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Regularity Versus Compactness

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Abstract. A well–known result by Cohen and Dunford ([1], 1937) characterizes the class of all bounded linear operators from the space of all convergent complex sequences into itself. It follows that a regular matrix transformation cannot be compact. We use the theory of BK spaces and the Hausdorff measure of noncompactness to present a new proof for these results and establish their extensions to the spaces of strongly summable and strongly convergent sequences, and of convergent series.

Key Words and Phrases: bounded linear operators, BK spaces, Hausdorff measure of noncompactness, regular and compact operators.

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1. Introduction and Notations

Measures of noncompactness are very useful tools in functional analysis, for instance in metric fixed point theory and the theory of operator equations in Banach spaces. They can also be used in the characterization of classes of compact bounded operators between BK spaces by establishing identities or estimates for their Hausdorff measures of noncompactness. This approach was initiated on a large scale in [2], and later was also presented in detail in [3, 4].

In this paper, we demonstrate how the theory of BK spaces can be applied to obtain the characterizations of some classes of bounded linear operators between certain sequence spaces related to convergence and strong summability. We also obtain the operator norms in each case.

More precisely, we study spaces of sequences that are strongly C_1–summable with index $p \geq 1$, and of strongly convergent sequences denoted by w^p and $[c]$, establish representations for the bounded linear operators and their operator norms from c, w^p and $[c]$ into c, and from the space of convergent series into itself. Furthermore, we obtain estimates for the Hausdorff measure of noncompactness in each case, which yield the characterizations of the subclasses of compact operators. It is also shown that matrix transformations between those spaces that preserve the associated limits cannot be compact. These results include the special case by Cohen and Dunford [1].

We denote, as usual, by ω the set of all complex sequences $(x_k)_{k=1}^\infty$, and write ℓ_∞, c, c_0 and ϕ for the subsets of all bounded, convergent, null and finite sequences in ω, $\ell_p = \{x \in \omega : \sum_{k=1}^\infty |x_k|^p < \infty\}$ for $1 \le p < \infty$, $bv = \{x \in \omega : \sum_{k=1}^\infty |x_k - x_{k+1}| < \infty\}$ for the set of all sequences of bounded variation, $bv_0 = bv \cap c_0$, and bs and cs for the sets of all bounded and convergent complex series. Let $e = (e_k)_{k=1}^\infty$ and $e^{(n)} = (e_k^{(n)})_{k=1}^\infty$ for $n \in \mathbb{N}$ denote the sequences with $e_k = 1$ for all k, and $e_n^{(n)} = 1$ and $e_k^{(n)} = 0$ for $k \neq n$.

We recall that a BK space X is a Banach sequence space with the property that all the coordinates $P_n : X \to \mathbb{C}$ with $P_n(x) = x_n$ $(x = (x_k)_{k=1}^\infty \in X)$ are continuous; a BK space X is said to have AK if $x = \lim_{m\to\infty} x^{[m]}$ for all $x = (x_k)_{k=1}^\infty \in X$, where $x^{[m]} = \sum_{k=1}^m x_k e^{(k)}$ denotes the m–section of the sequence $x = (x_k)_{k=1}^\infty \in X$; X is said to have AD if ϕ is dense in X. Clearly, a BK space with AK also has AD.

The following results are well known.

Example 1. *(a) The sets ℓ_∞, c, c_0, ℓ_p for $1 \le p < \infty$, bs, cs, bv and bv_0 are BK spaces with their natural norms $\|x\|_\infty = \sup_k |x_k|$ for ℓ_∞, c and c_0, $\|x\|_p = (\sum_{k=1}^\infty |x_k|^p)^{1/p}$ for ℓ_p, $\|x\|_{bs} = \sup_n |\sum_{k=1}^n x_k|$ for bs and cs ([5, Example 7.3.1]), and $\|x\|_{bv} = \sum_{k=1}^\infty |x_k - x_{k+1}| + |\lim_{k\to\infty} x_k|$ for $x \in bv, bv_0$ ([5, 7.3.4]).*
(b) The spaces c_0, ℓ_p for $1 \le p < \infty$, and cs ([5, Example 4.2.14]) and bv_0 ([5, Theorem 7.3.5 (i)]) have AK; ℓ_∞ and bs have no Schauder basis, and $x = \xi e + \sum_{k=1}^\infty (x_k - \xi) e^{(k)}$ for every $x = (x_k)_{k=1}^\infty \in c$, where $\xi = \lim_{k\to\infty} x_k$.

Let X and Y be Banach spaces. Then we write, as usual, $\mathcal{B}(X,Y)$ for the Banach space of all bounded linear operators $L : X \to Y$ with the operator norm $\|L\| = \sup\{\|L(x)\| : \|x\| = 1\}$; if $Y = \mathbb{C}$, then $X^* = \mathcal{B}(X,\mathbb{C})$ denotes the continuous dual of X with the norm $\|f\| = \sup\{|f(x)| : \|x\| = 1\}$.

Let X and Y be subsets of ω. Then the β– and γ–duals of X are the sets

$$X^\beta = \{a \in \omega : a \cdot x = (a_k x_k)_{k=1}^\infty \in cs \text{ for all } x \in X\} \text{ and}$$
$$X^\gamma = \{a \in \omega : a \cdot x \in bs \text{ for all } x \in X\}.$$

Remark 1. *Obviously $X^\beta \subset X^\gamma$, and if $X \supset \phi$ is a BK space with AD, then also $X^\gamma \subset X^\beta$ by* [5, Theorem 7.2.7].

The following relations between the continuous and β–duals of a BK space are well known.

Proposition 1. ([5, Theorem 7.2.9]) *Let $X \supset \phi$ be a BK space. Then $X^\beta \subset X^*$; this means that there exists a linear one–to–one map $T : X^\beta \to X^*$. If X has AK, then T is onto.*

We list the continuous and β–duals of some sequence spaces.

Example 2. *We have*
(a) $\omega^\beta = \phi$, $\phi^\beta = \omega$, $c_0^\beta = c^\beta = \ell_\infty^\beta = \ell_1$, $\ell_1^\beta = \ell_\infty$, $\ell_p^\beta = \ell_q$ for $1 \le p < \infty$ and $q = p/(p-1)$; $cs^\beta = bv$, $bs^\beta = bv_0$ and $bv^\beta = cs$ ([5, Theorem 7.3.5 (v),(vi),(iii)]);
(b) if $X \in \{c_0, \ell_p\ (1 \le p < \infty)\}$, then X^β and X^* are norm isomorphic ([6, Examples 6.4.4, 6.4.3]);
(c) ([6, Example 6.4.5]) $f \in c^*$ if and only if there exist $b \in \mathbb{C}$ and a sequence $a \in \ell_1$ with $f(x) = b\xi + \sum_{k=1}^\infty a_k x_k$ for all $x \in c$, where $\xi = \lim_{k\to\infty} x_k$; moreover $\|f\| = |b| + \|a\|_1$.

