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On Traces of Differential Operators with Unbounded Operator Coefficients

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Abstract. In this paper we present a review of some results on the calculation of regularized traces of differential operators. We mainly discuss the calculation techniques for differential operators with unbounded operator coefficients relying on our two works: for operator Sturm-Liouville equation and operator Bessel equation.

Key Words and Phrases: spectrum, operator differential equation, regularized trace.

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1. Introduction

The purpose of this paper is to give a comprehensive insight into the studies of regularized traces with unbounded operator coefficients, based on our two previous works [1, 2].

In [1, 2], we considered eigenvalue problems for Sturm–Liouville and Bessel operator equations with unbounded operator coefficients. The methods introduced therein to estimate regularized traces have been applied in a series of our later works.

The study of regularized traces of differential operators with discrete spectrum was initiated by I. Gelfand and B.M. Levitan in their famous work [3]. It should be noted that for operators with continuous spectrum, this question was first investigated by I.M. Lifshits [4], who applied the theory to compute the defect of energy of the crystal lattice after damage.

Regularized traces of differential operators with discrete spectrum were first applied by L.A. Dikiĭ [5], and later by other authors, to the computation of the first eigenvalues.

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For differential operators, the asymptotic behavior of eigenvalues is usually known. In quantum mechanics, however, in addition to the asymptotic behaviour of eigenvalues at infinity, the ground state of particles, corresponding to the first eigenvalues, is also of great importance.

In [3], the authors considered the next problem:

$$-y''(x) + q(x)y(x) = \lambda y(x), \tag{1}$$

$$y'(0) + h\,y(0) = 0, \tag{2}$$

$$y'(\pi) - H\,y(\pi) = 0, \tag{3}$$

where $q(x)$ is a continuous potential in $[0, \pi]$, and h, H are constants.

It is known that, under the assumption $q(x) \in C^1[0, \pi]$, the eigenvalues of operator L corresponding to problem (1)-(3) have the next asymptotic expansion

$$\lambda_n = n^2 + c + \mathcal{O}\left(\frac{1}{n^2}\right), \quad (n \to \infty). \tag{4}$$

Due to that relation the series $\sum_{n=1}^{\infty}(\lambda_n - n^2 - c)$ converges. The authors called it the *regularized trace* of the operator L and obtained the value of its sum as:

$$\sum_{n=1}^{\infty}(\lambda_n - n^2 - c) = -\frac{q(0) + q(\pi)}{4}, \tag{5}$$

which expresses it in terms of potential at endpoints 0 and π. This problem raised significant interest, attracting the attention of numerous researchers. A notable contribution to investigation of problem is due to Dikli [6], who investigated the Sturm–Liouville equation subject to Dirichlet boundary conditions.

In his work, the regularized trace was interpreted as the infinite sum of the differences between the eigenvalues of the perturbed operator (corresponding to $q(x) \in L^2(0, \pi)$) and those of the unperturbed operator (corresponding to $q(t) \equiv 0$). The main idea of this approach was applied for defining regularized traces of differential operators with operator coefficients [7], as well as to study this problem for abstract operators with given behavior of eigenvalues. In the latter cases, regularizations were given under different assumptions on the relative boundedness of the perturbation operator with respect to the principle part, [8]. In [9], the authors interpreted regularized traces as a generalization of the trace concept for trace-class operators. Recall that a trace-class operators σ_1 are compact operators A for which

$$\sum_{n=1}^{\infty}\langle A\varphi_n, \varphi_n\rangle,$$

converges independently of the choice of the orthonormal basis φ_n and coincides with the spectral trace, which is simply the sum of eigenvalues:

$$\sum_{n=1}^{\infty}(A\varphi_n, \varphi_n) = \sum_{n=1}^{\infty}\lambda_n. \tag{6}$$

Obviously, for unbounded operators, such infinite sums are divergent, and there are known different regularization techniques of such sums.

There is a huge amount of works on that theme for differential operators operators as well as for abstract operators. We will classify the studies of that question for operators with discrete spectrum as traces of

1. Ordinary differential operators;
2. Partial differential operators;
3. Differential operators with bounded operator coefficients;
4. Differential operators with unbounded operator coefficients;
5. Abstract operators with a prescribed distribution of eigenvalues.

Our studies correspond to the fourth category in this classification.

