&\leq C2^{-j\lambda}\|f_j\|_{L^{q(\cdot)}(\mathbb{R}^n)}\|(a(\cdot)-a_j)\chi_j\|_{L^{q'(\cdot)}(\mathbb{R}^n)}\\
&\quad +C2^{-j\lambda}|a_j-a(x)|\|f_j\|_{L^{q(\cdot)}(\mathbb{R}^n)}\|\chi_j\|_{L^{q'(\cdot)}(\mathbb{R}^n)}\\
&\leq C2^{-j\lambda}\|\chi_{B_{2^j}(0)}\|_{L^{q'(\cdot)}(\mathbb{R}^n)}\|f_j\|_{L^{q(\cdot)}(\mathbb{R}^n)}(\|a\|_* + |a_j-a(x)|).
\end{aligned}$$

Thus, for $j \geq k+3$, by Lemmas 3–5 we obtain that

$$\begin{aligned}
\||[a,T](f_j)|\chi_k\|_{L^{q(\cdot)}(\mathbb{R}^n)} &\leq C2^{-j\lambda}\|\chi_{B_{2^j}(0)}\|_{L^{q'(\cdot)}(\mathbb{R}^n)}\|f_j\|_{L^{q(\cdot)}(\mathbb{R}^n)}(\|a\|_*\|\chi_k\|_{L^{q(\cdot)}(\mathbb{R}^n)}\\
&\quad +\|(a_j-a(\cdot))\chi_k\|_{L^{q(\cdot)}(\mathbb{R}^n)})\\
&\leq C2^{-j\lambda}(j-k)\|a\|_*\|\chi_{B_{2^j}(0)}\|_{L^{q'(\cdot)}(\mathbb{R}^n)}\|f_j\|_{L^{q(\cdot)}(\mathbb{R}^n)}\|\chi_{B_{2^k}(0)}\|_{L^{q(\cdot)}(\mathbb{R}^n)}\\
&\leq C(j-k)\|a\|_*\|f_j\|_{L^{q(\cdot)}(\mathbb{R}^n)}\frac{\|\chi_{B_{2^k}(0)}\|_{L^{q(\cdot)}(\mathbb{R}^n)}}{\|\chi_{B_{2^j}(0)}\|_{L^{q(\cdot)}(\mathbb{R}^n)}}\\
&\leq C2^{(k-j)\lambda\delta_1}(j-k)\|a\|_*\|f_j\|_{L^{q(\cdot)}(\mathbb{R}^n)}.
\end{aligned}$$

Therefore, similar to the method of estimating I_3, by Hölder's inequality and using the fact that $\alpha > -\lambda\delta_1$ we have

$$J_3 \leq C\|a\|_* \sup_{\varepsilon>0}\left\{\varepsilon^\theta \sum_{k=-\infty}^{\infty}\left(\sum_{j=k+3}^{\infty} 2^{j\alpha}2^{(k-j)(\alpha+\lambda\delta_1)}(j-k)\|f_j\|_{L^{q(\cdot)}(\mathbb{R}^n)}\right)^{p(1+\varepsilon)}\right\}^{\frac{1}{p(1+\varepsilon)}}$$

$$
\begin{aligned}
&\le C\|a\|_* \sup_{\varepsilon>0}\left\{\varepsilon^\theta \sum_{j=-\infty}^{\infty} 2^{j\alpha p(1+\varepsilon)}\|f_j\|_{L^{q(\cdot)}(\mathbb{R}^n)}^{p(1+\varepsilon)}\right\}^{\frac{1}{p(1+\varepsilon)}} \\
&= C\|a\|_*\|f\|_{\dot K^{\alpha,p),\theta}_{q(\cdot)}(\mathbb{R}^n)}.
\end{aligned}
$$

By summing up J_1, J_2 and J_3, we finish the proof of Theorem 2.

To state some simple corollaries of Theorems 1 and 2, we still need to introduce more definitions.

Definition 3. *A function k is said to be a parabolic Calderón-Zygmund kernel (for short, constant PCZ kernel) on $\mathbb{R}^n$ endowed with the above parabolic metric ρ, if*

(i) $k \in C^\infty(\mathbb{R}^n \setminus \{0\})$;

(ii) $k(rx', r^2 t) = r^{-(n+1)}k(x', t)$ for any $r > 0$;

(iii) $\int_{S^{n-1}} k(x)d\sigma(x) = 0$, where $d\sigma(x)$ is the element of area of the ellipsoid $S^{n-1} = \{x \in \mathbb{R}^n : \rho(x) = 1\}$.