Let $A = (a_{nk})_{n,k=1}^\infty$ be an infinite matrix of complex entries, $X, Y \subset \omega$ and x be a sequence. We write $A_n = (a_{nk})_{k=1}^\infty$ $(n \in \mathbb{N})$ for the sequence in the n^{th} row of A, $A_n x = \sum_{k=1}^\infty a_{nk} x_k$ for $n \in \mathbb{N}$ and $Ax = (A_n x)_{n=1}^\infty$ for the A–transform of x (provided all the series converge); $X_A = \{x \in \omega : Ax \in X \text{ for all } x \in X\}$ denotes the matrix domain of A in X, and (X, Y) is the class of all matrix transformations from X into Y, that is, $A \in (X, Y)$ if and only if $X \subset Y_A$, or equivalently, $A \in (X, Y)$ if and only if $A_n \in X^\beta$ for all $n \in \mathbb{N}$ and $Ax \in Y$ for all $x \in X$.

An infinite matrix $T = (t_{nk})_{k=1}^\infty$ is said to be a triangle if $t_{nk} = 0$ for $k > n$ and $t_{nn} \neq 0$ for all n.

Let $a \in \omega$ and X be a BK space. Then we write $\|a\|_X^* = \sup\{|\sum_{k=1}^\infty a_k x_k| : \|x\| = 1\}$, provided the expression of the right exists and is finite, which is the case by Proposition 1, whenever $a \in X^\beta$.

We need the following result.

Proposition 2. *The continuous dual of cs is norm isomorphic to bv.*

Proof. Let $f \in cs^*$ be given. Since $cs^\beta = bv$ by Example 2(a), it follows by Proposition 1, that there exists sequence $a \in bv$ such that $f(x) = \sum_{k=1}^\infty a_k x_k$ for all $x \in cs$, and we obtain $\|f\| = \|a\|_{cs}^* = \|a\|_{bv}$ by [7, (2.3)].

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We recall the following well–known relations betwen (X, Y) and $\mathcal{B}(X, Y)$.

Theorem 1. ([3, Theorem 9.3.3]) *Let X and Y be BK spaces.*
(a) Then $(X, Y) \subset \mathcal{B}(X, Y)$, that is, every $A \in (X, Y)$ defines an operator $L_A \in \mathcal{B}(X, Y)$, where

$$L_A(x) = Ax \text{ for all } x \in X. \tag{1}$$

(b) If X has AK then $\mathcal{B}(X,Y) \subset (X,Y)$, that is, for each $L \in \mathcal{B}(X,Y)$, there exists a matrix $A \in (X,Y)$ such that

$$Ax = L(x) \text{ for all } x \in X; \tag{2}$$

in this case we say that the matrix A represents the operator L.
(c) We have $A \in (X, \ell_\infty)$ if and only if

$$\|A\|_{(X,\ell_\infty)} = \sup_n \|A_n\|_X^* < \infty; \tag{3}$$

moreover, if $A \in (X,Y)$, where $Y \in \{c_0, c, \ell_\infty\}$, then

$$\|L_A\| = \|A\|_{(X,\ell_\infty)}. \tag{4}$$

Since cs is a BK space with AK, every bounded linear operator L from cs into a BK space Y is represented by a matrix $A \in (cs, Y)$ as in (2).

Proposition 3. *(a) We have $L \in \mathcal{B}(cs, \ell_\infty)$ if and only if*

$$\sup_n \sum_{k=1}^\infty |a_{nk} - a_{n,k+1}| < \infty \tag{5}$$

and

$$\sup_n \left| \lim_{k\to\infty} a_{nk} \right| < \infty; \tag{6}$$

moreover, if $L \in \mathcal{B}(cs, \ell_\infty)$ then

$$\|L\| = \|A\|_{(cs,\ell_\infty)} = \sup_n \left(\sum_{k=1}^\infty |a_{nk} - a_{n,k+1}| + \left| \lim_{k\to\infty} a_{nk} \right| \right). \tag{7}$$

(b) If $A = (a_{nk})_{n,k=1}^\infty$ is any infinite matrix, then we write $C = (c_{nk})_{n,k=1}^\infty$ for the matrix with $c_{nk} = \sum_{j=1}^k a_{jk}$ for all n and k. We have $L \in \mathcal{B}(cs, bs)$ if and only if (5) and (6) hold a_{nk} and $a_{n+1,k}$ replaced by c_{nk} and $c_{n+1,k}$.
(c) We have $L \in \mathcal{B}(cs, cs)$ if and only if $L \in \mathcal{B}(cs, bs)$ and

$$\sum_{n=1}^\infty a_{nk} \text{ converges for all } k. \tag{8}$$

(d) If $L \in \mathcal{B}(cs, bs)$ or $L \in \mathcal{B}(cs, cs)$, then

$$\|L\| = \|A\|_{(cs,bs)} = \|C\|_{(cs,\ell_\infty)}. \tag{9}$$

Proof. (a) Part (a) follows by Theorem 1(b) and (c), and Proposition 2.
(b),(d) Let $\Sigma = (\sigma_{n,k=1}^{\infty})_{n,k=1}^{\infty}$ be the triangle with the rows $\Sigma_n = e^{[n]}$ for all n, then $C = \Sigma \cdot A$, and Parts (b) and (d) follow from Part (a), since $bs = (\ell_\infty)_\Sigma$, and by [2, Theorem 3.8(a) and (b)] $A \in (cs, bs)$ if and only if $C \in (cs, \ell_\infty)$ and $\|A\|_{(cs,bs)} = \|C\|_{(cs,\ell_\infty)}$.
(c) We observe that c is a closed subspace of ℓ_∞ ([5, Example 4.2.6]) and so cs is a closed subspace of bs by [5, Theorem 4.3.14]. Part (c) now follows from Part (b), since, by [5, 8.3.6], $A \in (cs, cs)$ if and only if $A \in (cs, bs)$ and $Ae \in cs$, the latter condition being that in (8).

