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Commutator of Sharp Maximal Operator with Lipschitz functions on Orlicz Spaces for the Dunkl Operator on the Real Line

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Abstract. On the real line, the Dunkl operators

$$D_\nu(f)(x) := \frac{df(x)}{dx} + (2\nu+1)\frac{f(x)-f(-x)}{2x}, \quad \forall\, x \in \mathbb{R},\ \forall\, \nu \geq -1/2$$

are differential-difference operators associated with the reflection group $\mathbb{Z}_2$ on $\mathbb{R}$. In this paper, in the setting $\mathbb{R}$ we establish necessary and sufficient conditions for the boundedness of the commutators of the sharp maximal operator $[b, M_\nu^\sharp]$ in Orlicz spaces $L^\Phi(\mathbb{R}, dm_\nu)$ for the Dunkl operator on the real line when b belongs to Lipschitz spaces $\dot{\Lambda}_\beta(\mathbb{R})$. We obtain some new characterizations for certain subclasses of Lipschitz spaces $\dot{\Lambda}_\beta(\mathbb{R})$.

Key Words and Phrases: Dunkl operator, Orlicz space, maximal function, sharp maximal function, commutator, Lipschitz function.

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1. Introduction

In [13], Dunkl introduced a family of first order differential-difference operators that play the role of the usual partial differentiation for the reflection group structure. For a real parameter $\nu \geq -1/2$, we consider the *Dunkl operator*, associated with the reflection group $\mathbb{Z}_2$ on $\mathbb{R}$:

$$D_\nu(f)(x) := \frac{df(x)}{dx} + (2\nu+1)\frac{f(x)-f(-x)}{2x}, \quad \forall\, x \in \mathbb{R}.$$

Note that $D_{-1/2} = d/dx$.

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Let $f \in L^1_{\rm loc}(\mathbb{R})$ and $0 \le \alpha < 2\nu+2$. The fractional maximal function $M_{\alpha,\nu} f$ is defined by

$$M_{\alpha,\nu} f(x) = \sup_{B \ni x} m_\nu(B)^{-1+\frac{\alpha}{2\nu+2}} \int_B |f(y)| \, dm_\nu(y),$$

and the sharp maximal function of Fefferman and Stein $M^\sharp_\nu f$ is defined by

$$M^\sharp_\nu f(x) = \sup_{B \ni x} m_\nu(B)^{-1} \int_B |f(y) - f_B| \, dm_\nu(y),$$

where the supremum is taken over all balls $B \subset \mathbb{R}$ containing x. When $\alpha = 0$, we simply denote by $M = M_0$.

The maximal commutator generated by $b \in L^1_{\rm loc}(\mathbb{R}, dm_\nu)$ and M_ν is defined by

$$M_{b,\nu}(f)(x) = \sup_{B \ni x} m_\nu(B)^{-1} \int_B |b(x) - b(y)||f(y)| \, dm_\nu(y).$$

The commutators generated by $b \in L^1_{\rm loc}(\mathbb{R}, dm_\nu)$ and $M^\sharp_\nu$ is defined by

$$[b, M^\sharp_\nu] f(x) = b(x) M^\sharp_\nu f(x) - M^\sharp_\nu(bf)(x).$$

In 1978, Janson [16] gave some characterizations of the Lipschitz space $\dot{\Lambda}_\beta(\mathbb{R}^n)$ via commutator $[b, T]$ and the author proved that $b \in \dot{\Lambda}_\beta(\mathbb{R}^n)$ if and only if $[b, T]$ is bounded from $L^p(\mathbb{R}^n)$ to $L^q(\mathbb{R}^n)$, where $1 < p < n/\beta$, $1/p - 1/q = \beta/n$ and T is the classical singular integral operator (see also [24]). The mapping properties of M_b, $[b, M]$ and $[b, M^\sharp]$ have been studied extensively by many authors. See, for instance, [4, 7, 8, 19, 20, 21, 28]. The operator M_b plays an important role in the study of commutators of singular integral operators with BMO symbols (see, for instance, [1, 2, 3, 7, 10, 14, 26]). The operators M, $[b, M]$, M_b and $[b, M^\sharp]$ play an important role in real and harmonic analysis and applications (see, for example [5, 6, 22, 23, 27, 28, 29]). Note that, the boundedness of the operator M_b on $L^p(\mathbb{R}^n)$ spaces was proved by Garcia-Cuerva et al. [7]. In [4] by Bastero et al. studied the necessary and sufficient condition for the boundedness of $[b, M]$ on $L^p(\mathbb{R}^n)$ spaces.