For differential operators with unbounded operator coefficients, usually only principal parts of asymptotics is known and there is not any remainder term whose infinite sum is convergent. That is, the main two questions that arise in this connection are:

1. How to define regularized trace?
2. How to evaluate it (find sum)?

The first of these questions was resolved in [7] with the help of perturbation theory and one relationship estimating difference of successive eigenvalues from below. In that work, the authors considered the Sturm-Liouville operator in $H_1 = L_2(0, \pi; H)$ generated by expression

$$ly \equiv -y'' + Ay + Q(t)\, y, \tag{7}$$

$$l_0 y \equiv -y'' + Ay, \tag{8}$$

and the Dirichlet boundary conditions:

$$y(0) = y(\pi) = 0. \tag{9}$$

where $A = A^* > I$, I is an identity operator in H, $A^{-1} \in \sigma_\infty$, $Q(t) = Q^*(t)$ is a bounded operator in H, for each $t \in [0, \pi]$. Corresponding to problems (7), (9) and (8), (9), operators are denoted by L and L_0, and their eigenvalues by $\lambda_1, \lambda_2, ...$ and $\mu_1, \mu_2, ...$ respectively. Eigenvalues and eigenvectors of A are denoted by $\gamma_1 \leq \gamma_2 \leq ...$ and $\varphi_1, \varphi_2, ...$ We will keep that notation throughout the text.

They proved the following lemma which played a crucial role in definition of regularized trace

Lemma 1. *If $\gamma_i \sim a \cdot i^\alpha$, when $i \to \infty$, $a > 0$, $\alpha > 2$, then there exists a subsequence $\mu_{n_1} < \mu_{n_2} < \ldots$ for which holds:*

$$\mu_k - \mu_{n_m} \geq d_0 \left(k^{\frac{2\alpha}{2+\alpha}} - n_m^{\frac{2\alpha}{2+\alpha}} \right), \quad k = n_m,\ n_m + 1, ... \tag{10}$$

In virtue of that lemma and the next relation of the perturbation theory

$$R_\lambda - R_\lambda^0 + \sum_{j=1}^{N} (-1)^j (Q R_\lambda^0)^j + R_\lambda (Q R_\lambda^0)^{N+1} = 0, \tag{11}$$

the authors proved that

$$\lim_{m\to\infty} \sum_{n=1}^{n_m} (\lambda_n - \mu_n) = \lim_{m\to\infty} \sum_{n=1}^{n_m} (Q\psi_n, \psi_n)_{H_1}, \tag{12}$$

where $\psi_n = \sqrt{\frac{2}{\pi}} \sin(k_n t) \varphi_{i_n}, \quad k_n, i_n \in \mathbb{N}$, are orthonormal eigenvectors of operator L_0 and a scalar product $(Q\psi_n, \psi_n)$ is in $L_2(0, \pi; H)$

Using this result in (12), the authors proved that the limit on the right-hand side of (12) is equal to

$$\frac{1}{\pi} \sum_{k=1}^{\infty} \sum_{i=1}^{\infty} \left(\int_0^\pi (Q(t)\varphi_i, \varphi_i) \cos kt \, dt \right), \tag{13}$$

which is the Fourier expansion of the scalar function $f_i(t) = (Q(t)\varphi_i, \varphi_i)$ at zero and π and is equal to

$$\sum_{i=1}^{\infty} \frac{(Q(0)\varphi_i, \varphi_i) + (Q(\pi)\varphi_i, \varphi_i)}{4}. \tag{14}$$

The latter expression is equal to $\frac{trQ(0) + trQ(\pi)}{4}$, provided that $Q(0)$ and $Q(\pi) \in \sigma_1$.

2. Trace of operator Sturm-Liouville equation

In our work [1], we considered the same problem as in [7], but with a different boundary condition at π:

$$y'(\pi) + hy(\pi) = 0 \tag{15}$$

The characteristic equation in our case is given by

$$z\cos(z\pi) + h\sin(z\pi) = 0. \tag{16}$$

The roots of this equation satisfy the following asymptotic relation as $k \to \infty$

$$\alpha_k = \pi k + \frac{1}{2} + O\left(\frac{1}{k}\right). \tag{17}$$

The eigenvalues of the unperturbed operator L_0 corresponding to these roots are $\gamma_i + \alpha_k^2$, and orthogonal eigenvectors are $\sin(\alpha_k t)\varphi_i$. After normalizing the eigenvectors, we obtain

$$\psi_n(t) = \sqrt{\frac{4\alpha_{k_n}}{2\alpha_{k_n}\pi - \sin(2\alpha_{k_n}\pi)}}\sin(\alpha_{k_n}t)\varphi_{i_n}. \tag{18}$$