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Remark 2. *(a) The characterization of the class* (cs, ℓ_∞) *can be found in* [8, **3.** (2.2), (3.1))]*, where an alternative characterization is given by* [8, **3.** (3.2))] *which can also be found in* [5, Example 8.4.5B].
(b) The characterization of the class (cs, bs) *can be found in* [8, **34.** (33.1), (34.1))]*, where an alternative characterization is given by* [8, **34.** (34.2))] *which can also be found in* [5, Example 8.4.6B].

2. The Hausdorff Measure of Noncompactness

We recall the definition of the Hausdorff measure of noncompactness on the class of bounded sets in complete metric spaces.

Definition 1. *([2, Definition 2.10] or [3, Definition 7.7.1])* Let X be a complete metric space and $\mathcal{M}_X$ denote the class of all bounded subsets of X. Then the function $\chi : \mathcal{M}_X \to [0, \infty)$ with $\chi(Q) = \inf\{\varepsilon > 0 : Q \text{ has a finite } \varepsilon\text{–net in } X\}$ is called the *Hausdorff* or *ball measure of noncompactness.*

The following well–known result gives an estimate for the Hausdorff measure of noncompactness of bounded sets in Banach spaces with a Schauder basis.

Theorem 2 (Goldenštein–Goh'berg–Markus)**.** ([2, Theorem2.25] or [3, Theorem 7.9.3]) *Let X be a Banach space with a Schauder basis $(b_n)_{n=1}^{\infty}$ and the operator $\mathcal{R}_n : X \to X$ for each $n \in \mathbb{N}$ be defined by $\mathcal{R}_n(x) = \sum_{k=n+1}^{\infty} \lambda_k b_k$ for all $x = \sum_{k=1}^{\infty} \lambda_k b_k \in X$. We put $\mu(Q) = \limsup_{n\to\infty}(\sup_{x\in Q} \|\mathcal{R}_n(x)\|)$ for all $Q \in \mathcal{M}_X$. Then the following inequalities hold for all $Q \in \mathcal{M}_X$*

$$\frac{1}{a} \cdot \mu(Q) \leq \chi(Q) \leq \inf_n \left(\sup_{x\in Q} \|\mathcal{R}_n(x)\| \right) \leq \mu(Q), \tag{10}$$

where $a = \limsup_{n\to\infty} \|\mathcal{R}_n\|$ is the basis constant of the Schauder basis.

Remark 3. ([3, Remark 7.9.4]) *The following inequalities also hold instead of* (10) *in* Theorem 2

$$\frac{1}{a} \cdot \inf_n \left(\sup_{x \in Q} \|\mathcal{R}_n(x)\| \right) \leq \|L\|_\chi \leq \inf_n \left(\sup_{x \in Q} \|\mathcal{R}_n(x)\| \right). \tag{11}$$

The next result is an immediate consequence of Theorem 2

Corollary 1. *(a)* ([2, Theorem 2.15]) *Let X be any of the spaces ℓ_p for $1 \leq p < \infty$ or c_0. Then we have*

$$\chi(Q) = \lim_{n\to\infty} \left(\sup_{x \in Q} \|\mathcal{R}_n(x)\| \right) \text{ for all } Q \in \mathcal{M}_X. \tag{12}$$

(b) ([3, Example 7.9.7]) *If $X = c$, then the limit on the right hand side in* (12) *exists for all $Q \in \mathcal{M}_c$ and*

$$\frac{1}{2} \cdot \lim_{n\to\infty} \left(\sup_{x \in Q} \|\mathcal{R}_n(x)\| \right) \leq \chi(Q) \leq \lim_{n\to\infty} \left(\sup_{x \in Q} \|\mathcal{R}_n(x)\| \right) \text{for all } Q \in \mathcal{M}_c. \tag{13}$$

(c) If $X = cs$, then

$$\frac{1}{2} \cdot \limsup_{n\to\infty} \left(\sup_{x \in Q} \|\mathcal{R}_n(x)\| \right) \leq \chi(Q) \leq \limsup_{n\to\infty} \left(\sup_{x \in Q} \|\mathcal{R}_n(x)\| \right) \text{for all } Q \in \mathcal{M}_c. \tag{14}$$

Proof. We need only prove Part (c).
Let $n \in \mathbb{N}$ be given. Then we have for all $x \in cs$

$$\|\mathcal{R}_n(x)\|_{bs} = \sup_m \left| \sum_{k=n+1}^{m} x_k \right| \leq \|x\|_{bs} + \left| \sum_{k=1}^{n} x_k \right| \leq 2 \cdot \|x\|_{bs},$$

that is, $\|\mathcal{R}_n\| \leq 2$.
On the other hand, if we choose $x = e^{(n)} - 2 \cdot e^{(n+1)}$, then $\|x\|_{bs} = 1$ and $\|\mathcal{R}(x)_n\|_{bs} = 2 \leq \|\mathcal{R}_n\|$.
Thus we have shown $a = \lim_{n\to\infty} \|\mathcal{R}_n\| = 2$ and the inequalities in (14) follow immediately from (10).

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Remark 4. *The proof of he existence in* (12) *and* (13) *uses the property of the norms $\|\cdot\|_p$ and $\|\cdot\|_\infty$, namely that if $|x_k| \leq |y_k|$ for all k, then $\|x\|_p \leq \|y\|_p$ and $\|x\|_\infty \leq \|y\|_\infty$* (cf. [9, Lemma 1.10], [4, Theorem 5.16] *and* [3, Example 7.9.7])*. The norm $\|\cdot\|_{bs}$ on cs, however, does not have this property, as is easily seen by taking the sequences $x = 2(e^{(1)} + e^{(2)})$ and $y = 3(e^{(1)} - e^{(2)})$.*

Now we recall the definition of the Hausdorff measure of noncompactness of operators between Banach spaces.

Definition 2. *([2, Definition 2.24] or [3, Definition 7.11.1])* Let X and Y be Banach spaces and χ be the Hausdorff measure of noncompactness.
(a) An operator $L : X \to Y$ is said to be *χ–bounded*, if $L(Q) \in \mathcal{M}_Y$ for all $Q \in \mathcal{M}_X$, and if there exists a nonnegative real number c such that

$$\chi(L(Q)) \leq c \cdot \chi(Q) \text{ for all } Q \in \mathcal{M}_X. \tag{15}$$

(b) If an operator L is χ–bounded, then the number

$$\|L\|_\chi = \inf\{c \geq 0 : (15) \text{ holds}\} \tag{16}$$

is called the *Hausdorff measure of noncompactness of L.*

For the reader's convenience, we list some important properties of the Hausdorff measure of noncompactness of operators.