The aim of this paper is to study the commutators of the sharp maximal operator $[b, M^\sharp_\nu]$ in the Orlicz spaces $L^\Phi(\mathbb{R}, dm_\nu)$ for the Dunkl operator on the real line.

Inspired by the above literature, our main aim is to characterize the commutator functions b, involved in the boundedness on Orlicz spaces of the sharp maximal operator $[b, M^\sharp_\nu]$ (Theorem 4), see [12].

The notation $A \lesssim B$ means that means that there exists a positive constant C, independent of the relevant quantities, such that $A \le CB$. If $A \lesssim B$ and $B \lesssim A$, we write $A \approx B$ and say that A and B are equivalent.

2. Preliminaries in the Dunkl setting on $\mathbb{R}$

Let $\nu > -1/2$ be a fixed number and m_ν be the *weighted Lebesgue measure* on $\mathbb{R}$ given by

$$dm_\nu(x) := \left(2^{\nu+1}\Gamma(\nu+1)\right)^{-1} |x|^{2\nu+1}\, dx, \quad \forall\, x \in \mathbb{R}.$$

For any $x \in \mathbb{R}$ and $r > 0$, let $B(x,r) := \{y \in \mathbb{R} : |y| \in]\max\{0, |x| - r\}, |x| + r[\ \}$. Then $B(0,r) =]-r, r[$ and

$$m_\nu(B(0,r)) = c_\nu\, r^{2\nu+2},$$

where $c_\nu := \left[2^{\nu+1}\,(\nu+1)\,\Gamma(\nu+1)\right]^{-1}$.

To introduce the notion of Orlicz spaces in the Dunkl setting on $\mathbb{R}$, we first recall the definition of Young functions.

Definition 1. A function $\Phi : [0,\infty) \to [0,\infty]$ is called a *Young function* if Φ is convex, left-continuous, $\lim_{r\to+0}\Phi(r) = \Phi(0) = 0$ and $\lim_{r\to\infty}\Phi(r) = \infty$.

From the convexity and $\Phi(0) = 0$ it follows that any Young function is increasing. If there exists $s \in (0,\infty)$ such that $\Phi(s) = \infty$, then $\Phi(r) = \infty$ for all $r \geq s$. The set of Young functions such that

$$0 < \Phi(r) < \infty \qquad \text{for all} \qquad 0 < r < \infty$$

is denoted by $\mathcal{Y}$. If $\Phi \in \mathcal{Y}$, then Φ is absolutely continuous on every closed interval in $[0,\infty)$ and bijective from $[0,\infty)$ to itself.

For a Young function Φ and $0 \leq s \leq \infty$, let

$$\Phi^{-1}(s) := \inf\{r \geq 0 : \Phi(r) > s\}.$$

If $\Phi \in \mathcal{Y}$, then Φ^{-1} is the usual inverse function of Φ. It is well known that

$$r \leq \Phi^{-1}(r)\widetilde{\Phi}^{-1}(r) \leq 2r \qquad \text{for any } r \geq 0, \tag{1}$$

where $\widetilde{\Phi}(r)$ is defined by

$$\widetilde{\Phi}(r) := \begin{cases} \sup\{rs - \Phi(s) : s \in [0,\infty)\}, & r \in [0,\infty) \\ \infty, & r = \infty. \end{cases}$$

A Young function Φ is said to satisfy the Δ_2-*condition*, denoted also as $\Phi \in \Delta_2$, if

$$\Phi(2r) \leq C\,\Phi(r), \qquad \forall\, r > 0$$

for some $C > 1$. If $\Phi \in \Delta_2$, then $\Phi \in \mathcal{Y}$. A Young function Φ is said to satisfy the ∇_2-*condition*, denoted also by $\Phi \in \nabla_2$, if

$$\Phi(r) \leq \frac{1}{2C}\Phi(Cr), \qquad \forall\, r \geq 0$$

for some $C > 1$. In what follows, for any subset E of $\mathbb{R}$, we use χ_E to denote its *characteristic function*.