By virtue of the eigenvalue asymptotics

$$\lambda_n \sim cn^{\frac{2\alpha}{2+\alpha}}, \quad n \to \infty \tag{19}$$

the above statement of Lemma 1 from [7] holds in our case too. Thus

$$\lim_{m\to\infty}\sum_{n=1}^{n_m}(\lambda_n - \mu_n) = \lim_{m\to\infty}\sum_{n=1}^{n_m}(Q\psi_n, \psi_n)_{H_1} =$$

$$= \sum_{i=1}^{\infty}\sum_{k=1}^{\infty}\int_0^{\pi}\frac{2\,\alpha_k}{2\alpha_k\pi - \sin(2\alpha_k\pi)}\cos(2\alpha_k t)(Q(t)\varphi_i, \varphi_i)_H dt. \tag{20}$$

The convergence of the double series is justified due to asymptotics of α_k, as $k \to \infty$, and requirement on $Q(t)$ to be twice differentiable in weak sense, that is due to differentiability of scalar function $f_i(t) = (Q(t)g, g)$, where $g \in H$.

In the present work, the main issue is finding the sum of the double series, in contrast to [7], where the primary goal was to define the regularized trace. The series (13) is the Fourier expansions along $cos(nt)$ of $f_i(t)$ at endpoints of the interval. But in our case, it is necessary to determine the sum of the series in (20).

There we offer the next method which includes the use of Cauchy theorem about residues. Namely, for partial sums:

$$L_N(t) \equiv \sum_{k=1}^{N} \frac{2\alpha_k}{2\alpha_k\pi - \sin 2\alpha_k\pi} \cos 2\alpha_k x, \tag{21}$$

we construct the following function of the complex variable z:

$$g(z) = \frac{z\cos(2zx)}{\sin^2(z\pi)\,(zctg(z\pi)+h)}, \tag{22}$$

where

$$\Delta(z) \equiv \sin(z\pi)(2ctg(z\pi)+h) = z\cos z\pi + h\sin z\pi,$$

is, in fact, a characteristic determinant of operator L_0.

The denominator of $g(z)$ is the product of $\Delta(z)$ and $\sin(z\pi)$, thus, $g(z)$ has poles at their zeros. It is easy to verify that residues at α_k are terms of (21) taken with negative sign and resideus at k (zeros of $\sin(z\pi)$) are $\frac{\cos(2kt)}{\pi}$.

As the contour of integration, we choose the rectangle with vertices at $\pm iB$, $A_N \pm iB$, where B tends to infinity and $A_N = N + \frac{1}{4}$ we have

$$\alpha_N < A_N < \alpha_{N+1}$$

$$N < A_N < N+1$$

Since the function $g(z)$ is odd, the integral along the left sides of the rectangle vanishes. If $z = u + iv$, then for sufficiently large $|v|$ and $u \geq 0$ the order of the integrand is

$$O(e^{(2t-2\pi)|v|}),$$

thus, the integrals along upper and lower sides of the integral vanish as $B \to \infty$.

Thus, we obtain the formula

$$\frac{1}{2\pi i} \lim_{B\to\infty} \int_{A_N-iB}^{A_N+iB} \frac{z\cos(2zt)dz}{\sin^2(z\pi)\,(2ctg(z\pi)+h)} = K_N(t) - L_N(t), \tag{23}$$

where $K_N(t) = \sum_{k=1}^{N} \frac{cos2kt}{\pi}$.

We proved there

$$\lim_{N\to\infty} \frac{1}{2\pi i} \lim_{B\to\infty} \int_0^{\pi} \int_{A_N-iB}^{A_N+iB} \frac{zcos(2zt)f_i(t)dzdt}{sin(2z\pi)(zctgz\pi+h)} = \frac{f_i(\pi)}{2} \tag{2.10}$$

which together with (23), (13), (14) yields

$$\lim_{m\to\infty} \sum_{n=1}^{n_m} (\lambda_n - \mu_n) = \frac{trQ(0) - trQ(1)}{4}.$$

This method of calculation has subsequently been applied in a series of our works, for example, in [11, 12], etc., where we studied second- and fourth-order differential operators with unbounded operator coefficients and with various involvement of spectral parameter in the boundary condition, see [10, 11, 12, 13, 14].