Let X and Y be Banach spaces and $\mathcal{K}(X, Y)$ denote the class of compact operators in $\mathcal{B}(X, Y)$.

Theorem 3. *Let X and Y be BK spaces and $L \in \mathcal{B}(X, Y)$. Then we have*
(a) ([2, Theorem 2.25] or [3, Theorem 7.11.4])

$$\|L\|_\chi = \chi(L(S_X)) = \chi(L(\bar{B}_X)) = \chi(L(B_X)), \tag{17}$$

where S_X, $\bar{B}_X$ and B_X are the unit sphere, the closed and open unit balls in X;
(b) ([2, Theorem 2.26] or [3, Theorem 7.11.5 (7.68)])

$$\|L\|_\chi = 0 \textit{ if and only if } L \in \mathcal{K}(X, Y). \tag{18}$$

Corollary 2. *We use the notations of* Proposition 3(b)*. Let $L \in \mathcal{B}(cs, cs)$. We put $M_m(cs, cs) = \sup_{n\geq m}(\sum_{k=1}^{\infty} |c_{nk} - c_{n,k+1}| + |\lim_{k\to\infty} c_{nk}|)$ for all $m \in \mathbb{N}$. Then*

$$\frac{1}{2} \cdot \limsup_{m\to\infty} M_m(cs, cs) \leq \|L\|_\chi \leq \limsup_{m\to\infty} M_m(cs, cs). \tag{19}$$

Furthermore, $L \in \mathcal{B}(cs, cs)$ is compact if and only if

$$\lim_{m\to\infty} M_m(cs, cs) = 0. \tag{20}$$

Proof. If A is an infinite matrix and $m \in \mathbb{N}$, then we write $A^{<m>}$ for the matrix with the first m rows replaced by the zero sequence. We obviously have $(\mathcal{R}_m \circ L)(x) = A^{<m>}x$ for all x, and so, by (9),

$$\sup_{x\in S_{cs}} \|(\mathcal{R}_m \circ L)(x)\|_{cs} = \|\mathcal{R}_m \circ L\|^*_{cs} = \left\|A^{<m>}\right\|_{(cs,bs)} = \left\|C^{<m>}\right\|_{(cs,\ell_\infty)}.$$

Now (19) follows by (7), (17) and (14).
Finally the condition in (20) for the compactness of L follows from (20) by (18).

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Remark 5. *Using* (11) *instead* (10) *we obtain*

$$\frac{1}{2}\cdot\limsup_{n\to\infty}\left(\sum_{k=1}^{\infty}|c_{nk}-c_{n,k+1}|+\left|\lim_{k\to\infty}c_{nk}\right|\right)\le\|L\|_{\chi}\le \limsup_{n\to\infty}\left(\sum_{k=1}^{\infty}|c_{nk}-c_{n,k+1}|+\left|\lim_{k\to\infty}c_{nk}\right|\right). \quad (21)$$

3. The Hausdorff Measure of Noncompactness of Some Operators

In this section, we establish some identities and estimates for the Hausdorff measure of noncompactness of operators from arbitrary BK spaces with AK into the spaces c_0 and c. We need the following general result.

Theorem 4. ([3, Theorem 9.8.4 (a), (b)]) *Let X and Y be BK spaces, X have AK, $L\in\mathcal{B}(X,Y)$ and A be the matrix that represents L as in* (2)*.*
(a) If $Y=c$, then we have

$$\frac{1}{2}\cdot\lim_{m\to\infty}\left(\sup_{n\ge m}\left\|\tilde{A}_n\right\|_X^*\right)\le\|L\|_{\chi}\le\lim_{m\to\infty}\left(\sup_{n\ge m}\left\|\tilde{A}_n\right\|_X^*\right), \quad (22)$$

where

$$\alpha_k=\lim_{n\to\infty}a_{nk}\ \textit{for all } k \quad (23)$$

and $\tilde{A}=(\tilde{a}_{nk})_{n,k=1}^{\infty}$ is the matrix with $\tilde{a}_{nk}=a_{nk}-\alpha_k$ for all n and k.
(b) If $Y=c_0$, then we have

$$\|L\|_{\chi}=\lim_{m\to\infty}\left(\sup_{n\ge m}\|A_n\|_X^*\right). \quad (24)$$

Proof. We write $\|\cdot\|^*=\|\cdot\|_X^*$, for short.
(a) Let $A=(a_{n,k})_{n,k=1}^{\infty}\in(X,c)$. Then $\|L\|=\|A\|_{(X,\ell_\infty)}<\infty$ by (3) in Theorem 1(c) and the limits α_k exist for all k by [5, 8.3.6].

(a.i) We show $(\alpha_k)_{k=1}^{\infty}\in X^{\beta}$.

Let $x \in X$ be given. Since X has AK, it is easy to see that there exists a positive constant C such that $\|x^{[m]}\| \leq C\|x\|$ for all $m \in \mathbb{N}$ and it follows that

$$\left|\sum_{k=1}^{m} a_{nk}x_k\right| = \left|A_n x^{[m]}\right| \leq C\|A_n\|^* \cdot \|x\| \leq C\|A\|_{(X,\ell_\infty)} \cdot \|x\| \text{ for all } n \text{ and } m,$$

hence by (23)

$$\left|\sum_{k=1}^{m} \alpha_k x_k\right| = \lim_{n\to\infty}\left|\sum_{k=1}^{m} a_{nk}x_k\right| \leq C\|A\|_{(X,\ell_\infty)} \cdot \|x\| \text{ for all } m. \tag{25}$$

This implies $(\alpha_k x_k)_{k=1}^\infty \in bs$. Since $x \in X$ was arbitrary, we conclude $(\alpha_k)_{k=1}^\infty \in X^\gamma$, and so $(\alpha_k)_{k=1}^\infty \in X^\beta$ by Remark 1, since X has AK and so AD.
Also $(\alpha_k)_{k=1}^\infty \in X^\beta$ implies $\|(\alpha_k)_{k=1}^\infty\|^* < \infty$ by Proposition 1.