Definition 2. (Orlicz Space). For a Young function Φ, the set

$$L^{\Phi}(\mathbb{R}, dm_\nu) := \left\{ f \in L^1_{\text{loc}}(\mathbb{R}, dm_\nu) : \int_{\mathbb{R}} \Phi(k|f(x)|)\, dm_\nu(x) < \infty \text{ for some } k > 0 \right\}$$

is called the *Orlicz space*. If $\Phi(r) := r^p$ for all $r \in [0, \infty)$, $1 \leq p < \infty$, then $L^{\Phi}(\mathbb{R}, dm_\nu) = L^p(\mathbb{R}, dm_\nu)$. If $\Phi(r) := 0$ for all $r \in [0, 1]$ and $\Phi(r) := \infty$ for all $r \in (1, \infty)$, then $L^{\Phi}(\mathbb{R}, dm_\nu) = L^{\infty}(\mathbb{R}, dm_\nu)$. The *space* $L^{\Phi}_{\text{loc}}(\mathbb{R}, dm_\nu)$ is defined as the set of all functions f such that $f\chi_B \in L^{\Phi}(\mathbb{R}, dm_\nu)$ for all balls $B \subset \mathbb{R}$.

$L^{\Phi}(\mathbb{R}, dm_\nu)$ is a Banach space with respect to the norm

$$\|f\|_{L^{\Phi,\nu}} := \inf \left\{ \lambda > 0 : \int_{\mathbb{R}} \Phi\Big(\frac{|f(x)|}{\lambda}\Big)\, dm_\nu(x) \leq 1 \right\}.$$

For a measurable function f on $\mathbb{R}$ and $t > 0$, let

$$m(f, t)_\nu := m_\nu(\{x \in \mathbb{R} : |f(x)| > t\}).$$

Definition 3. The *weak Orlicz space*

$$WL^{\Phi}(\mathbb{R}, dm_\nu) := \{f \in L^{1,\nu}_{\text{loc}}(\mathbb{R}) : \|f\|_{WL^{\Phi,\nu}} < \infty\}$$

is defined by the norm

$$\|f\|_{WL^{\Phi,\nu}} := \inf \Big\{ \lambda > 0 \ : \ \sup_{t>0} \Phi(t)\, m\Big(\frac{f}{\lambda},\ t\Big)_\nu \ \leq 1 \Big\}.$$

We note that $\|f\|_{WL^{\Phi,\nu}} \leq \|f\|_{L^{\Phi,\nu}}$,

$$\sup_{t>0} \Phi(t)\, m(f, t)_\nu = \sup_{t>0} t\, m(f, \Phi^{-1}(t))_\nu = \sup_{t>0} t\, m(\Phi(|f|), t)_\nu$$

and

$$\int_{\mathbb{R}} \Phi\Big(\frac{|f(x)|}{\|f\|_{L^{\Phi,\nu}}}\Big)\, dm_\nu(x) \leq 1, \qquad \sup_{t>0} \Phi(t) m\Big(\frac{f}{\|f\|_{WL^{\Phi,\nu}}},\ t\Big)_\nu \leq 1. \tag{2}$$

The following analogue of the Hölder inequality is well known (see, for example, [25]).

Theorem 1. *Let the functions f and g be measurable on $\mathbb{R}$. For a Young function Φ and its complementary function $\widetilde{\Phi}$, the following inequality is valid*

$$\int_{\mathbb{R}} |f(x)g(x)| \, dm_\nu(x) \le 2\|f\|_{L^{\Phi,\nu}} \|g\|_{L^{\tilde{\Phi},\nu}}.$$

By elementary calculations we have the following property.

Lemma 1. *Let Φ be a Young function and B be a ball in $\mathbb{R}$. Then*

$$\|\chi_B\|_{L^{\Phi,\nu}} = \|\chi_B\|_{WL^{\Phi}(\mathbb{R},dm_\nu)} = \frac{1}{\Phi^{-1}\left(m_\nu(B)^{-1}\right)}.$$

By Theorem 1, Lemma 1 and (1) we obtain the following estimate.

Lemma 2. *For a Young function Φ and for the ball B the following inequality is valid:*

$$\int_B |f(y)| \, dm_\nu(y) \le 2m_\nu(B) \ \Phi^{-1}\left(m_\nu(B)^{-1}\right) \|f\,\chi_B\|_{L^{\Phi,\nu}}.$$

The known boundedness statement for M_ν in Orlicz spaces on spaces of homogeneous type runs as follows.