Definition 4. *Let Ω be an open subset of $\mathbb{R}^n$. A function $k(x, y)$, $x \in \Omega$, $y \in \{\mathbb{R}^n \setminus \{0\}\}$ is said to be a variable PCZ kernel on Ω, if*

(i) $k(x, \cdot)$ is a constant PCZ kernel for a.e. $x \in \Omega$;

(ii) $\max_{|j|\le 2n} \|(\partial^j/\partial y^j)k(x, y)\|_{L^\infty(\Omega\times S^{n-1})} = M < \infty$.

For a constant PCZ kernel k on $\mathbb{R}^n$ or a variable PCZ kernel on Ω. We define the corresponding operator T by

$$
Tf(x) = \text{p.v.} \int_{\mathbb{R}^n} k(x-y)f(y)dy \quad \text{for } x \in \mathbb{R}^n
$$

or

$$
Tf(x) = \text{p.v.} \int_{\Omega} k(x, x-y)f(y)dy \quad \text{for } x \in \Omega.
$$

And similar to the methods of [6] and [14], by Theorems 1, 2 and the extension theorem for BMO function on a domain (see [20]) we can get the following corollaries.

Corollary 1. *Let $0 < p < \infty$, $q(\cdot) \in \mathcal{B}(\mathbb{R}^n)$ and $-\lambda\delta_1 < \alpha < \lambda\delta_2$. If k is a constant PCZ kernel or a variable PCZ kernel on $\mathbb{R}^n$, T is the corresponding operator and $a \in \mathrm{BMO}(\mathbb{R}^n)$, then T and $[a, T]$ are bounded, respectively, on $\dot K^{\alpha,p),\theta}_{q(\cdot)}(\mathbb{R}^n)$ and $K^{\alpha,p),\theta}_{q(\cdot)}(\mathbb{R}^n)$. That is, there exists a constant C independent of f such that*

$$
\|Tf\|_{\dot K^{\alpha,p),\theta}_{q(\cdot)}(\mathbb{R}^n)} \le C\|f\|_{\dot K^{\alpha,p),\theta}_{q(\cdot)}(\mathbb{R}^n)}, \quad \|Tf\|_{K^{\alpha,p),\theta}_{q(\cdot)}(\mathbb{R}^n)} \le C\|f\|_{K^{\alpha,p),\theta}_{q(\cdot)}(\mathbb{R}^n)}
$$

and

$$\|[a,T]f\|_{\dot{K}^{\alpha,p),\theta}_{q(\cdot)}(\mathbb{R}^n)} \leq C\|f\|_{\dot{K}^{\alpha,p),\theta}_{q(\cdot)}(\mathbb{R}^n)}, \quad \|[a,T]f\|_{K^{\alpha,p),\theta}_{q(\cdot)}(\mathbb{R}^n)} \leq C\|f\|_{K^{\alpha,p),\theta}_{q(\cdot)}(\mathbb{R}^n)}.$$

Corollary 2. *Let $m \in \mathbb{Z}$, $0 < p < \infty$, $q(\cdot) \in \mathcal{B}(B_{2^m}(0))$ and $-\lambda\delta_1 < \alpha < \lambda\delta_2$. If k is a variable PCZ kernel on $B_{2^m}(0)$, T is the corresponding operator and $a \in \mathrm{BMO}(B_{2^m}(0))$, then there exists a constant C independent of f such that*

$$\|Tf\|_{\dot{K}^{\alpha,p),\theta}_{q(\cdot)}(B_{2^m}(0))} \leq C\|f\|_{\dot{K}^{\alpha,p),\theta}_{q(\cdot)}(B_{2^m}(0))}$$

and

$$\|[a,T]f\|_{\dot{K}^{\alpha,p),\theta}_{q(\cdot)}(B_{2^m}(0))} \leq C\|f\|_{\dot{K}^{\alpha,p),\theta}_{q(\cdot)}(B_{2^m}(0))}.$$

For $a \in \mathrm{VMO}(\mathbb{R}^n)$, by a similar method in [6, 8] and Corollary 2, we can obtain the following corollary.