(a.ii) Now we show

$$\lim_{n\to\infty} A_n x = \sum_{k=1}^{\infty} \alpha_k x_k \text{ for all } x \in X. \tag{26}$$

Let $x \in X$ and $\varepsilon > 0$ be given. Since X has AK, there exists $k_0 \in \mathbb{N}$ such that

$$\|x - x^{[k_0]}\| \leq \frac{\varepsilon}{2(M+1)}, \text{ where } M = \|A\|_{(X,\ell_\infty)} + \|(\alpha_k)_{k=1}^\infty\|^*. \tag{27}$$

It also follows from (23) that there exists $n_0 \in \mathbb{N}$ such that

$$\left|\sum_{k=1}^{k_0}(a_{nk} - \alpha_k)x_k\right| < \frac{\varepsilon}{2} \text{ for all } n \geq n_0. \tag{28}$$

Let $n \geq n_0$ be given, Then it follows from (27) and (28) that

$$\begin{aligned}\left|A_n x - \sum_{k=1}^{\infty}\alpha_k x_k\right| &\leq \left|\sum_{k=1}^{k_0}(a_{nk} - \alpha_k)x_k\right| + \left|\sum_{k=k_0+1}^{\infty}(a_{nk} - \alpha_k)x_k\right| \\ &< \frac{\varepsilon}{2} + \|A_m - (\alpha_k)_{k=1}^\infty\|^* \cdot \|x - x^{[k_0]}\| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.\end{aligned}$$

Thus we have shown (26).

(a.iii) Now we show the inequalities in (22).
Let $y = (y_n)_{n=1}^\infty \in c$ be given. Then, by Example 1(b), y has a unique representation $y = \eta e + \sum_{n=1}^\infty (y_n - \eta)e^{(n)}$, where $\eta = \lim_{n\to\infty} y_n$, and so $\mathcal{R}_m y = \sum_{n=m+1}^\infty (y_n - \eta)e^{(n)}$ for all m. We write $y_n = A_n x$ for $n = 1, 2, \ldots$, and obtain

$$\|\mathcal{R}_m(Ax)\|_\infty = \sup_{n\geq m+1}|y_n - \eta| = \sup_{n\geq m+1}\left|A_n x - \sum_{k=1}^{\infty}\alpha_k x_k\right| = \sup_{n\geq m+1}|\tilde{A}_n x|$$

whence $\sup_{x\in\bar{B}_X}\|\mathcal{R}_m(Ax)\|_\infty = \sup_{n\geq m+1}\|\tilde{A}_n\|^*$.
Now the inequalities in (22) follow from (17), Corollary 1(b) and (13).
Thus we have shown Part (a).
(b) Part (b) follows from (a) with $\alpha_k = 0$ for all k and since $\|a\| = \lim_{m\to\infty}\|\mathcal{R}_m\| = 1$ by (12) in Corollary 1(a).

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Remark 6. *By* (11)*, the term* $\lim_{n\to\infty}\sup_{n\geq m}\|\cdot\|_X^*$ *in* (22) *and* (23) *can be replaced by* $\limsup_{n\to\infty}\|\cdot\|_X^*$.

Example 3. *We write* $q=\infty$ *for* $p=1$ *and* $q=p/(p-1)$ *for* $1<p<\infty$*. Since* ℓ_p *has AK for* $1\leq p<\infty$ *by Example 1(a), we have by Example 2(a), (b), (22) and Remark 6*

$$\|\tilde{A}_n\|_{\ell_p}^* = \|\tilde{A}_n\|_q = \begin{cases} \sup\limits_k |a_{nk}-\alpha_k| & (p=1) \\ \left(\sum\limits_{k=1}^{\infty}|a_{nk}-\alpha_k|^q\right)^{1/q} & (1<p<\infty) \end{cases},$$

$$\frac{1}{2}\cdot\limsup_{n\to\infty}\left\|\tilde{A}_n\right\|_q \leq \|L\|_\chi \leq \limsup_{n\to\infty}\left\|\tilde{A}_n\right\|_q$$

and so $L\in\mathcal{K}(\ell_p,c)$ *by (18) if and only if* $\lim_{n\to\infty}\|\tilde{A}_n\|_q = 0$.

4. Regularity and Strong regularity

In this section, we use the Hausdorff measure of noncompactness to obtain the well–known result that a regular operator cannot be compact. We also establish the analogue of this result for series–to–series transformations.

Furthermore, we generalize the concept of regularity to two versions of *strong regularity* involving the concepts of strong summability and strong convergence to obtain similar results, namely that a *strongly regular matrix transformation cannot be compact*.

Throughout, let $1\leq p<\infty$. We use the convention that every term with a subsrcipt ≤ 0 is equal to zero.

The sets $w_0^p = \{x\in\omega : \lim_{n\to\infty}(1/n)\sum_{k=1}^n|x_k|^p = 0\}$, $w^p = \{x\in\omega : x-\xi e\in w_0^p$ for some $\xi\in\mathbb{C}\}$ and $w_\infty^p = \{x\in\omega : \sup_n(1/n)\sum_{k=1}^n|x_k|^p<\infty\}$ of sequences that are strongly summable to zero, strongly summable and strongly bounded, respectively, with index p by the Cesàro method of order 1 were defined and studied by *Maddox* [10].

The sets $[c]_0 = \{x\in\omega : \lim_{n\to\infty}(1/n)\sum_{k=1}^n|kx_k-(k-1)x_{k-1}| = 0\}$, $[c] = \{x\in\omega : x-\xi e\in[c_0]$ for some $\xi\in\mathbb{C}\}$ and $[c_\infty] = \{x\in\omega : \sup_n(1/n)\sum_{k=1}^n|kx_k-$

$(k-1)x_{k-1}| < \infty\}$ of all sequences that are strongly convergent to zero, strongly convergent and strongly bounded were studied by *Kuttner* and *Thorpe* [11].

Let X be any of the sets c, w^p for $1 \le p < \infty$ and $[c]$, and X_0 and X_∞ the corresponding sets c_0, w_0^p, $[c_0]$, and ℓ_∞, w_∞^p and $[c_\infty]$. If $x = (x_k)_{k=1}^\infty \in X$, then we write ξ_X for the *X–limit of* x, that is, for the unique complex number ξ_X with $\xi_c = \lim_{k\to\infty} x_k$, the usual limit, $\lim_{n\to\infty}(1/n)\sum_{k=1}^n |x_k - \xi_{w^p}|^p = 0$, the *$w^p$–limit*, and $\lim_{n\to\infty}(1/n)\sum_{k=1}^n |kx_k - (k-1)x_{k-1} - \xi_{[c]}| = 0$, the *$[c]$–limit*.