Theorem 2. [*17*] *Let Φ be any Young function. Then the maximal operator M_ν is bounded from $L^\Phi(\mathbb{R}, dm_\nu)$ to $WL^\Phi(\mathbb{R}, dm_\nu)$ and for $\Phi \in \nabla_2$ bounded in $L^\Phi(\mathbb{R}, dm_\nu)$.*

For a given ball B and $0 \le \alpha < 2\nu + 2$, we define the following maximal function:

$$M_{\alpha,B,\nu} f(x) = \sup_{B \supseteq B' \ni x} m_\nu(B')^{-1+\frac{\alpha}{2\nu+2}} \int_{B'} |f(y)| \, dm_\nu(y),$$

where the supremum is taken over all balls B' such that $x \in B' \subseteq B$. Moreover, we denote by $M_{B,\nu} = M_{0,B,\nu}$ when $\alpha = 0$.

The following result completely characterizes the boundedness of $M_{\alpha,\nu}$ on Orlicz spaces.

Theorem 3. [*18*] *Let $0 < \alpha < 2\nu + 2$, Φ, Ψ be Young functions and $\Phi \in \mathcal{Y}$. The condition*

$$r^\alpha \Phi^{-1}\big(r^{-2\nu-2}\big) \le C\Psi^{-1}\big(r^{-2\nu-2}\big) \tag{3}$$

for all $r > 0$, where $C > 0$ does not depend on r, is necessary and sufficient for the boundedness of $M_{\alpha,\nu}$ from $L^\Phi(\mathbb{R}, dm_\nu)$ to $WL^\Psi(\mathbb{R}, dm_\nu)$. Moreover, if $\Phi \in \nabla_2$, the condition (3) is necessary and sufficient for the boundedness of $M_{\alpha,\nu}$ from $L^\Phi(\mathbb{R}, dm_\nu)$ to $L^\Psi(\mathbb{R}, dm_\nu)$.

In this section, as an application of Theorem 3 we consider the boundedness of $M_{b,\alpha}$ on Orlicz spaces when b belongs to the Lipschitz space, by which some new characterizations of the Lipschitz spaces are given.

Next we give the definition of the Lipschitz spaces on $\mathbb{R}$, and state some basic properties and useful lemmas.

Definition 4. *(Lipschitz-type spaces on $\mathbb{R}$) Let $0<\beta<1$.*
(1) *We say a function b belongs to the Lipschitz space $\dot{\Lambda}_\beta(\mathbb{R})$ if there exists a constant C such that for all $x, y \in \mathbb{R}$,*

$$|b(x)-b(y)| \le C|x-y|^\beta.$$

The smallest such constant C is called the $\dot{\Lambda}_\beta(\mathbb{R})$ norm of b and is denoted by $\|b\|_{\dot{\Lambda}_\beta(\mathbb{R})}$.

(2) [*15*] *The space $\mathrm{Lip}_\beta(\mathbb{R}, dm_\nu)$ is defined to be the set of all locally integrable functions b, i.e., there exists a positive constant C, such that*

$$\sup_B \frac{1}{m_\nu(B)^{1+\beta/(2\nu+2)}} \int_B |b(x)-b_B|\, dm_\nu(x) \le C,$$

where the supremum is taken over every ball $B \subset \mathbb{R}$ containing x and $b_{B,\nu} = \frac{1}{m_\nu(B)}\int_B b(y)\, dm_\nu(y)$. The smallest such constant C is called the $\mathrm{Lip}_\beta(\mathbb{R}, dm_\nu)$ norm of b and is denoted by $\|b\|_{\mathrm{Lip}_\beta(\mathbb{R},dm_\nu)}$.

To prove the theorems, we need auxiliary results. The first one is the following characterizations of Lipschitz space (see [15]).