Corollary 3. *Suppose that $0 < p < \infty$, $q(\cdot) \in \mathcal{B}(\mathbb{R}^n)$ and $-\lambda\delta_1 < \alpha < \lambda\delta_2$. Let $m_0 \in \mathbb{Z}$, $a \in \mathrm{VMO}(\mathbb{R}^n)$ and η be its VMO modulus, and T be the corresponding operator. Then for any $\nu > 0$, there exists an integer $m_1 = m_1(\nu, \eta) \leq m_0$ such that for any $m < m_1$ and any $f \in \dot{K}^{\alpha,p),\theta}_{q(\cdot)}(B_{2^m}(0))$,*

$$\|[a,T]f\|_{\dot{K}^{\alpha,p),\theta}_{q(\cdot)}(B_{2^m}(0))} \leq C\nu\|f\|_{\dot{K}^{\alpha,p),\theta}_{q(\cdot)}(B_{2^m}(0))},$$

where C is independent of f, ν and m.

4. Some applications

In this section, we will study the regularity of strong solutions to nondivergence parabolic equations with VMO coefficients in parabolic grand Herz spaces with variable exponent by applying the estimates about singular integral operators and commutators obtained on the above section.

Let $m_0 \in \mathbb{Z}$ and L be a linear parabolic operator of the form

$$Lu(x) = u_t(x) - \sum_{i,j=1}^{n-1} a_{ij}(x)u_{x_ix_j}(x),$$

where $x = (x', t) = (x_1, ..., x_{n-1}, t) \in \mathbb{R}^n$. The principal part of the operator is symmetric and uniformly elliptic, that is,

$$a_{ij}(x) = a_{ji}(x)\,(i,j = 1, ..., n-1) \quad \text{and} \quad \lambda^{-1}|a|^2 \leq \sum_{i,j=1}^{n-1} a_{ij}(x)a_ia_j \leq \lambda|a|^2 \quad (4)$$

for some $\lambda \geq 1$ and for every $a \in \mathbb{R}^n$, *a.e.* $x \in Q_{2^{m_0}} = \overline{B}_{2^{m_0}}(0) \times (0, 2^{2m_0})$, where $\overline{B}_{2^{m_0}}(0) = \{x \in \mathbb{R}^{n-1} : |x| < 2^{m_0}\}$. Let $a_{ij} \in \mathrm{VMO}(Q_{2^{m_0}})$. Without loss of generality, we may assume that $a_{ij} \in \mathrm{VMO}(\mathbb{R}^n)$; see Theorem 1.2 in [4]. We are interested in the regularity of strong solutions to the Cauchy-Dirichlet problem

$$\begin{cases} Lu = f & \text{in } Q_{2^{m_0}}, \\ u = 0 & \text{on } \partial B_{2^{m_0}}(0) \times (0, 2^{m_0}), \\ u(x', 0) = 0 & \text{in } \overline{B}_{2^{m_0}}(0), \end{cases} \tag{5}$$

where f is in parabolic grand Herz spaces with variable exponent.

Let

$$\Gamma(x, y) = \begin{cases} (4\pi t)^{-\frac{n-1}{2}} (D(x))^{-1/2} \exp\left[-\frac{1}{4t} A^{ij}(x)\, y_i y_j\right], & t > 0, \\ 0, & t \leq 0, \end{cases}$$

where $D(x)$ is the determinant of the matrix $\{a_{ij}(x)\}$ of the coefficients of L and the $A^{ij}(x)$'s are the elements of the inverse of the same matrix. We also set

$$\Gamma_i(x, y) = \frac{\partial \Gamma(x, y)}{\partial y_i}, \qquad \Gamma_{ij}(x, y) = \frac{\partial^2 \Gamma(x, y)}{\partial y_i\, \partial y_j},$$

$i, j = 1, \ldots, n$, $A = \mathbb{R}^n \cap \{t > 0\}$, and

$$\mathcal{C}_t = \{u \in C_0^\infty(A) : \ u(x', 0) = 0\},$$

where $C_0^\infty(A)$ denotes the space of infinitely differentiable functions with compact support on A.

Now, we can state the interior representation formula established in [4].