We write $\max_\nu$ and $\sum_\nu$ for the maximum and sum taken over all indices $k \in [2^\nu, 2^{\nu+1}-1]$ $(\nu = 0, 1, \dots)$ and put

$$\|x\|_{X_\infty} = \begin{cases} \sup\limits_k |x_k| & (x \in c_0, c, \ell_\infty) \\ \sup\limits_{\nu\in\mathbb{N}_0} \left(\dfrac{1}{2^\nu}\sum_\nu |x_k|^p\right)^{1/p} & (x \in w_0^p, w^p, w_\infty^p) \\ \sup\limits_\nu \left(\dfrac{1}{2^\nu}\sum_\nu |kx_k - (k-1)x_{k-1}|\right) & (x \in [c_0], [c], [c_\infty]). \end{cases}$$

and $\mathcal{X} = \{a \in \omega : \|a\|_{\mathcal{X}} < \infty\}$, where

$$\|a\|_{\mathcal{X}} = \begin{cases} \|a\|_1 & (x \in c_0, c, \ell_\infty) \\ \begin{cases} \sum\limits_{\nu=0}^\infty 2^\nu \max_\nu |a_k| \\ \qquad (p = 1) \\ \sum\limits_{\nu=0}^\infty 2^{\nu/p} \left(\sum_\nu |a_k|^q\right)^{1/q} \\ \qquad (1 < p < \infty;\ q = p/(p-1)) \end{cases} & (x \in w_0^p, w^p, w_\infty^p) \\ \sum\limits_{\nu=0}^\infty 2^\nu \max_\nu \left|\sum\limits_{j=k}^\infty \dfrac{a_j}{j}\right| & (x \in [c_0], [c], [c_\infty]). \end{cases}$$

The following results are well known (Examples 1 and 2 for $X_0 = c_0$, $X = c$ and $X_\infty = \ell_\infty$; [10], [2, Proposition 3.44] for $X_0 = w_0^p$, $X = w^p$ and $X_\infty = w_\infty^p$; [12, Theorem 2], [13, Theorem 2.2],[14, Theorem 2] for $X_0 = [c_0]$, $X = [c]$ and $X_\infty = [c_\infty]$).

Proposition 4. *(a) The sets X_0, X and X_∞ are BK spaces with respect to the norm $\|\cdot\|_{X_\infty}$; X_0 is a closed subspace of X and X is a closed subspace of X_∞; X_0 has AK, and every sequence $x = (x_k)_{k=1}^\infty \in X$ has a unique representation*

$$x = \xi e + \sum_{k=1}^\infty (x_k - \xi_X)e^{(k)},\ \textit{where } \xi_X \textit{ is the } X\textit{–limit of } x; \tag{29}$$

finally, X_∞ has no Schauder basis.
(b) We have $(X_0)^\beta = (X)^\beta = (X_\infty)^\beta = \mathcal{X}$; X_0^* is norm isomorphic to $(\mathcal{X}, \|\cdot\|_{\mathcal{X}})$; $f \in X^*$ if and only if there exist $b \in \mathbb{C}$ and a sequence $a = (a_k)_{k=1}^\infty \in \mathcal{X}$ such that

$$\begin{gathered} f(x) = \xi_X b + \sum_{k=1}^\infty a_k x_k \text{ for all } x \in w^p, \text{ where } \xi_X \text{ is the } X\text{–limit of } x, \\ a = (f(e^{(n)}))_{n=1}^\infty \text{ and } b = f(e) - \sum_{n=1}^\infty a_n; \end{gathered} \tag{30}$$

moreover

$$\|f\| = |b| + \|a\|_{\mathcal{X}} \text{ for all } f \in X^*; \tag{31}$$

finally, $\|a\|_X^* = \|a\|_{\mathcal{X}}$ for all $a \in X_\infty^\beta$.

Now we establish a representation of $L \in \mathcal{B}(X, c)$ and an estimate for $\|L\|_\chi$ when $X \in \{c, w^p, [c]\}$.

Theorem 5. ([3, Theorem 9.9.1] for $X = c$, [15, Theorem 6.3] for $X = w^p$ and [15, Theorem 6.13] for $X = [c]$)
Let $X \in \{c, w^p, [c]\}$. We write $L_n = P_n \circ L$ for all n.
(a) We have $L \in \mathcal{B}(X, c)$ if and only if there exists a sequence $b \in \ell_\infty$ and a matrix $A \in (X_0, c)$ such that

$$L(x) = b\xi_X + Ax \text{ for all } x \in X, \tag{32}$$

$$a_{nk} = L_n(e^{(k)}),\ b_n = L_n(e) - \sum_{k=1}^\infty a_{nk} \text{ for all } n \text{ and } k, \tag{33}$$

$$\beta = \lim_{n\to\infty} \left(b_n + \sum_{k=1}^\infty a_{nk} \right) \text{ exists,} \tag{34}$$

and

$$\|L\| = \sup_n \left(|b_n| + \|A_n\|_{\mathcal{X}} \right). \tag{35}$$

(b) If $L \in \mathcal{B}(X, c)$, then

$$\frac{1}{2} \cdot \limsup_{n\to\infty} \left(|\gamma_n| + \|\tilde{A}_n\|_{\mathcal{X}} \right) \le \|L\|_\chi \le \limsup_{n\to\infty} \left(|\gamma_n| + \|\tilde{A}_n\|_{\mathcal{X}} \right), \tag{36}$$

where

$$\lim_{n\to\infty} a_{nk} = \alpha_k \text{ for all } k \in \mathbb{N}, \tag{37}$$

$\tilde{A} = (\tilde{a}_{nk})_{n,k=1}^\infty$ is the matrix with $\tilde{a}_{nk} = a_{nk} - \alpha_k$ for all n and k, and

$$\gamma_n = b_n - \beta + \sum_{k=1}^\infty \alpha_k \text{ for all } n;$$

we also have

$$\lim_{n\to\infty} L_n(x) = \left(\beta - \sum_{k=1}^{\infty} \alpha_k\right)\xi + \sum_{k=1}^{\infty} \alpha_k x_k. \tag{38}$$