Lemma 3. *Let $0<\beta<1$ and $b \in L^1_{\mathrm{loc}}(\mathbb{R}, dm_\nu)$, then*

(1)

$$\|b\|_{\dot{\Lambda}_\beta(\mathbb{R})} \approx \|b\|_{\mathrm{Lip}_\beta(\mathbb{R},dm_\nu)}.$$

(2) *Let $B_1 \subset B_2 \subset \mathbb{R}$ and $b \in \mathrm{Lip}_\beta(\mathbb{R}, dm_\nu)$, where B_1 and B_2 are balls. Then there exists a constant C depending only on B_1 and B_2 such that*

$$|b_{B_1} - b_{B_2}| \le C\, \|b\|_{\mathrm{Lip}_\beta(\mathbb{R},dm_\nu)}\, m_\nu(B_2)^{\frac{\beta}{2\nu+2}}.$$

(3) *There exists a constant C depending only on β such that*

$$|b(x)-b(y)| \le C\, \|b\|_{\mathrm{Lip}_\beta(\mathbb{R},dm_\nu)}\, m_\nu(B_2)^{\frac{\beta}{2\nu+2}}$$

holds for any ball B containing x and y.

Lemma 4. *Let $0<\beta<1$ and $b \in \dot{\Lambda}_\beta(\mathbb{R})$. Then the following pointwise estimate holds*

$$M_{b,\nu}f(x) \le C\|b\|_{\dot{\Lambda}_\beta(\mathbb{R})}\, M_{\beta,\nu}f(x).$$

Proof. If $b \in \dot{\Lambda}_\beta(\mathbb{R})$, then

$$\begin{aligned} M_{b,\nu}(f)(x) &= \sup_{B\ni x} m_\nu(B)^{-1}\int_B |b(x)-b(y)||f(y)|\, dm_\nu(y) \\ &\le C\|b\|_{\dot{\Lambda}_\beta(\mathbb{R})} \sup_{B\ni x} m_\nu(B)^{-1+\frac{\beta}{2\nu+2}}\int_B |f(y)|\, dm_\nu(y) \\ &= C\|b\|_{\dot{\Lambda}_\beta(\mathbb{R})}\, M_\beta f(x). \end{aligned}$$

◄

Lemma 5. *If $b \in L^1_{\rm loc}(\mathbb{R}, dm_\nu)$ and $B_0 := B(x_0, r_0)$, then*

$$|b(x) - b_{B_0}| \le M_{b,\nu}\chi_{B_0}(x) \quad \textit{for every} \quad x \in B_0.$$

Proof. For $x \in B_0$, we get

$$\begin{aligned} &M_{b,\nu}\chi_{B_0}(x) = \sup_{B\ni x} m_\nu(B)^{-1}\int_B |b(x)-b(y)|\chi_{B_0}(y)\, dm_\nu(y) \\ &= \sup_{B\ni x} m_\nu(B)^{-1}\int_{B\cap B_0} |b(x)-b(y)|\, dm_\nu(y) \\ &\ge m_\nu(B_0)^{-1}\int_{B_0\cap B_0} |b(x)-b(y)|\, dm_\nu(y) \\ &\ge \Big|m_\nu(B_0)^{-1}\int_{B_0} (b(x)-b(y))\, dm_\nu(y)\Big|\, |b(x)-b_{B_0}|. \end{aligned}$$

◄

3. Commutators of sharp maximal function in Orlicz Spaces

For a function b defined on $\mathbb{R}$, we denote

$$b^-(x) := \begin{cases} 0\ , & \text{if } b(x) \ge 0 \\ |b(x)|, & \text{if } b(x) < 0 \end{cases}$$

and $b^+(x) := |b(x)| - b^-(x)$. Obviously, $b^+(x) - b^-(x) = b(x)$.

The following relations between $[b, M_\nu]$ and $M_{b,\nu}$ are valid. Let b be any non-negative locally integrable function. Then for all $f \in L^1_{\rm loc}(\mathbb{R}, dm_\nu)$ and $x \in \mathbb{R}$ the following inequality is valid

$$\big|[b, M_\nu]f(x)\big| = \big|b(x)M_\nu f(x) - M_\nu(bf)(x)\big|$$

$$= \big|M_\nu(b(x)f)(x) - M_\nu(bf)(x)\big| \le M_\nu(|b(x)-b|f)(x) = M_{b,\nu}(f)(x).$$

If b is any locally integrable function on $\mathbb{R}$, then

$$|[b,M_\nu]f(x)| \le M_{b,\nu}(f)(x) + 2b^-(x)M_\nu f(x), \qquad x\in\mathbb{R} \tag{4}$$

holds for all $f \in L^1_{\rm loc}(\mathbb{R}, dm_\nu)$ (see, for example, [6, 9, 28]).