Lemma 6. *Let $u \in \mathcal{C}_t$. Then for x in the support of u, the following holds*

$$\begin{aligned} u_{x_i x_j}(x) \;\; &= \;\; \text{p.v.}\int_{\mathbb{R}^n} \Gamma_{ij}(x, x - y) \left\{ \sum_{h,k=1}^{n-1} \left[a_{hk}(y) - a_{hk}(x)\right] u_{y_h y_k}(y) + Lu(y) \right\} dy \\ &\quad + Lu(x) \int_{E_n} \Gamma_i(x, y)\, n_i\, d\sigma(y), \end{aligned}$$

where n_i is the i-th component of the outer normal to the surface E_n.

Now, let η_{ij} be the VMO modulus of a_{ij} and $\eta(r) = \left(\sum_{i,j=1}^{n-1} \eta_{ij}^2(r)\right)^{1/2}$. We also set

$$\max_{i,j=1,\dots,n-1} \max_{|u|\le 2n} \left\| \left(\frac{\partial}{\partial y_i}\right)^u \Gamma_{ij}(x,y) \right\|_{L^\infty(Q_{2^{m_0}}\times E_n)} \equiv M.$$

On the regularity of the solutions to (5), we have

Theorem 3. *Let m_0 and L be as above. Suppose $1 \le p < \infty$, $q(\cdot) \in \mathcal{B}(\mathbb{R}^n)$ and $-\lambda\delta_1 < \alpha < \lambda\delta_2$. Then there exist positive numbers C and $m_1 = m_1(C,\eta)$ such that for any $m < m_1$ and $u \in \mathcal{C}_t$, we have*

$$\left\|u_{x_ix_j}\right\|_{\dot K_{q(\cdot)}^{\alpha,p),\theta}\left(B_{2^m}^+(0)\right)} \le C\left\|Lu\right\|_{\dot K_{q(\cdot)}^{\alpha,p),\theta}\left(B_{2^m}^+(0)\right)}, \qquad i,j=1,\dots,n-1 \tag{6}$$

and

$$\|u_t\|_{\dot K_{q(\cdot)}^{\alpha,p),\theta}\left(B_{2^m}^+(0)\right)} \le C\|Lu\|_{\dot K_{q(\cdot)}^{\alpha,p),\theta}\left(B_{2^m}^+(0)\right)}, \tag{7}$$

where $B_{2^m}^+(0) = B_{2^m}(0) \cap \{t \ge 0\}$.

Proof. Similar to the method in [6], we only need to prove (6), because (7) is easily deduced from (6) by

$$u_t = Lu + \sum_{i,j=1}^{n-1} a_{ij}(x)\, u_{x_ix_j}$$

and the fact that $a_{ij} \in L^\infty(\mathbb{R}^n)$ since they satisfy uniformly elliptic condition (4). By the definition of $\Gamma_i(x,y)$ and the uniformly elliptic condition, we can obtain that

$$\left\| \int_{E_n} \Gamma_i(x,y)\, n_i\, d\sigma(y) \right\|_{L^\infty(Q_{2^{m_0}})} \le C(n,\lambda).$$

On the other hand, it is easy to verify that $\Gamma_{ij}(x,y)$ is a variable PCZ kernel on $B_{2^{m_0}}(0)$. Thus by Lemma 6, Corollaries 2 and 3, we can prove that for any $\nu > 0$, there exists $m_1 = m_1(\nu,\eta)$ such that for any $m < m_1$,

$$\|u_{x_ix_j}\|_{\dot K_{q(\cdot)}^{\alpha,p),\theta}\left(B_{2^m}^+(0)\right)} \le C_1 \sum_{h,k=1}^{n-1} \nu\, \|u_{y_hy_k}\|_{\dot K_{q(\cdot)}^{\alpha,p),\theta}\left(B_{2^m}^+(0)\right)} + C_1\|Lu\|_{\dot K_{q(\cdot)}^{\alpha,p),\theta}\left(B_{2^m}^+(0)\right)},$$

where C_1 is independent of ν and m. Choosing $\nu = \frac{1}{4(n-1)}$, we obtain that

$$\sum_{h,k=1}^{n-1} \|u_{x_ix_j}\|_{\dot K_{q(\cdot)}^{\alpha,p),\theta}\left(B_{2^m}^+(0)\right)} \le C\|Lu\|_{\dot K_{q(\cdot)}^{\alpha,p),\theta}\left(B_{2^m}^+(0)\right)}.$$

This finishes the proof of Theorem 3.

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