Proof. (a.i) First we assume that $L \in \mathcal{B}(X,c)$ and show that L has the given representation and satisfies (35).
We assume $L \in \mathcal{B}(X,c)$. Since c is a BK space, $L_n = P_n \circ L \in X^*$ for all $n \in \mathbb{N}$, that is, by (30) (with b and a_k replaced by b_n and a_{nk})

$$L_n(x) = b_n \cdot \xi_X + \sum_{k=1}^{\infty} a_{nk}x_k \text{ for all } x \in X, \tag{39}$$

with b_n and a_{nk} from (33); we also have by (31)

$$\|L_n\| = |b_n| + \|A_n\|_{\mathcal{X}} \text{ for all } n. \tag{40}$$

Now (39) yields the representation of the operator in (32).
Furthermore, since $L(x^{(0)}) = Ax^{(0)}$ for all $x^{(0)} \in X_0$, we have $A \in (X_0, c)$ and so by (3) and (4) $\|A\| = \sup_n \|A\|_X^* = \sup_n \|A_n\|_{\mathcal{X}} < \infty$. Also $L(e) = b + Ae$ by (32), and so $L(e) \in c$ yields (34), and we obtain $\|b\|_\infty \leq \|L(e)\|_\infty + \|A\| < \infty$, that is, $b \in \ell_\infty$. Consequently we have by (40) $\sup_n \|L_n\| = \sup_n (|b_n| + \|A\|) < \infty$. It is easy to see that $|\xi_X| \leq \|x\|_X$ for all $x \in X$, and we obtain by (39) and (40)

$$\begin{aligned}\|L(x)\|_\infty &= \sup_n \left| b_n\xi_X + \sum_{k=1}^{\infty} a_{nk}x_k \right| \\ &\leq \left[\sup_n (|b_n| + \|A_n\|_{\mathcal{X}})\right] \cdot \|x\|_\infty = \sup_n \|L_n\| \cdot \|x\|_\infty,\end{aligned}$$

hence $\|L\| \leq \sup_n \|L_n\|$. We also have $|L_n(x)| \leq \|L(x)\|_\infty \leq \|L\|$ for all $x \in S_X$ and all n, that is, $\sup_n \|L_n\| \leq \|L\|$, and we have shown (35).
(a.ii) Now we show that if L has the given representation, then $L \in \mathcal{B}(X,c)$.
We assume $A \in (c_0, c)$ and $b \in \ell_\infty$ and that the conditions in (32), (34) and (35) are satisfied. First $A \in (X_0, c)$ implies $\|A\| = \sup_n \|A_n\|_{\mathcal{X}} < \infty$ by (3) and Proposition 4(b). This and $b \in \ell_\infty$ yield $\|L\| < \infty$, hence $L \in \mathcal{B}(X, \ell_\infty)$. Also it follows from $A \in (X_0, c)$ that $A_n \in X_0^\beta$ for all n. Since $X_0^\beta = X^\beta$ by Proposition 29 (b) and $e \in X$, the series $\sum_{k=1}^{\infty} a_{nk}$ converges for each n. Therefore, if $x \in X$ is given and ξ_X is the X–limit of x, then $x - \xi_X \cdot e \in X_0$, and, by (32),

$$L_n(x) = b_n\xi_X + \sum_{k=1}^{\infty} a_{nk}x_k = \left(b_n + \sum_{k=1}^{\infty} a_{nk}\right) \cdot \xi_X + A_n(x - \xi_X) \text{ for all } n. \tag{41}$$

Now it follows from (34) and $A \in (X_0, c)$ that $\lim_{n\to\infty} L_n(x)$ exists. Since $x \in X$ was arbitrary, we have $L \in (\mathcal{B}(X, c)$.
(b) Now we show that if $L \in \mathcal{B}(X, c)$, then $\|L\|_\chi$ satisfies the inequalities in (36). We assume $L \in (X, c)$.
First we note that the limits β and α_k exist for all k by Part (a). Also, since $A \in (X_0, c)$ and X_0 is a BK space with AK by Proposition 4(b), it follows that $(\alpha_k)_{k=1}^\infty \in X_0^\beta = X^\beta$ by Part (a.i) of the proof of Theorem 4, and so $(\alpha_k)_{k=1}^\infty \in cs$. Therefore γ_n is defined for each n.
Now let $x \in X$ be given, and $y = L(x)$. It follows from (26) that

$$\lim_{n\to\infty} A_n(x - e\cdot\xi_X) = \sum_{k=1}^\infty \alpha_k(x_k - \xi_X) = \sum_{k=1}^\infty \alpha_k x_k - \xi_X \sum_{k=1}^\infty \alpha_k, \tag{42}$$

and so by (41), (34) and (42)

$$\begin{aligned}\eta = \lim_{n\to\infty} y_n &= \lim_{n\to\infty}\left[\xi_X\left(b_n + \sum_{k=1}^\infty a_{nk}\right) + A_n(x - \xi_X\cdot e)\right] \\ &= \xi_X\cdot\lim_{n\to\infty}\left(b_n + \sum_{k=1}^\infty a_{nk}\right) + \lim_{n\to\infty} A_n(x-\xi_X\cdot e) \\ &= \xi_X\cdot\beta + \sum_{k=1}^\infty \alpha_k x_k - \xi_X\sum_{k=1}^\infty \alpha_k = \xi_X\cdot\left(\beta - \sum_{k=1}^\infty \alpha_k\right) + \sum_{k=1}^\infty \alpha_k x_k,\end{aligned} \tag{43}$$

that is, we have shown (38).
For each m, we have $\mathcal{R}_m(y) = \sum_{n=m+1}^\infty (y_n - \eta)e^{(n)}$ for $y \in c$ and $\eta = \lim_{n\to\infty} y_n$. Writing $f_n^{(m)}(x) = (\mathcal{R}_m(L(x)))_n$, we obtain for $n \geq m+1$ by (41) and (43)

$$\begin{aligned}f_n^{(m)}(x) = y_n - \eta &= b_n\xi_X + A_n x - \left[\xi_X\left(\beta - \sum_{k=1}^\infty \alpha_k\right) + \sum_{k=1}^\infty \alpha_k x_k\right] \\ &= \xi_X\cdot\left(b_n - \beta + \sum_{k=1}^\infty \alpha_k\right) + \sum_{k=1}^\infty (a_{nk} - \alpha_k)x_k = \xi_X\gamma_n + \sum_{k=1}^\infty \tilde{a}_{nk}x_k.\end{aligned}$$

Since $f_n^{(m)} \in X^*$, we have by (31) that $\|f_n^{(m)}\| = |\gamma_n| + \|\tilde{A}_n\|_{\mathcal{X}}$, and it follows that $\sup_{x\in S_X}\|\mathcal{R}_m(L(x))|_\infty = \sup_{n\geq m+1}\|f_n^{(m)}\| = \sup_{n\geq m+1}(|\gamma_n| + \|\tilde{A}_n\|_X)$. Now the inequalities in (36) follow from (17) and (11) with $a = 2$.