3. Trace of operator Bessel equation

In [2], we considered in $L_2(0,1;H)$ the problem

$$l[y] = -y''(t) + \frac{\nu^2 - \frac{1}{4}}{t^2} y(t) + Ay(t) + q(t)y(t) = \lambda y(t), \quad \nu \geq 1, \tag{24}$$

$$y'(1) - \lambda y(1) = 0, \tag{25}$$

where we investigated the asymptotics of eigenvalue distribution and the regularized trace of corresponding to problem operator L defined in the direct sum space $L_2 = L_2(0,1;H) \oplus H$. The pperator corresponding to the unperturbed case $q(t) \equiv 0$ is denoted, as in previous section, by L_0. Requirements on operator A are the same as above. In L_2 the inner product of elements $Y = \{y(t), y_1\}, Z = (z(t), z_1)$ is defined by:

$$(Y, Z)_{L_2} = \int_0^1 (y(t), z(t))\, dt + (y_1, z_1). \tag{26}$$

Eigenvalues of operator L_0 are defined from the characteristic equation:

$$\frac{1}{2} J_\nu(\sqrt{\lambda - \gamma_k}) + \sqrt{\lambda - \gamma_k}\, J'_\nu(\sqrt{\lambda - \gamma_k}) - \lambda J_\nu(\sqrt{\lambda - \gamma_k}) = 0, \tag{27}$$

where γ_k-s are eigenvalues of operator A, $J_\nu(z)$ is a Bessel's function.

By setting $z = \sqrt{\lambda - \gamma_k}$ equation (27) can be rewritten in the form

$$\Delta(z) \equiv z J'_\nu(z) + \left(\tfrac{1}{2} - z^2 - \gamma_k\right) J_\nu(z) = 0. \tag{28}$$

$\Delta(z)$ is the characteristic determinant of L_0. $\Delta(z)$ has series representation and methods of differential calculus used in cases when $\Delta(z)$ is an elementary function are not applicable to study of roots of (28).

We proved that equation (28) possesses only real and purely imaginary roots.

The eigenvalues corresponding to these roots are $\gamma_k + z^2$. Imaginary roots we looked for in form $z = 2i\sqrt{y}, y > 0$. Substituting this expression into the series representation of (28) yields

$$\sum_{n=0}^{\infty} \frac{y^n}{n!} \frac{(4n+2)(\nu+n) - \left(\gamma_k + \nu - \frac{1}{2}\right)}{\Gamma(\nu+n)(\nu+n)} = 0. \tag{29}$$

By Descartes' rule of signs, it was justified that (29) starting with some k has exactly one positive root y, which corresponds to an imaginary root of (28).

Thus, the following lemma is proved.

Lemma 2. *The eigenvalues of L_0 form two sequences behaving as*

$$\lambda_k \sim \frac{2\nu^2 + \frac{7}{2}}{4}\sqrt{\gamma_k},$$

$$\lambda_{m,k} = \gamma_k + z_{m,k}^2, \quad z_{m,k} \sim \left(\pi m + \tfrac{\nu\pi}{2} - \tfrac{\pi}{4}\right), \quad m \to \infty$$

Arguments used in proof of that lemma were successfully applied in our further works repeatedly.

Unlike the situation considered in [1], the choice of the contour of integration must also take into account the imaginary roots of equation (28). To this end, we choose the rectangular contour C with vertices at

$$\pm iB, \quad \pm iB + \pi m + \tfrac{1}{2}\nu\pi + \tfrac{\pi}{4} = \pm iB + A_m,$$

which bypasses imaginary root of (28) is denoted by $-ix_{0,k}$ along small semicircles on the right of the imaginary axis and $ix_{0,k}$ on the left of it.

The following important statement, which determines the number of roots of equation (28) enclosed by the contour C is proved.