Obviously, the $M_{b,\nu}$ and $[b,M_\nu]$ operators are essentially different from each other because $M_{b,\nu}$ is positive and sublinear and $[b,M_\nu]$ is neither positive.

Theorem 4. *Let $0<\beta<1$ and b be a locally integrable function. Suppose that Φ,Ψ be Young functions, $\Phi\in\mathcal{Y}\cap\nabla_2$ and $\Psi^{-1}(t)\approx\Phi^{-1}(t)t^{-\frac{\beta}{2\nu+2}}$. Then the following statements are equivalent:*

1. *$b\in\dot{\Lambda}_\beta(\mathbb{R})$ and $b\ge 0$.*
2. *$[b,M_\nu^\sharp]$ is bounded from $L^\Phi(\mathbb{R},dm_\nu)$ to $L^\Psi(\mathbb{R},dm_\nu)$.*
3. *There exists a constant $C>0$ such that*

$$\sup_B m_\nu(B)^{-\frac{\beta}{2\nu+2}}\,\Psi^{-1}\big(m_\nu(B)^{-1}\big)\,\big\|b(\cdot)-2M_\nu^\sharp\big(b\,\chi_B\big)(\cdot)\big\|_{L^\Psi(B)} \le C. \tag{5}$$

4. *There exists a constant $C>0$ such that*

$$\sup_B m_\nu(B)^{-1-\frac{\beta}{2\nu+2}}\,\big\|b(\cdot)-2M_\nu^\sharp\big(b\,\chi_B\big)(\cdot)\big\|_{L^1(B)} \le C. \tag{6}$$

Proof. We only need to prove (1) ⇒ (2), (2) ⇒ (3), (3) ⇒ (4) and (4) ⇒ (1).

(1) ⇒ (2). Since $b\in\dot{\Lambda}_\beta(\mathbb{R})$ and $b\ge 0$, then for any locally integrable function f and a.e. $x\in\mathbb{R}$

$$\begin{aligned}
&\big|[b,M_\nu^\sharp]f(x)\big| = \Big|\sup_{B\ni x}\frac{b(x)}{m_\nu(B)}\int_B |f(y)-f_B|\,dm_\nu(y) \\
&-\sup_{B\ni x}\frac{1}{m_\nu(B)}\int_B |b(y)f(y)-(bf)_B|\,dm_\nu(y)\Big| \\
&\le \sup_{B\ni x}\frac{1}{m_\nu(B)}\int_B \big|(b(y)-b(x))f(y)+b(x)f_B-(bf)_B\big|\,dm_\nu(y) \\
&\le \sup_{B\ni x}\Big(\frac{1}{m_\nu(B)}\int_B |b(y)-b(x)|\,|f(y)| + \big|b(x)f_B-(bf)_B\big|\Big) \\
&\lesssim \|b\|_{\dot{\Lambda}_\beta} M_\beta f(x) + \sup_{B\ni x}\Big|\frac{b(x)}{m_\nu(B)}\int_B f(z)\,dm_\nu(z) - \frac{1}{m_\nu(B)}\int_B b(z)f(z)\,dm_\nu(z)\Big| \\
&\lesssim \|b\|_{\dot{\Lambda}_\beta} M_\beta f(x) + \sup_{B\ni x}\frac{1}{m_\nu(B)}\int_B |b(x)-b(z)||f(z)|\,dm_\nu(z) \\
&\lesssim \|b\|_{\dot{\Lambda}_\beta} M_\beta f(x).
\end{aligned}$$

Then, it follows from Theorem 3 that $[b, M_\nu^\sharp]$ is bounded from $L^\Phi(\mathbb{R}, dm_\nu)$ to $L^\Psi(\mathbb{R}, dm_\nu)$.