◀

Corollary 3. *Let $X \in \{c, w^p, [c]\}$ and $L \in \mathcal{B}(X, c)$. Then $L \in \mathcal{K}(X, c)$ if and only if*

$$\lim_{n\to\infty}\left(|\gamma_n| + \|\tilde{A}_n\|_{\mathcal{X}}\right) = 0. \tag{44}$$

Proof. The condition in (44) follows from (36) by (18).

◄

As before, let $X \in \{c, w^p, [c]\}$. Then an operator $L \in \mathcal{B}(X, c)$ is said to be *X–regular*, if $\lim_{n\to\infty} L_n(x) = \xi_X$ for all $x \in X$, where ξ_X is the X–limit of x. A matrix $A \in (X, c)$ is said to be *X–regular*, if the operator L_A is X–regular. If $X = c$, then X–regularity is the usual regularity.

Corollary 4. *Let $L \in \mathcal{B}(X, c)$ be an X–regular transformation. Then L is compact if and only if*

$$\lim_{n\to\infty} \left(|b_n - 1| + \|A\|_{\mathcal{X}}\right) = 0. \tag{45}$$

Proof. If $L \in \mathcal{B}(X, c)$ is regular, then by (33) and (34)

$$\alpha_k = \lim_{n\to\infty} a_{nk} = \lim_{n\to\infty} L_n(e^{(k)}) = 0 \text{ for each } k \text{ and } \beta = \lim_{n\to\infty} L_n(e) = 1,$$

hence $\gamma_n = b_n - 1$ for all n, and so (45) follows from (44).

◄

Remark 7. *The case $X = c$ of* Corollary 4 *is the classical characterisation of the compact regular operators by Cohen and Dunford* [1]. *The condition in this case is* $\lim_{n\to\infty}(|b_n - 1| + \sum_{k=1}^{\infty} |a_{nk}|) = 0$.

Corollary 5. *Let $X \in \{c, w^p, [c]\}$ and $A \in (X, c)$. Then*

$$\frac{1}{2} \cdot \limsup_{n\to\infty} \left(|\beta - \sum_{k=1}^{\infty} \alpha_k| + \|\tilde{A}_n\|_{\mathcal{X}} \right) \leq \|L_A\|_{\chi} \leq \limsup_{n\to\infty} \left(|\beta - \sum_{k=1}^{\infty} \alpha_k| + \|\tilde{A}_n\|_{\mathcal{X}} \right), \tag{46}$$

where $\beta = \lim_{n\to\infty} \sum_{k=1}^{\infty} a_{nk}$, and $L_A \in \mathcal{K}(X, c)$ if and only if

$$\lim_{n\to\infty} \left(|\beta - \sum_{k=1}^{\infty} \alpha_k| + \|\tilde{A}_n\|_{\mathcal{X}} \right) = 0. \tag{47}$$

Proof. If $A \in (X, c)$, then $b_n = 0$ for all n, and the the conditions in (46) and (47) follow from (36) and (45).

◄

Remark 8. *(a) If $A \in (X,c)$ is X–regular, then $\beta = 1$ and $\alpha_k = 0$ for all k then $|\beta - \sum_{k=1}^{\infty} \alpha_k| = 1$, and so L_A cannot be X–regular by (47).*
(b) If $A \in (X,c)$ and L_A is compact, then it follows from (47) that

$$\lim_{n\to\infty} \sum_{k=1}^{\infty} a_{nk} - \sum_{k=1}^{\infty} \alpha_k = 0.$$

In the case $X = c$ this means that a compact conservative matrix A is conull.

Finally, we say that an operator $L \in \mathcal{B}(cs, cs)$ is (cs, cs)*–regular* if

$$\lim_{n\to\infty} \sum_{k=1}^{n} L_n(x) = \lim_{n\to\infty} \sum_{k=1}^{n} x_k \text{ for all } (x_k)_{k=1}^{\infty} \in cs.$$

and establish an analogous result to the case of $X = c$ in Remark 8(a).

Remark 9. *Let $L \in \mathcal{B}(cs, cs)$ and A be the matrix that represents L. Then L is (cs, cs)–regular if and only if $A \in (cs, cs)$ and*

$$\sum_{n=1}^{\infty} a_{nk} = 1 \textit{ for all } k. \tag{48}$$

Proof. (i) First, we show the sufficiency of the condition in (48).
We put $f(x) = \sum_{n=1}^{\infty} L_n(x) = \sum_{n=1}^{\infty} A_n x$. Since cs is has AK, we have by (48)

$$f(x) = f\left(\sum_{k=1}^{\infty} x_k e^{(k)}\right) = \sum_{k=1}^{\infty} f(e^{(k)} x_k) = \sum_{k=1}^{\infty}\sum_{n=1}^{\infty} a_{nk} x_k = \sum_{k=1}^{\infty} x_k.$$

(ii) Now, we show the necessity of the condition in (48).
If $f(x) = \sum_{k=1}^{\infty} x_k$ for all $x \in cs$, then we obtain for $x = e^{(j)}$

$$f(e^{(j)}) = \sum_{n=1}^{\infty} A_n e^{(j)} = \sum_{n=1}^{\infty} a_{nj} = \sum_{k=1}^{\infty} e_k^{(j)} = 1.$$

◀

The following result is the series–to–series version of a classical result by Cohen and Dunford [1] stated in the case of $X = c$ in Remark 8.

Corollary 6. *A (cs, cs)–regular operator cannot be compact.*

Proof. If $L \in \mathcal{B}(cs, cs)$, then we have for all r and all m by (48)

$$\sup_{n \ge r} \left| \sum_{j=1}^{n} a_{jm} \right| \ge \left| \sum_{j=0}^{\infty} a_{jm} \right| = 1 = \left| \lim_{m\to\infty} \sum_{j=0}^{\infty} a_{jm} \right|,$$

hence $M_r(cs, cs) \ge 1$ for all r by (19) and so $L \notin \mathcal{K}(cs, cs)$ by (20).

◀

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