Lemma 3. *For a sufficiently large integer m the function*

$$z^{-\nu}\big(zJ'_\nu(z) + \big(\tfrac{1}{2} - z^2 + \gamma_k\big)J_\nu(z)\big)$$

has exactly m zeros between the imaginary axis and the line

$$Rez = \pi m + \tfrac{\nu\pi}{2} + \tfrac{\pi}{4}.$$

Denoting eigenvalues of L by μ_n, we have

Theorem 1. *If $\gamma_n \sim an^\alpha$ with $0 < a$ and $\alpha = const$, then*

$$\lambda_n(L_0) \sim \mu_n(L) \sim dn^\delta,$$

$$\delta = \begin{cases} \dfrac{2\alpha}{\alpha+2}, & \text{for } \alpha > 2, \\ \dfrac{\alpha}{2}, & \text{for } \alpha < 2, \\ 1, & \text{for } \alpha = 2. \end{cases}$$

In virtue of that theorem the following relation is valid:

$$\lim_{m\to\infty}\sum_{n=1}^{nm}\big(\mu_n-\lambda_n-(Q\psi_n,\psi_n)_{H_1}\big)=0, \tag{30}$$

where ψ_n are orthonormal eigenvectors of L_0. After normalizing orthogonal eigenvectors we found them in the following form:

$$\psi_{m,k}=\frac{1}{J_\nu(x_{m,k})}\times$$
$$\sqrt{\frac{2x_{m,k}^2}{x_{m,k}^4+2x_{m,k}^2\gamma_k+2x_{m,k}^2+\gamma_k^2-\gamma_k-\nu^2+\frac14}}\Big\{\sqrt{t}\,J_\nu(x_{m,k}t),\ J_\nu(x_{m,k})\Big\}\varphi_k, \tag{31}$$

where $x_{m,k}$ are roots of (28).

From (30) and (31) the next relationship was established:

$$\lim_{m\to\infty}\sum_{n=1}^{n_m}(\mu_n-\lambda_n)=$$
$$=\sum_{k=1}^{N-1}\sum_{m=1}^{\infty}\int_0^1\frac{2x_{m,k}^2tJ_\nu^2(x_{m,k}t)q_k(t)dt}{J_\nu^2(x_{m,k})(x_{m,k}^4+2x_{m,k}^2+2\gamma_kx_{m,k}^2+\gamma_k^2-\gamma_k-\nu^2+\frac14)}+$$
$$+\sum_{k=N}^{\infty}\sum_{m=0}^{\infty}\int_0^1\frac{2tJ_\nu^2(x_{m,k}t)q_k(t)dt}{J_\nu^2(x_{m,k})(x_{m,k}^4+2x_{m,k}^2+2\gamma_kx_{m,k}^2+\gamma_k^2-\gamma_k-\nu^2+\frac14)} \tag{32}$$

To find its exact value one has to find the sum of double series on the right of (32).

To do that we successfully apply as in [1] the Cauchy theorem about residues. So, to study the asymptotic behavior as $N\to\infty$ of the function

$$R_N(t)=\sum_{m=0}^{N-1}\frac{2tx_{m,k}^2I_\nu^2(x_{m,k}t)}{J_\nu^2(x_{m,k})(x_{m,k}^4+2x_{m,k}^2+2x_{m,k}^2+2\gamma_kx_{m,k}^2+\gamma_k^2-\gamma_k-\nu^2+\frac14)}, \tag{33}$$

we constructed the function

$$g(z)=\frac{2tzJ_\nu^2(tz)}{J_\nu(z)\Big(z^2J_\nu'(z)+\big(\frac12-z^2-\gamma_k\big)J_\nu(z)\Big)}, \tag{34}$$

having poles at $x_{m,k}$ and $j_n(J_\nu(j_n))=0, n=\overline{1,\infty}$ residues of which at that poles equal to terms of (33).

As the contour of integration, we chose the contour C described above. It should be noted that existence of imaginary roots of (28) was solved by making left-hand side of contour by pass $ix_{m,k}, (x_{m,k} > 0)$ on left and $-ix_{m,k}, (x_{m,k} < 0)$ on right side. Due to that choice and oddness of $g(z)$, the integral along left hand side of contour of integration vanishes.

In virtue of asymptotics of $g(z)$ for large in modulus z values, also as $z \to 0$, it was established that

$$\lim_{m\to\infty} \sum_{n=1}^{n_m} (\lambda_n - \mu_n) = \sum_{k=1}^{\infty} \lim_{N\to\infty} \int_0^1 q_k(t) R_N(t)\, dt = \tag{35}$$

$$= -\sum_{k=1}^{\infty} \frac{2\nu q_k(0) + q_k(1)}{4} = -\frac{2\nu trq(0) + trq(1)}{4}.$$

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