(2) $\Rightarrow$ (3). Assume $[b, M_\nu^\sharp]$ is bounded from $L^\Phi(\mathbb{R}, dm_\nu)$ to $L^\Psi(\mathbb{R}, dm_\nu)$, we will prove (5). For any fixed ball B, we have (see [4, page 3333] or [28, page 1383] for details)

$$M_\nu^\sharp(\chi_B)(x) = \frac{1}{2} \quad \text{for all} \quad x \in B.$$

Then, for all $x \in B$,

$$\begin{aligned} b(x) - 2M_\nu^\sharp(b\,\chi_B)(x) &= 2\Big(\frac{b(x)}{2} - M_\nu^\sharp(b\,\chi_B)(x)\Big) \\ &= 2\Big(b(x)M_\nu^\sharp(\chi_B)(x) - M_\nu^\sharp(b\,\chi_B)(x)\Big) \\ &= [b, M_\nu^\sharp](\chi_B)(x). \end{aligned}$$

Since $[b, M_\nu^\sharp]$ is bounded from $L^\Phi(\mathbb{R}, dm_\nu)$ to $L^\Psi(\mathbb{R}, dm_\nu)$, then by applying Lemma 1 and noting that $\Psi^{-1}(t) \approx \Phi^{-1}(t)t^{-\frac{\beta}{2\nu+2}}$, we have

$$\begin{aligned} &m_\nu(B)^{-\frac{\beta}{2\nu+2}}\,\Psi^{-1}\big(m_\nu(B)^{-1}\big)\,\big\|b(\cdot) - 2M_\nu^\sharp(b\,\chi_B)(\cdot)\big\|_{L^\Psi(B)} \\ &= 2m_\nu(B)^{-\frac{\beta}{2\nu+2}}\Psi^{-1}\big(m_\nu(B)^{-1}\big)\,\big\|[b, M_\nu^\sharp](\chi_B)\big\|_{L^\Psi(B)} \\ &\lesssim m_\nu(B)^{-\frac{\beta}{2\nu+2}}\Psi^{-1}\big(m_\nu(B)^{-1}\big)\,\big\|\chi_B\big\|_{L^\Phi} \lesssim 1. \end{aligned}$$

which implies (5).

(3) $\Rightarrow$ (4): We deduce (5) from (6). Assume (5) holds, then for any fixed ball B, it follows from Lemma 2 and (5) that

$$\begin{aligned} &m_\nu(B)^{-1-\frac{\beta}{2\nu+2}}\,\Big\|b(\cdot) - 2M_\nu^\sharp(b\,\chi_B)(\cdot)\Big\|_{L^1(B)} \\ &\le 2\,m_\nu(B)^{-\frac{\beta}{2\nu+2}}\,\Psi^{-1}\big(m_\nu(B)^{-1}\big)\big\|b(\cdot) - 2M_\nu^\sharp(b\,\chi_B)(\cdot)\big\|_{L^\Psi(B)} \le C\,, \end{aligned}$$

where the constant C is independent of B. So we obtain (6).

(4) $\Rightarrow$ (1). We first prove $b \in \dot{\Lambda}_\beta(\mathbb{R})$. For any fixed ball B, we have (see (2) in [4] for details)

$$\big|b_B\big| \le 2M_\nu^\sharp(b\,\chi_B)(x), \quad \text{for any} \quad x \in B. \tag{7}$$

For any ball B, let $E = \{y \in B : b(y) \le b_B\}$ and $F = \{y \in B : b(y) > b_B\}$. The following equality is true (see [4, page 3331]):

$$\int_E |b(y) - b_B|\; dm_\nu(y) = \int_F |b(y) - b_B|\; dm_\nu(y).$$

Since $b(y) \le b_B \le |b_B| \le 2M_\nu^\sharp\big(b\,\chi_B\big)(y)$ for any $y \in E$, we obtain

$$|b(y) - b_B| \le \big|b(y) - 2M_\nu^\sharp\big(b\,\chi_B\big)(y)\big|,\ y \in E.$$

Then we have

$$\begin{aligned}
&\frac{1}{m_\nu(B)^{1+\frac{\beta}{2\nu+2}}} \int_B |b(y) - b_B|\, dm_\nu(y) = \frac{2}{m_\nu(B)^{1+\frac{\beta}{2\nu+2}}} \int_E |b(y) - b_B|\, dm_\nu(y) \\
&\le \frac{2}{m_\nu(B)^{1+\frac{\beta}{2\nu+2}}} \int_E \big|b(y) - 2M_\nu^\sharp\big(b\,\chi_B\big)(y)\big|\, dm_\nu(y) \\
&\le \frac{2}{m_\nu(B)^{1+\frac{\beta}{2\nu+2}}} \int_B \big|b(y) - 2M_\nu^\sharp\big(b\,\chi_B\big)(y)\big|\, dm_\nu(y).
\end{aligned}$$

Applying Lemma 3 we get $b \in \dot{\Lambda}_\beta(\mathbb{R})$.

In order to prove $b \ge 0$, it suffices to show $b^- = 0$.

Then, for all $x \in B$,

$$2M_\nu^\sharp\big(b\,\chi_B\big)(x) - b(x) \ge \big|b_B\big| - b(x) = \big|b_B\big| - b^+(x) + b^-(x).$$

By (6), there exists a constant $C > 0$ such that for any ball B

$$\begin{aligned}
C &\ge \frac{1}{m_\nu(B)^{1+\frac{\beta}{2\nu+2}}} \int_B \big|b(y) - 2M_\nu^\sharp\big(b\,\chi_B\big)(y)\big|\, dm_\nu(y) \\
&\ge \frac{1}{m_\nu(B)^{1+\frac{\beta}{2\nu+2}}} \int_B \big(2M_\nu^\sharp\big(b\,\chi_B\big)(y) - b(y)\big)\, dm_\nu(y) \\
&\ge \frac{1}{m_\nu(B)^{1+\frac{\beta}{2\nu+2}}} \int_B \big(\big|b_B\big| - b^+(y) + b^-(y)\big)\, dm_\nu(y) \\
&= \frac{1}{m_\nu(B)^{\frac{\beta}{2\nu+2}}} \Big(\big|b_B\big| - \frac{1}{m_\nu(B)} \int_B b^+(y)\, dm_\nu(y) + \frac{1}{m_\nu(B)} \int_B b^-(y)\, dm_\nu(y)\Big).
\end{aligned}$$

This gives

$$\big|b_B\big| - \frac{1}{m_\nu(B)} \int_B b^+(y)\, dm_\nu(y) + \frac{1}{m_\nu(B)} \int_B b^-(y)\, dm_\nu(y) \le C m_\nu(B)^{\frac{\beta}{2\nu+2}} \quad (8)$$

for all balls B and the constant C is independent of B.

Let the radius of ball B tends to 0 (then $m_\nu(B) \to 0$) with $x \in B$, Lebesgue differentiation theorem assures that the limit of the left-hand side of (8) equals to

$$|b(x)| - b^+(x) + b^-(x) = 2b^-(x) = 2|b^-(x)|.$$

Moreover, the right-hand side of (8) tends to 0. Thus, we have $b^- = 0$. Then $b \in \dot{\Lambda}_\beta(\mathbb{R})$ and $b \geq 0$. The proof of Theorem 4 is completed. ◄

If we take $\Phi(t) = t^p$ and $\Psi(t) = t^q$ with $1 \leq p < \infty$ and $1 \leq q \leq \infty$ at Theorem 4, we have the following result.

Corollary 1. *Let $0 < \beta < 1$, $b \in L^1_{\mathrm{loc}}(\mathbb{R}, dm_\nu)$, b be a locally integrable function, $1 < p < q \leq \infty$ and $\frac{1}{p} - \frac{1}{q} = \frac{\beta}{2\nu+2}$. Then the following statements are equivalent:*

1. *$b \in \dot{\Lambda}_\beta(\mathbb{R})$ and $b \geq 0$.*
2. *$[b, M_\nu^\sharp]$ is bounded from $L^p(\mathbb{R}, dm_\nu)$ to $L^q(\mathbb{R}, dm_\nu)$.*
3. *There exists a constant $C > 0$ such that*

$$\sup_B \frac{1}{m_\nu(B)^{\frac{\beta}{2\nu+2}}} \left(\frac{1}{m_\nu(B)} \int_B \left| b(x) - 2M_\nu^\sharp\big(b\,\chi_B\big)(x) \right|^q \, dm_\nu(x) \right)^{1/q} \leq C.$$

4. *There exists a constant $C > 0$ such that*

$$\sup_B \frac{1}{m_\nu(B)^{1+\frac{\beta}{2\nu+2}}} \int_B \left| b(x) - 2M_\nu^\sharp\big(b\,\chi_B\big)(x) \right| \, dm_\nu(x) \leq C.$$

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Conflicts of interest

The author declare that there is no conflict of interest.

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